Sensitivity Analysis of Risk Measures for Discrete Distributions

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Abstract

We consider the computation of quantiles and spectral risk measures for discrete distributions. This accounts for the empirical distributions of portfolio returns or outcomes of Monte Carlo simulations. We study the differentiability of quantiles with respect to portfolio allocation. We show that quantiles and spectral risk measures are piecewise linear with respect to portfolio allocation. We also provide differentiability conditions for a given allocation and relate the gradient to conditional expectations. Eventually, we extend our results to spectral or distortion risk measures.

The theory of risk measures is expanding quite quickly with contributions coming for the actuarial community, financial mathematicians and specialists of optimisation. For example, Artzner et al [1999], Delbaen [2000] introduce the concept of coherent measures of risk, which Acerbi [2002] further specializes to spectral measures of risk. On the insurance side Denneberg [1990], Wang et al [1997] introduce the closely related concept of distortion risk measures based on earlier work by Yaari [1987], Schmeidler [1986, 1989]. Amongst popular risk measures being considered by the financial community, one may quote the expected shortfall (see Pflug [2000], Acerbi et al [2001], Acerbi & Tasche [2001, 2002], Rockafellar & Uryasev [2000, 2002]). Chateauneuf et al [1996], Hodges [1998], Frittelli [2000], Föllmer & Schied [2002], have related such non linear approaches to pricing rules in markets with frictions. Amongst the actuarial community, the same kind of approaches have been applied to study the PH transform of Wang [1995], or other risk measures such as the absolute deviation of Denneberg [1990]. Wang & Young [1998],

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Wirch & Hardy [1999], Landsman & Sherris [2001], Dhaene et al [2003] provide other applications of the risk measure theory to insurance problems.

Among significant contributions for practitioners, Rockafellar & Uryasev [2000, 2002] have proposed a very fast algorithm for computing efficient frontiers under Expected Shortfall constraints. They exhibit an underlying piecewise linear structure that allows the use of linear programming techniques. The use of discrete distributions is a key aspect of the technique. On the other hand, while the sensitivity of VaR measures has been explored for continuous distributions, we do not have such results for discrete distributions. Discrete distributions are important for practical purpose since there are involved through empirical measures or outcomes of Monte Carlo simulations. For such discrete distributions, we show that quantiles are piecewise linear functions of portfolio allocations. This property is inherited by spectral risk measures or using the actuarial terminology by (concave) distortion risk measures. As a consequence, one may use linear programming techniques as was already suggested by Acerbi & Simonetti [2002]. In case of differentiability with respect to portfolio allocation, we show that the result stated by Gouriéroux, Laurent & Scaillet [2000], Tasche [2000] is still valid. Non differentiability is related to the possibility of multiple scenarios leading to the same portfolio value.

The paper is organized as follows. For the paper to be self-contained, the first section reviews some basic results about quantiles. The second section considers quantiles of discrete distributions. The third section studies quantiles of portfolio returns with applications towards sensitivity analysis with respect to portfolio allocation. The third section shows how the results can be extended to spectral risk measures (or distortion risk measures).

1 Quantiles

Let us start with some common definitions about quantiles.

Définition 1.1 Quantile set
Let us consider a real random variable X on a probabilistic space $(\Omega, \mathcal{A}, P)$ and $\alpha \in]0,1[$. The set:

$$Q_\alpha(X) = \{x \in \mathbb{R}, P(X < x) \leq \alpha \leq P(X \leq x)\}. \tag{1.1}$$

defines the α quantiles of X.

It can be readily checked that $Q_\alpha(X)$ is an interval[1]: let x_1 and x_2 be in $Q_\alpha(X)$, $x_1 < x_2$ and $y \in]x_1, x_2[$. Since $x_1 < y$, $P(X \leq x_1) \leq P(X \leq y)$, we get $\alpha \leq P(X \leq y)$. Since $y < x_2$, $P(X < y) \leq P(X < x_2)$ and since $P(X < x_2) \leq \alpha$, we get $P(X < y) \leq \alpha$. This shows that y is in $Q_\alpha(X)$. From the definition, it is clear that $Q_\alpha(X)$ depends on X only through its distribution (invariance in distribution). We can also immediately state some invariance with respect to location and scale: let $a \in \mathbb{R}$, then $Q_\alpha(X + a) = Q_\alpha(X) + a$, where $Q_\alpha(X) + a = \{x + a, x \in Q_\alpha(X)\}$; let $\lambda > 0$, then $Q_\alpha(\lambda X) = \lambda Q_\alpha(X)$, where $\lambda Q_\alpha(X) = \{\lambda x, x \in Q_\alpha(X)\}$.

[1] We will see later on that $Q_\alpha(X)$ is closed and non empty.

1.1 Higher quantile

Définition 1.2 higher quantile
Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space and $\alpha \in]0,1[$. For X being a real random variable defined on $(\Omega, \mathcal{A}, P)$,

$$q_\alpha^+(X) = \sup\{x \in \mathbb{R}, P(X < x) \leq \alpha\} \tag{1.2}$$

is the higher quantile of order α of X.

Let us check that the set $\{x \in \mathbb{R}, P(X < x) \leq \alpha\}$ is non empty and thus $q_\alpha^+(X)$ is well defined: the sets $X^{-1}(]-\infty, n[)$, $n \in \mathbb{N}$ are decreasing and $\cap_{n\in\mathbb{N}} X^{-1}(]-\infty, -n[) = \emptyset$. Thus, $\lim_{n\to\infty} P(X < -n) = 0$. As a consequence, $\exists n_0 \in \mathbb{N}$, $P(X < -n_0) \leq \alpha$. Let us now check that for $\alpha \in]0,1[$, $q_\alpha^+(X) \in \mathbb{R}$ or equivalently the set $\{x \in \mathbb{R}, P(X < x) \leq \alpha\}$ is upper bounded. If the set $\{x \in \mathbb{R}, P(X < x) \leq \alpha\}$ was not bounded, we could construct an increasing series $(x_n)_{n\in\mathbb{N}}$ converging to infinity, with $\forall n \in \mathbb{N}$, $P(X < x_n) \leq \alpha$. From the left continuity of $x \to P(X < x)$, we vould obtain $1 \leq \alpha$ and thus a contradiction. We can also notice that for $\alpha = 1$, we always have $q_\alpha^+(X) = \infty$.

The following proposition relates $q_\alpha^+(X)$ and the set of quantiles $Q_\alpha(X)$.

Proposition 1.1 *Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space and $\alpha \in]0,1[$. $q_\alpha^+(X)$ is an α quantile of X and is the right end of $Q_\alpha(X)$.*

Proof:
Let us now firstly that $q_\alpha^+(X)$ is in $Q_\alpha(X)$:

- let us firstly show that $\alpha \leq P\left(X \leq q_\alpha^+(X)\right)$. Let us assume that $\alpha > P\left(X \leq q_\alpha^+(X)\right)$. We consider a decreasing series $(x_n)_{n\in\mathbb{N}}$ converging to $q_\alpha^+(X)$. Since $x \to P(X \leq x)$ is right continuous, $P(X \leq x_n) \to P\left(X \leq q_\alpha^+(X)\right)$ and there exists n such that $\alpha > P(X \leq x_n)$. Then $\alpha > P(X < x_n)$ and since $x_n > q_\alpha^+(X)$, $q_\alpha^+(X)$ cannot be the higher quantile.
- let us now check that $P(X < q_\alpha^+(X)) \leq \alpha$. Let us consider an increasing series $(x_n)_{n\in\mathbb{N}}$ in the set $\{x \in \mathbb{R}, P(X < x) \leq \alpha\}$ converging towards $q_\alpha^+(X)$. Since $s \in \mathbb{R} \to P(X < s)$ is left continuous, we obtain $P(X < x_n) \to P(X < q_\alpha^+(X))$. Since $\forall n \in \mathbb{N}$, $P(X < x_n) \leq \alpha$, $\lim_{n\to\infty} P(X < x_n) \leq \alpha$. Eventually, $P(X < q_\alpha^+(X)) \leq \alpha$, which shows that $q_\alpha^+(X) \in Q_\alpha(X)$ and thus the set $Q_\alpha(X)$ is not empty.

Let us now show that $x \in Q_\alpha(X)$ implies that $x \leq q_\alpha^+(X)$.

- since x is an α quantile, we get $P(X < x) \leq \alpha$. From the definition of $q_\alpha^+(X)$, x is smaller than $q_\alpha^+(X)$. $q_\alpha^+(X)$ is then on the right of $Q_\alpha(X)$.

$q_\alpha^+(X)$ is then the right end of the quantile set $Q_\alpha(X)$. Since $Q_\alpha(X)$ is an interval, we have just shown that it was closed on the right. We can also provide an alternative characterization of the higher quantile that is often used:

Proposition 1.2 higher quantile
Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space, $\alpha \in]0,1[$ and X a random variable. Then,

$$q_\alpha^+(X) = \inf\{x \in \mathbb{R}, P(X \leq x) > \alpha\}. \tag{1.3}$$

Proof: let us denote $q^{\alpha}(X) = \inf\{x \in \mathbb{R}, P(X \leq x) > \alpha\}$.

- let us firstly show that $q^{\alpha}(X) \leq q_{\alpha}^{+}(X)$. Let us assume that $q^{\alpha}(X) > q_{\alpha}^{+}(X)$. Let $z \in]q_{\alpha}^{+}(X), q^{\alpha}(X)[$.
  - $z < q^{\alpha}(X)$. This implies $P(X \leq z) \leq \alpha$. Indeed, if $P(X \leq z) > \alpha$, we would have $q^{\alpha}(X) \leq z$, by definition of $q^{\alpha}(X)$.
  - $z > q_{\alpha}^{+}(X)$. We then get $P(X < z) > \alpha$. If $P(X < z) \leq \alpha$, we would have $z \leq q_{\alpha}^{+}(X)$ by definition of $q_{\alpha}^{+}(X)$. But $P(X \leq z) \geq P(X < z)$, then $P(X \leq z) > \alpha$. This provides the inconsistency we were looking for.
- let us now show that $q^{\alpha}(X) \geq q_{\alpha}^{+}(X)$. Let x_1 be such that $P(X < x_1) \leq \alpha$ and x_2 such that $P(X \leq x_2) > \alpha$. Since $P(X < x_1) < P(X \leq x_2)$, $x_1 \leq x_2$[2]. This shows $q_{\alpha}^{+}(X) \leq q^{\alpha}(X)$. ■

1.2 Characterization of higher quantile

Proposition 1.3 Characterization of higher quantile
Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space, $\alpha \in]0,1[$ and X a random variable. Let $s \in \mathbb{R}$. Then,

$$q_{\alpha}^{+}(X) \geq s \Leftrightarrow P(X < s) \leq \alpha.$$

Proof:

- $P(X < s) \leq \alpha$ implies $s \leq q_{\alpha}^{+}(X)$ since $q_{\alpha}^{+}(X) = \sup\{s \in \mathbb{R}, P(X < s) \leq \alpha\}$.
- let us assume that $s \leq q_{\alpha}^{+}(X)$. Then, $P(X < s) \leq P(X < q_{\alpha}^{+}(X))$. Since $q_{\alpha}^{+}(X)$ is a quantile, $P(X < q_{\alpha}^{+}(X)) \leq \alpha$, which shows that $P(X < s) \leq \alpha$). ■

We can remark that the previous proposition can also be written as: $q_{\alpha}^{+}(X) < s \Leftrightarrow P(X < s) > \alpha$. As a consequence, we readily obtain the following:

Corollary 1.1 Simulating X
Let U be a $[0,1]$ uniform random variable[3]. Then, the random variable $q_{U}^{+}(X)$ has the same distribution as X.

Proof: from the previous proposition, we can state, $P\left(q_{U}^{+}(X) < s\right) = P\left(U < P(X < s)\right) = P(X < s)$. $q_{U}^{-}(X)$ and X have the same distribution function and thus the same distribution. ■

1.3 Lower quantile

Définition 1.3 lower quantile
Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space and $\alpha \in]0,1[$. For X being a real random variable defined on $(\Omega, \mathcal{A}, P)$,

$$q_{\alpha}^{-}(X) = \inf\{x \in \mathbb{R}, P(X \leq x) \geq \alpha\} \tag{1.4}$$

is the lower quantile of order α of X

[2] Indeed if $x_1 > x_2$, then $]-\infty, x_2] \subset]-\infty, x_1]$, which implies $P(X \leq x_2) \leq P(X < x_1)$.
[3] U is not necessarily defined on the same probability space as X. Indeed, it may be that no uniform random variable can be defined on $(\Omega, \mathcal{A}, P)$.

Let us firstly check that for $\alpha \in]0,1[$, the set $\{x \in \mathbb{R}, P(X \leq x) \geq \alpha\}$ is not empty. From the basic properties of distribution functions, we get that $\lim_{x\to\infty} P(X \leq x) = 1$. As a consequence, there exists some $x \in \mathbb{R}$ such that $P(X \leq x) \geq \alpha$. Let us now check that the set $\{x \in \mathbb{R}, P(X \leq x) \geq \alpha\}$ is bounded from below that is $q_\alpha^-(X) \in \mathbb{R}$. If the previous set was not bounded from below, we could construct a decreasing to $-\infty$ series $(x_n)_{n\in\mathbb{N}}$ with $P(X \leq x_n) \geq \alpha$. From the right continuity of $x \to P(X \leq x)$, we would obtain $0 \geq \alpha$ and thus a contradiction. We can also notice that for $\alpha = 0$, we always have $q_\alpha^-(X) = -\infty$.

The following proposition provides an alternative characterization of the lower quantile.

Proposition 1.4 lower quantile
Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space, $\alpha \in]0,1[$ and X a random variable. Then,

$$q_\alpha^-(X) = \sup\{x \in \mathbb{R}, P(X < x) < \alpha\}. \tag{1.5}$$

While the definition provides a right approximation of the lower quantile, the latter expression provides a left approximation.

Proof: let us denote $q_\alpha = \sup\{x \in \mathbb{R}, P(X < x) < \alpha\}$.

- firstly, we show that $q_\alpha \leq q_\alpha^-(X)$. Let x_1 such that $P(X < x_1) < \alpha$ and x_2 such that $P(X \leq x_2) \geq \alpha$. Since $P(X < x_1) < P(X \leq x_2)$, $x_1 \leq x_2$. Then:

$$\sup_{\{x_1, P(X<x_1)<\alpha\}} x_1 \leq \inf_{\{x_2, P(X\leq x_2)\geq\alpha\}} x_2,$$

  which means $q_\alpha \leq q_\alpha^-(X)$.

- let us now show that $q_\alpha^-(X) \leq q_\alpha$. Let us assume $q_\alpha < q_\alpha^-(X)$. Let z be such that $q_\alpha < z < q_\alpha^-(X)$.
  - $z > q_\alpha$. We cannot have $P(X < z) < \alpha$, otherwise from the definition of q_α, $z \leq q_\alpha$. Thus, $P(X < z) \geq \alpha$.
  - $z < q_\alpha^-(X)$. We cannot have $P(X \leq z) \geq \alpha$ otherwise from the definition of $q_\alpha^-(X)$, $z \geq q_\alpha^-(X)$. Thus, $P(X \leq z) < \alpha$.

  we then get $P(X \leq z) < P(X < z)$, thus the required inconsistency. ■

We can now relate $q_\alpha^-(X)$ and the set of quantiles $Q_\alpha(X)$.

Proposition 1.5 *Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space, $\alpha \in]0,1[$ and X a random variable. $q_\alpha^-(X)$ is an α quantile of X and is the left end of $Q_\alpha(X)$.*

Proof: let us firstly check that $q_\alpha^-(X)$ is an α quantile. The starting point is $q_\alpha^-(X) = \inf\{x \in \mathbb{R}, P(X \leq x) \geq \alpha\}$. There exists a decreasing series $(x_n)_{n\in\mathbb{N}}$, such that $P(X \leq x_n) \geq \alpha$, $\forall n \in \mathbb{N}$ and converging towards $q_\alpha^-(X)$. By using right continuity of $x \to P(X \leq x)$, we get $P(X \leq q_\alpha^-(X)) \geq \alpha$. We now use $q_\alpha^-(X) = \sup\{x \in \mathbb{R}, P(X < x) < \alpha\}$. There exists an increasing series $(y_n)_{n\in\mathbb{N}}$ with $P(X < y_n) < \alpha$ converging towards $q_\alpha^-(X)$. Using left continuity of $x \to P(X < x)$, we get $P(X < q_\alpha^-(X)) \leq \alpha$. Eventually, $P(X < q_\alpha^-(X)) \leq \alpha \leq P(X \leq q_\alpha^-(X))$, shows that $q_\alpha^-(X) \in Q_\alpha(X)$. Now, let x be an α quantile of X. By definition of quantiles, we have $P(X \leq x) \geq \alpha$. By the definition of the lower quantile, $q_\alpha^-(X) = \inf\{x \in \mathbb{R}, P(X \leq x) \geq \alpha\}$, we get $x \geq q_\alpha^-(X)$, which shows that the lower quantile is the left end of the quantile set. Eventually, for $\alpha \in]0,1[$, we get $Q_\alpha(X) = [q_\alpha^-(X), q_\alpha^+(X)]$.

1.4 Characterization of lower quantile

Proposition 1.6 Characterization of lower quantile
Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space, $\alpha \in]0,1[$ and X a random variable. Let $s \in \mathbb{R}$. Then,

$$q_\alpha^-(X) \leq s \Leftrightarrow P(X \leq s) \geq \alpha.$$

Proof:

- $P(X \leq s) \geq \alpha$ implies $s \geq q_\alpha^-(X)$ since $q_\alpha^-(X) = \inf\{s \in \mathbb{R}, P(X \leq s) \geq \alpha\}$.
- let us assume that $s \geq q_\alpha^-(X)$. Then, $P(X \leq s) \geq P(X \leq q_\alpha^-(X))$. Since $q_\alpha^-(X)$ is a quantile, $P(X \leq q_\alpha^-(X)) \geq \alpha$, which shows $P(X \leq s) \geq \alpha$). ■

As for the higher quantile, we can simulate X from its lower quantile.

Corollary 1.2 simulating X
Let U be a uniform random variable on $[0,1]$. Then the random variable $q_U^-(X)$ has the same distribution as X.

Proof: From the previous proposition, $P\left(q_U^-(X) \leq s\right) = P\left(U \leq P(X \leq s)\right) = P(X \leq s)$. $q_U^-(X)$ and X have the same distribution function and thus the same distribution. ■

1.5 Quantiles of $-X$

Proposition 1.7 Quantiles of $-X$
Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space, $\alpha \in]0,1[$ and X a random variable. The α quantiles of $-X$ are the opposites of the $1-\alpha$ quantiles of X. If A is a subset of $\mathbb{R}$, let us denote by $-A = \{-x, x \in A\}$. Then,

$$Q_\alpha(-X) = -Q_{1-\alpha}(X),$$

Proof: Let x be an α quantile of $-X$. Then $P(-X < x) \leq \alpha \leq P(-X \leq x)$. We can write these two inequalities as:

$$1 - P(-X \geq x) \leq \alpha \leq 1 - P(-X > x),$$

or equivalently as: $1 - P(X \leq -x) \leq \alpha \leq 1 - P(X < -x)$, thus $P(X < -x) \leq 1 - \alpha \leq P(X \leq -x)$, which shows that $-x$ is an $1-\alpha$ quantile of X. We have indeed stated some equivalences, which shows the property ■

Corollary 1.3 *Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space, $\alpha \in]0,1[$ and X a random variable.*

$$q_\alpha^-(-X) = -q_{1-\alpha}^+(X), \quad q_\alpha^+(-X) = -q_{1-\alpha}^-(X).$$

Proof: let us recall that $Q_\alpha(-X) = [q_\alpha^-(-X), q_\alpha^+(-X)]$ and $Q_{1-\alpha}(X) = [q_{1-\alpha}^-(X), q_{1-\alpha}^+(X)]$[4]. Since $-Q_{1-\alpha}(X) = [-q_{1-\alpha}^+(X), -q_{1-\alpha}^-(X)]$, we obtain the required result, thanks to previous proposition.

[4]These two intervals may be singletons.

2 Quantiles for discrete distributions

We are now going to consider the case where X is a discrete random variable taking values $x_1, \dots, x_n$, $n \in \mathbb{N}$. To each x_i, we associate $p_i > 0$, with $\sum_{i=1}^n p_i = 1$. To ease notations, we assume that $x_1 \leq \ldots \leq x_n$. In case of equality, we choose an arbitrary ordering. We denote by $P_k = P(X \leq x_k) = \sum_{j, x_j \leq x_k} p_j$, for $k = 1, \dots, n$ the distribution function of X.

2.1 Quantile Characterization

Let us denote $A_k = [P_{k-1}, P_k[$ for $k = 2, \dots, n$[5] and $A_1 =]0, P_1[$. The sets $A_1, \dots, A_k, \dots, A_n$ form a partition of $]0,1[$, possibly with some empty sets. We consider the step function i defined over $]0,1[$ taking values in the set $\{1, \dots, n\}$ by:

$$\forall \alpha \in]0,1[,\ i(\alpha) = \sum_{1 \leq i \leq n} i 1_{A_i}(\alpha). \tag{2.6}$$

We also define the sets $A_1^*, \dots, A_k^*, \dots, A_n^*$ by: $A_k^* =]P_{k-1}, P_k]$ for $k = 2, \dots, n-1$ and $A_1^* =]0, p_1], A_n^* =]P_{n-1}, 1[$. The sets $A_1^*, \dots, A_k^*, \dots, A_n^*$ form a partition of $]0,1[$. We also consider the step function i^* defined on $]0,1[$, taking values in the set $\{1, \dots, n\}$ by:

$$\forall \alpha \in]0,1[,\ i^*(\alpha) = \sum_{1 \leq i \leq n} i 1_{A_i^*}(\alpha). \tag{2.7}$$

We can then state the following result for $\alpha \in]0,1[$:

$$Q_\alpha(X) = [x_{i^*(\alpha)}, x_{i(\alpha)}]. \tag{2.8}$$

Proof:

- we firstly set for $\alpha \in]0,1[$[6]:

$$\begin{cases} P(X \leq x_{i(\alpha)-1}) & \leq \alpha < \quad P(X \leq x_{i(\alpha)}), \\ P(X \leq x_{i^*(\alpha)-1}) & < \alpha \leq \quad P(X \leq x_{i^*(\alpha)}). \end{cases}$$

  By definition of functions i and i^* and because the sets A_k, $k = 1, \dots, n$ and A_k^*, $k = 1, \dots, n$ form partitions of $]0,1[$, we get $\alpha \in A_{i(\alpha)}$ and $\alpha \in A^*_{i^*(\alpha)}$. From the definition of $A_k = [P_{k-1}, P_k[$, $P_{i(\alpha)-1} \leq \alpha < P_{i(\alpha)}$. From the definition of $P_k = P(X \leq x_k)$, $P(X \leq x_{i(\alpha)-1}) \leq \alpha < P(X \leq x_{i(\alpha)})$. We treat the inequalities regarding i^* similarly. Let us also remark that as a consequence $x_{i(\alpha)-1} < x_{i(\alpha)}$. Indeed, if $x_{i(\alpha)-1} = x_{i(\alpha)}$, then we would have $P\left(X \leq x_{i(\alpha)}\right) < P\left(X \leq x_{i(\alpha)}\right)$. We can also state $x_{i^*(\alpha)-1} < x_{i^*(\alpha)}$. This allows to write $P\left(X < x_{i(\alpha)}\right) = P\left(X \leq x_{i(\alpha)-1}\right) + P\left(X \in]x_{i(\alpha)-1}, x_{i(\alpha)}[\right)$. The latter term being equal to zero, we get $P\left(X < x_{i(\alpha)}\right) = P\left(X \leq x_{i(\alpha)-1}\right)$. Similarly, we have $P\left(X < x_{i^*(\alpha)}\right) = P\left(X \leq x_{i^*(\alpha)-1}\right)$.

- let us show that $[x_{i^*(\alpha)}, x_{i(\alpha)}] \subset Q_\alpha(X)$. We write $P(X \leq x_{i^*(\alpha)-1}) = P(X < x_{i^*(\alpha)})$ and $P(X \leq x_{i(\alpha)-1}) = P(X < x_{i(\alpha)})$ and using the stated inequalities, we obtain:

[5] If $x_{k-1} = x_k$ then A_k is an empty set.

[6] If $i(\alpha) = 1$, we then define $P(X \leq x_{i(\alpha)-1}) = 0$, we treat the case $i^*(\alpha) = 1$ similarly by setting $P(X \leq x_{i^*(\alpha)-1}) = 0$.

$$\begin{cases} P(X < x_{i(\alpha)}) & \leq \alpha < & P(X \leq x_{i(\alpha)}), \\ P(X < x_{i^*(\alpha)}) & < \alpha \leq & P(X \leq x_{i^*(\alpha)}). \end{cases}$$

This shows that $x_{i(\alpha)}$ and $x_{i^*(\alpha)}$ are α quantiles of X. As a consequence $Q_\alpha(X)$ is not empty. Since $Q_\alpha(X)$ is convex, it also contains the interval $[x_{i^*(\alpha)}, x_{i(\alpha)}]$ (we are going to show in the next point that indeed, $x_{i^*(\alpha)} \leq x_{i(\alpha)}$).

- let us show now that $Q_\alpha(X) \subset [x_{i^*(\alpha)}, x_{i(\alpha)}]$. We recall that $Q_\alpha(X) = \{x \in \mathbb{R}, P(X < x) \leq \alpha \leq P(X \leq x)\}$. Let $x \in Q_\alpha(X)$[7]. We can write $P(X < x) \leq \alpha$; since $\alpha < P(X \leq x_{i(\alpha)})$, we get $P(X < x) < P(X \leq x_{i(\alpha)})$. This implies $x \leq x_{i(\alpha)}$. Indeed if we had $x_{i(\alpha)} < x$, then $P(X \leq x_{i(\alpha)}) \leq P(X \leq x)$, which would lead to $P(X < x) < P(X \leq x)$. By taking $x = x_{i^*(\alpha)}$, we have also that $x_{i^*(\alpha)} \leq x_{i(\alpha)}$, which was stated in the previous step.

  Let us now write $\alpha \leq P(X \leq x)$. We have $P(X \leq x_{i^*(\alpha)-1}) < \alpha \leq P(X \leq x)$. Since $P(X \leq x_{i^*(\alpha)-1}) = P(X < x_{i^*(\alpha)})$, we obtain $P(X < x_{i^*(\alpha)}) < P(X \leq x)$. This implies that $x_{i^*(\alpha)} \leq x$. Indeed, if we had $x < x_{i^*(\alpha)}$, then $P(X \leq x) \leq P(X < x_{i^*(\alpha)})$ would contradict $P(X < x_{i^*(\alpha)}) < P(X < x_{i^*(\alpha)})$. We have thus shown that $x \in [x_{i^*(\alpha)}, x_{i(\alpha)}]$, which shows $Q_\alpha(X) \subset [x_{i^*(\alpha)}, x_{i(\alpha)}]$.

2.2 Higher quantile

Since we already know that $Q_\alpha(X) = [q_\alpha^-(X), q_\alpha^+(X)]$, we can state the higher quantile for discrete distributions as:

$$q_\alpha^+(X) = x_{i(\alpha)} = \sum_{1 \leq i \leq n} x_i 1_{A_i}(\alpha). \tag{2.9}$$

Let us now give a further characterization of the higher quantile for discrete distributions. We firstly remark that higher (and lower) quantiles of a discrete distribution correspond to points with positive probability.

Proposition 2.8 higher quantile, discrete distribution
Let X be a random variable taking values among $x_i \in \mathbb{R}$, $i = 1, \ldots, n$. Then,

$$x_i = q_\alpha^+(X) \;\Leftrightarrow\; P(X < x_i) \leq \alpha < P(X \leq x_i). \tag{2.10}$$

Proof: we already know that $q_\alpha^+(X) = x_{i(\alpha)}$ and $P(X < x_{i(\alpha)}) \leq \alpha < P(X \leq x_{i(\alpha)})$, which shows the direct implication. Let us now show the converse. The set $E = \{x_i, i \in \{1, \ldots, n\},\ P(X < x_i) \leq \alpha < P(X \leq x_i)\}$ is non empty since it contains $x_{i(\alpha)}$. Let us show that E is a singleton. Let us assume that x_{i_1} and x_{i_2} are in E and $x_{i_1} < x_{i_2}$. This implies $P(X \leq x_{i_1}) \leq P(X < x_{i_2}) \leq \alpha$, where the second inequality comes from $x_{i_2} \in E$. Since x_{i_1} is also in E, we obtain $\alpha < P(X \leq x_{i_1})$, and thus the impossibility. ■

2.3 Lower quantile

Similarly to the higher quantile, we obtain the lower quantile for discrete distributions as:

$$q_\alpha^-(X) = x_{i^*(\alpha)} = \sum_{1 \leq i \leq n} x_i 1_{A_i^*}(\alpha). \tag{2.11}$$

We get a further characterization of the lower quantile, similar to that of the higher quantile:

[7] From the previous step, we know that $Q_\alpha(X)$ is not empty since $x_{i^*(\alpha)}$ and $x_{i(\alpha)}$ are in $Q_\alpha(X)$.

Proposition 2.9 lower quantile, discrete distribution
Let X be a random variable taking values among $x_i \in \mathbb{R}$, $i = 1, \ldots, n$. Then,

$$x_i = q_\alpha^-(X) \Leftrightarrow P(X < x_i) < \alpha \leq P(X \leq x_i). \tag{2.12}$$

Proof: we already know that $q_\alpha^+(X) = x_{i^*(\alpha)}$ and $P(X < x_{i^*(\alpha)}) < \alpha \leq P(X \leq x_{i^*(\alpha)})$, which shows the direct implication. Let us show the converse. The set $E = \{x_i, i \in \{1, \ldots, n\}, P(X < x_i) < \alpha \leq P(X \leq x_i)\}$ is not empty since it contains $x_{i^*(\alpha)}$. Let us show that E is a singleton. Let us assume that x_{i_1} and x_{i_2} are in E with $x_{i_1} < x_{i_2}$. Then, $P(X \leq x_{i_1}) \leq P(X < x_{i_2}) < \alpha$, where the second inequality comes from $x_{i_2} \in E$. Since x_{i_1} is also in E, we obtain $\alpha \leq P(X \leq x_{i_1})$, which contradicts the previous inequality. ■

3 Portfolio quantiles, discrete distributions

We consider a set of p assets and we denote by $x_1, \ldots, x_n$, where the x_i are in $\mathbb{R}^p$, the values taken by the asset returns. X will here denote the vector of returns and we set $P(X = x_i) = p_i > 0$ for $i = 1, \ldots, n$ the joint distribution of the returns. We denote by $a \in \mathbb{R}^p$ the portfolio allocation and by a' the transpose of a. Then, for a portfolio allocation a, the set of possible returns are $a'x_1, \ldots, a'x_n$. The portfolio distribution is such that: $P(a'X \leq a'x_i) = \sum_{j=1}^n p_j 1_{a'x_j \leq a'x_i}$. Let us remark that we may have $a'x_i = a'x_j$ for $i \neq j$. In this case, we talk of multiple scenarios. If there are no $j \in \{1, \ldots, n\}$, $j \neq i$, such that $a'x_i = a'x_j$, we will say that i is an isolated scenario (for portfolio allocation a). For $i \in \{1, \ldots, n\}$, we will further denote $x_i = (x_i^1, \ldots, x_i^p)$ where x_i^j, $j \in \{1, \ldots, p\}$ corresponds to the return of asset j in scenario i.

3.1 Scenarios associated with a given quantile

We recall that higher or lower quantiles of discrete distributions take their values in the support of the distribution. Let us firstly state a technical lemma:

Lemma 3.1 *Let $i \in \{1, \ldots, n\}$ be a scenario such that $a'x_i \neq q_\alpha^+(a'X)$. Then, there exists a neighbourhood $v(a)$ of a such that $\forall b \in v(a)$, $b'x_i \neq q_\alpha^+(b'X)$.*

Proof:

- let us firstly assume that $a'x_i < q_\alpha^+(a'X)$. We recall that there exists at least a portfolio value, that we denote here $a'x_{i_a(\alpha)}$ such that $q_\alpha^+(a'X) = a'x_{i_a(\alpha)}$[8]. Let us choose z in $]a'x_i, a'x_{i_a(\alpha)}[$, z not being a portfolio value[9]. Let us choose a neighbourhood of a, $v(a)$, such that $\forall b \in v(a)$, $b'x_i < z$[10]. This implies that on $v(a)$, $\{j \in 1, \ldots, n,\ b'x_j \leq b'x_i\} \subset \{j \in 1, \ldots, n,\ b'x_j < z\}$. Thus, for all b in $v(a)$, $P(b'X \leq b'x_i) \leq P(b'X < z)$. We now look at the set $\{j \in 1, \ldots, n,\ a'x_j < z\}$. Since z is not a portfolio value, for $j = 1, \ldots, n$, $a'x_j > z$ or $a'x_j < z$. We conclude the existence of a neighbourhood of a, $v^*(a)$ such that for all b in $v^*(a)$, $\{j \in 1, \ldots, n,\ b'x_j < z\} = \{j \in 1, \ldots, n,\ a'x_j < z\}$. Thus, for all $b \in v^*(a)$, $P(b'X < z) = P(a'X < z)$. Eventually, for b in $v(a) \cap v^*(a)$,

$$P(b'X \leq b'x_i) \leq P(a'X < z) \leq P(a'X < q_\alpha^+(a'X)) \leq \alpha,$$

[8] Several scenarios $i \in \{1, \ldots, n\}$ may correspond to that portfolio value.
[9] $\forall j = 1, \ \ldots, n, a'x_j \neq z$.
[10] This is possible since $\{a \in \mathbb{R}^n,\ a'x_i < z\}$ is open.

the latter inequality coming from proposition (2.8) characterizing the higher quantile. But, if $b'x_i = q^+_\alpha(b'X)$, always by the same proposition (2.8), $P(b'X \leq b'x_i) > \alpha$[11] ; the scenario i cannot be associated with a α quantile for any portfolio b taken in $v(a) \cap v^*(a)$.

- let us now show that $a'x_i > q^+_\alpha(a'X)$. Let us choose z in $]a'x_{i_a(\alpha)}, a'x_i[$, z not being a portfolio value. We take a neighbourhood of a, $v(a)$, such that $\forall b \in v(a)$, $b'x_i > z$[12]. Thus, on $v(a)$, $\{j \in 1,\dots,n,\ b'x_j \geq b'x_i\} \subset \{j \in 1,\dots,n,\ b'x_j > z\}$. Thus, for all b in $v(a)$, $P(b'X \geq b'x_i) \leq P(b'X > z)$. We now look at the set $\{j \in 1,\dots,n,\ a'x_j > z\}$. As before there exists a neighbourhood of a, $v^*(a)$ such that for all b in $v^*(a)$, $\{j \in 1,\dots,n,\ b'x_j > z\} = \{j \in 1,\dots,n,\ a'x_j > z\}$. Thus, for all $b \in v^*(a)$, $P(b'X > z) = P(a'X > z)$. Eventually, for all b in $v(a) \cap v^*(a)$,

$$P(b'X \geq b'x_i) \leq P(a'X > z) \leq P(a'X > q^+_\alpha(a'X)).$$

We can then state:

$$P(b'X < b'x_i) \geq P(a'X \leq q^+_\alpha(a'X)) > \alpha,$$

the second inequality coming from proposition (2.8) characterizing $q^+_\alpha(a'X)$. As $P(b'X < b'x_i) > \alpha$, $b'x_i$ cannot be an α quantile of $b'X$[13]. This shows the lemma.

Définition 3.4 *We denote by $I_a(\alpha) = \{i \in \{1,\dots,n\},\ a'x_i = q^+_\alpha(a'X)\}$ the scenarios $i \in \{1,\dots,n\}$ associated with the higher quantile $q^+_\alpha(a'X)$.*

We then state the following proposition:

Proposition 3.10 quantile differentiation, isolated scenarios
Let us assume that the α quantile $q^+_\alpha(a'X)$ is associated with a unique scenario, that is:

$$\#I_a(\alpha) = 1,$$

where $I_a(\alpha)$ is the set of scenarios associated with the quantile $q^+_\alpha(a'X)$. We denote by $i_a(\alpha)$ the corresponding scenario and $q^+_\alpha(a'X) = a'x_{i_a(\alpha)}$. Then, the α higher quantile is differentiable with respect to a and:

$$\frac{\partial q^+_\alpha(a'X)}{\partial a_j} = x^j_{i_a(\alpha)},\ j = 1,\dots,n. \tag{3.13}$$

Proof: let $j \in 1,\dots,n$, $j \neq i_a(\alpha)$. From the previous lemma, there exists a neighbourhood of a, $v_j(a)$ such that all b in $v_j(a)$, j is not associated with a higher α quantile of $b'X$. In this neighbourhood of a, $\cap_{j \neq i_a(\alpha)} v_j(a)$, the scenarios $j \neq i_a(\alpha)$ are not associated with higher α quantile of $b'X$. Since $q^+_\alpha(b'X)$ is associated with at least one scenario, it must be $i_a(\alpha)$. As a consequence, on this neighbourhood of a, $q^+_\alpha(b'X) = a'x_{i_a(\alpha)}$, which gives the stated result. ■

[11] Let us remark that if $b'x_i \geq q^+_\alpha(b'X)$, then $P(b'X \leq b'x_i) \geq P(b'X \leq q^+_\alpha(b'X)) > \alpha$ and we get a contradiction. Thus, $b'x_i < q^+_\alpha(b'X)$.

[12] This can be done since $\{a \in \mathbb{R}^n,\ a'x_i > z\}$ is open.

[13] Let us assume that $b'x_i \leq q^+_\alpha(b'X)$. Then $P(b'X < q^+_\alpha(b'X)) \geq P(b'X < b'x_i) \geq P(a'X \leq q^+_\alpha(a'X)) > \alpha$. Eventually, $P(b'X < q^+_\alpha(b'X)) > \alpha$, which is not possible from proposition (2.8). Thus, $b'x_i > q^+_\alpha(b'X)$.

The only cases where the quantile is not differentiable with respect to portfolio allocation thus correspond to multiple scenarios. The set of portfolios $a \in \mathbb{R}^n$ associated with multiple scenarios is included in $\cup_{i \neq j}\{a \in \mathbb{R}^n, a'(x_i - x_j) = 0\}$. This union of hyperplanes has zero Lebesgue measure by Fubini's theorem. Higher quantiles are then almost surely differentiable with respect to portfolio allocation in the case of discrete distributions. Let us also remark that, in the differentiability case, the partial derivatives are locally constant.

3.2 Continuity of higher quantiles

Let us further study how higher α quantiles depend on portfolio allocation. We firstly state a simple characterization of quantiles for discrete distributions. We recall that for $a \in \mathbb{R}^p$, $q_\alpha^+(a'X) = \sup\{x \in \mathbb{R}, P(a'X < x) \leq \alpha\}$ and $P(a'X < q_\alpha^+(a'X)) \leq \alpha$. On the other hand, for a discrete distribution, the quantiles correspond to portfolio values: $q_\alpha^+(a'X) \in \{a'x_i, i = 1, \dots, n\}$. Thus,

$$q_\alpha^+(a'X) = \sup\{a'x_i, i = 1, \dots, n,\ P(a'X < a'x_i) \leq \alpha\}. \tag{3.14}$$

Let us notice that the α higher quantile is the superior envelope of affine functions. We are now going to show the continuity of $q_\alpha^+(a'X)$ with respect to a.

Proposition 3.11 continuity of α quantiles
Let X be a p dimensional random vector following a discrete distribution and $\alpha \in]0,1[$. Then, $a \in \mathbb{R}^p \to q_\alpha^+(a'X)$ is continuous.

Proof: from the previous lemma, there exists a neighbourhood of a, $v(a)$, such that for all portfolios b in $v(a)$, the quantiles of $b'X$ are associated with scenarios in $I_a(\alpha)$. For $b \in \mathbb{R}^p$, we can then write:

$$q_\alpha^+(b'X) = \sup\{(a + (b-a))'x_i, i \in I_a(\alpha),\ P(b'X < b'x_i) \leq \alpha\},$$

where the set of scenarios i such that $i \in I_a(\alpha)$ and $P(b'X < b'x_i) \leq \alpha$ is non empty. For $i \in I_a(\alpha)$, $a'x_i = q_\alpha^+(a'X)$. We can then write:

$$q_\alpha^+(b'X) = q_\alpha^+(a'X) + \sup\{(b-a)'x_i, i \in I_a(\alpha),\ P(b'X < b'x_i) \leq \alpha\}. \tag{3.15}$$

We can write $\sup\{(b-a)'x_i, i \in I_a(\alpha),\ P(b'X < b'x_i) \leq \alpha\} \leq \sup\{(b-a)'x_i, i \in \{1, \dots, n\}\}$. By Cauchy-Schwarz inequality, $(b-a)'x_i \leq \parallel b-a \parallel \times \parallel x_i \parallel$. We can then state that for b in $v(a)$,

$$q_\alpha^+(b'X) - q_\alpha^+(a'X) \leq \parallel b-a \parallel \times \max_{i=1,\dots,n} \parallel x_i \parallel .$$

Equation (3.15) also allows to state:

$$q_\alpha^+(b'X) \geq q_\alpha^+(a'X) + \inf\{(b-a)'x_i,\ i = 1, \dots, n\}.$$

By Cauchy-Schwarz inequality, we get: $(b-a)'x_i \geq - \parallel b-a \parallel \times \parallel x_i \parallel$ and moreover, $- \parallel b-a \parallel \times \parallel x_i \parallel \geq - \parallel b-a \parallel \times \sup_{i=1,\dots,n} \parallel x_i \parallel$. Eventually, $\inf\{(b-a)'x_i,\ i = 1, \dots, n\} \geq - \parallel b-a \parallel \times \sup_{i=1,\dots,n} \parallel x_i \parallel$ and:

$$q_\alpha^+(b'X) - q_\alpha^+(a'X) \geq - \parallel b-a \parallel \times \max_{i=1,\dots,n} \parallel x_i \parallel .$$

We have then stated:

$$\mid q_\alpha^+(b'X) - q_\alpha^+(a'X) \leq \mid \parallel b-a \parallel \times \max_{i=1,\dots,n} \parallel x_i \parallel, \tag{3.16}$$

which shows that higher quantiles are continuous with respect to portfolio allocation.

3.3 Right and left differentiability of higher quantiles

We now study the general case where the α quantile of portfolio $a \in \mathbb{R}^p$ may not be associated with an isolated quantile.

Proposition 3.12 left and right differentiability of α quantiles
Let $a \in \mathbb{R}^p$ be a portfolio allocation and $q_\alpha^+(a'X)$ the corresponding quantile. Let $\delta \in \mathbb{R}^p$ be a modification in the portfolio allocation. $q_\alpha^+(a'X)$ is left and right differentiable in the direction δ, and the derivatives are in between $\min_{i\in I_a(\alpha)} \delta' x_i$ and $\max_{i\in I_a(\alpha)} \delta' x_i$, where $I_a(\alpha)$ is the set of scenarios associated with the quantile $q_\alpha^+(a'X)$.

In the special case of an isolated scenario, $\min_{i\in I_a(\alpha)} \delta' x_i = \max_{i\in I_a(\alpha)} \delta' x_i$ and we find back the stated differentiability result.

Proof: let $\varepsilon > 0$. We recall that $q_\alpha^+\left((a+\varepsilon\delta)'X\right) = \sup\{x \in \mathbb{R}, P\left((a+\varepsilon\delta)'X < x\right) \leq \alpha\}$ and that the sup is reached. On the other hand, for a discrete distribution, the quantiles belong to the support of the distribution, and there exists a scenario $i \in \{1, \ldots, n\}$ such that $q_\alpha^+\left((a+\varepsilon\delta)'X\right) = (a+\varepsilon\delta)'x_i$. We consider a scenario i that is not in $I_a(\alpha)$. In this case, $a'x_i \neq q_\alpha^+(a'X)$. If $a'x_i < q_\alpha^+(a'X)$, by going back to the proof of the previous lemma, there exists a neighbourhood of a, $v_i(a)$, such that for any portfolio b in $v_i(a)$, $b'x_i < q_\alpha^+(b'X)$. Similarly, if i is a scenario such that $a'x_i > q_\alpha^+(a'X)$, there exists a neighbourhood of a, $v(a)$, such that for any portfolio b in $v_i(a)$, $b'x_i > q_\alpha^+(b'X)$. Let us now look at the neighbourhood of a, $v(a)$, where $v(a) = \cup_{i\neq I_a(\alpha)} v_i(a)$. Let us choose ε such that $a+\varepsilon\delta$ is in $v(a)$. All the scenarios associated with $q_\alpha^+\left((a+\varepsilon\delta)'X\right)$ must be in $I_a(\alpha)$. We can then write the α quantile of portfolio $a+\varepsilon\delta$ as:

$$q_\alpha^+\left((a+\varepsilon\delta)'X\right) = \max\{(a+\varepsilon\delta)'x_i, i \in I_a(\alpha), P\left((a+\varepsilon\delta)'X < (a+\varepsilon\delta)'x_i\right) \leq \alpha\}.$$

Let $0 < \varepsilon < \inf_{j\notin I_a(\alpha)} \left|\frac{a'(x_j-x_i)}{\delta'(x_j-x_i)}\right|$ and $i \in I_a(\alpha)$. Then,

$$\{j\notin I_a(\alpha), (a+\varepsilon\delta)'x_j < (a+\varepsilon\delta)'x_i\} = \{j\notin I_a(\alpha), a'x_j < a'x_i\}.$$

On the other hand, $\{j \in I_a(\alpha), (a+\varepsilon\delta)'x_j < (a+\varepsilon\delta)'x_i\} = \{j \in I_a(\alpha), \delta'x_j < \delta'x_i\}$. Eventually,

$$\{j \in \{1, \ldots, n\}, (a+\varepsilon\delta)'x_j < (a+\varepsilon\delta)'x_i\} = \{j \in I_a(\alpha), \delta'x_j < \delta'x_i\} \cup \{j\notin I_a(\alpha), a'x_j < a'x_i\}$$

does not depend on ε. We conclude that for $\varepsilon > 0$ and small enough, $P\left((a+\varepsilon\delta)'X < (a+\varepsilon\delta)'x_i\right)$ does not depend on ε. We can then write:

$$\frac{q_\alpha^+\left((a+\varepsilon\delta)'X\right) - q_\alpha^+(a'X)}{\varepsilon} = \max\{\delta'x_i, i \in I_a(\alpha), P\left((a+\varepsilon\delta)'X < (a+\varepsilon\delta)'x_i\right) \leq \alpha\},$$

and this quantity does not depend on ε. This shows that $q_\alpha^+(a'X)$ is right differentiable in the direction δ and the derivative is locally constant. Similarly, $q_\alpha^+(a'X)$ is left differentiable in the direction δ and the derivative is locally constant and equal to:

$$\min\{\delta'x_i, i \in I_a(\alpha), P\left((a+\varepsilon\delta)'X < (a+\varepsilon\delta)'x_i\right) \leq \alpha\},$$

where:

$$\{j \in \{1, \ldots, n\}, (a+\varepsilon\delta)'x_j < (a+\varepsilon\delta)'x_i\} = \{j \in I_a(\alpha), \delta'x_j > \delta'x_i\} \cup \{j\notin I_a(\alpha), a'x_j < a'x_i\},$$

for $i \in I_a(\alpha)$ and $-\inf_{j \notin I_a(\alpha)} \left| \frac{a'(x_j - x_i)}{\delta'(x_j - x_i)} \right| < \varepsilon < 0$. Let us remark that the right and left derivatives are in the interval $\left[\min\{\delta' x_i, i \in I_a(\alpha)\}, \max\{\delta' x_i, i \in I_a(\alpha)\}\right]$. Differentiability with respect to direction δ means that $\delta' x_i$ does not depend on the scenario i in $I_a(\alpha)$. Differentiability in all directions then implies that all scenarios in $I_a(\alpha)$ are equal (that is we deal with an isolated scenario associated with the α quantile).

3.4 Quantile differentiation and conditional expectations

It is well known that for absolutely continuous distributions, quantile derivatives are related to conditional expectations of the asset returns. Under this assumption, from Gouriéroux, Laurent & Scaillet [2000], Tasche [2000]: $\dfrac{\partial q_\alpha^+(a'X)}{\partial a_j} = E\left[X^j \mid a'X = q_\alpha^+(a'X)\right]$, for $j = 1, \ldots, n$ where $X = (X^1, \ldots, X^p)$ denotes the random vector of asset returns.

We can now consider the case of discrete distributions. We have shown that differentiability only occurs when the quantile is associated with a unique scenario which we have denoted by $i_a(\alpha)$ for an allocation a and a risk level α. We can then readily check that $q_\alpha^+(a'X) = a'x_{i_a(\alpha)}$ implies that $X = x_{i_a(\alpha)}$ and then $E\left[X^j \mid a'X = q_\alpha^+(a'X)\right] = x^j_{i_a(\alpha)}$. As a consequence, the relationship between partial derivatives and conditional expectations still holds in the discrete case, provided that the quantile is associated with a unique scenario. In the case of multiple scenarios associated with $q_\alpha^+(a'X)$[14], we know that $q_\alpha^+(a'X)$ is not differentiable with respect to a. However, we can still compute:

$$E\left[X^j \mid a'X = q_\alpha^+(a'X)\right] = \frac{\sum_{i \in I_a(\alpha)} p_i x_i^j}{\sum_{i \in I_a(\alpha)} p_i},$$

which is between $\min_{i \in I_a(\alpha)} x_i^j$ and $\max_{i \in I_a(\alpha)} x_i^j$ corresponding to the stated bounds on left and right derivatives.

4 Risk Measures for Discrete Distributions

We thereafter consider risk measures ρ which be written as $\rho(X) = \int_0^1 q_\alpha^+(X) dF(\alpha)$, where F is a non decreasing function with $F(0) = 0$ and $F(1) = 1$. This set corresponds to the distortion risk measures and also includes the spectral risk measures. The purpose of this section is to provide some characterization of such risk measures for portfolios when the distribution of returns is discrete. We use the same notations as above. X will denote the vector of portfolio returns, $x_i \in \mathbb{R}^p$, $i = 1, \ldots, n$ the possible returns, where $P(X = x_i) = p_i > 0$ and $a \in \mathbb{R}^p$, the portfolio allocation. We denote by $(a'x)_{i:n}$, $i = 1, \ldots, n$ the sorted portfolios values; thus $(a'x)_{1:n} \leq \ldots \leq (a'x)_{n:n}$. As above, $P_k = P(a'X \leq (a'x)_{k:n}) = \sum_{j=1}^n p_j 1_{(a'x)_{j:n} \leq (a'x)_{k:n}}$ for $k = 1, \ldots, n$ where we omit the dependence in a for notational simplicity; $A_k = [P_{k-1}, P_k[$ for $k = 2, \ldots, n$, $A_1 =]0, P_1[$ and $i(\alpha) = \sum_{i=1}^n i 1_{A_i}(\alpha)$. For simplicity, we further denote $P_0 = 0$. We are then going to consider $\rho(a'X) = \int_0^1 q_\alpha^+(a'X) dF(\alpha)$.

Proposition 4.13 Risk measure representation

[14] That is $\#I_a(\alpha) > 1$.

Under the standing notations, we have:

$$\rho(a'X) = \sum_{i=1}^{n} (a'x)_{i:n} \left(F(P_i) - F(P_{i-1}) \right), \tag{4.17}$$

where $(a'x)_{i:n}$, $i = 1, \ldots, n$, *denotes the sorted portfolio returns associated with portfolio allocation* a, *i.e.* $(a'x)_{1:n} \leq \ldots \leq (a'x)_{n:n}$.

Proof: since the A_i, $i = 1, \ldots, n$ form a partition of $]0,1[$ and $q_\alpha^+(a'X) = (a'x)_{i(\alpha):n}$,

$$\rho(a'X) = \int_0^1 q_\alpha^+(a'X) dF(\alpha) = \sum_{i=1}^{n} \int_{A_i} (a'x)_{i(\alpha):n} dF(\alpha).$$

On A_i, $i(\alpha) = i$, thus $\rho(a'X) = \sum_{i=1}^n \int_{A_i} (a'x)_{i:n} dF(\alpha) = \sum_{i=1}^n \int_{P_{i-1}}^{P_i} (a'x)_{i:n} dF(\alpha)$, which provides the stated result ■

As can be seen from the previous proposition, the risk measure can be locally decomposed as a weighted average of sorted portfolio values (or quantiles). Moreover, it can be immediately checked that the P_i are order zero homogeneous in a, and thus the weights $F(P_{i-1}) - F(P_i)$ are also order zero homogeneous in a. Let us emphasize that in most practical applications (empirical measure, outcome of Monte Carlo simulations), we get $P(X = x_i) = p_i = \frac{1}{n}$, that is scenarios have equal probabilities. We can then state:

Proposition 4.14 Risk measure representation, uniform distributions
Under the standing notations, and assuming that $P(X = x_i) = \frac{1}{n}$, $i = 1, \ldots, n$, *we have:*

$$\rho(a'X) = \sum_{i=1}^{n} (a'x)_{i:n} \left(F\left(\frac{i}{n}\right) - F\left(\frac{i-1}{n}\right) \right), \tag{4.18}$$

where $(a'x)_{i:n}$, $i = 1, \ldots, n$, *denotes the sorted portfolio returns associated with portfolio allocation* a, *i.e.* $(a'x)_{1:n} \leq \ldots \leq (a'x)_{n:n}$.

Proof: we write,

$$\rho(a'X) = \int_0^1 q_\alpha^+(a'X) dF(\alpha) = \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} q_\alpha^+(a'X) dF(\alpha).$$

We now check that for $\alpha \in \left[\frac{i-1}{n}, \frac{i}{n}\right[$, $q_\alpha^+(a'X) = (a'x)_{i:n}$. From proposition (2.8), we need to check that:

$$P(a'X < (a'x)_{i:n}) \leq \alpha < P(a'X \leq (a'x)_{i:n}).$$

These inequalities hold since $P(a'X < (a'x)_{i:n}) \leq \frac{i-1}{n}$ and $P(a'X \leq (a'x)_{i:n}) \geq \frac{i}{n}$ (strict inequalities may occur in case of multiple scenarios) ■

Let us remark that the weights $F\left(\frac{i}{n}\right) - F\left(\frac{i-1}{n}\right)$ no longer depend on portfolio allocation a.

From now on, we will assume that all scenarios are isolated, which means that $(a'x)_{1:n} < \ldots < (a'x)_{n:n}$. We consider the differentiability of the risk measure ρ with respect to portfolio allocation. We denote by ζ_a the function defined on $\{1, \ldots, n\}$ by $(a'x)_{i:n} = a'x_{\zeta_a(i)}$; $\zeta_a(i)$ is the scenario associated with the rank i portfolio value. For isolated scenarios ζ_a is locally invariant on a. We can then write $\rho(a'X) =$

$\sum_{i=1}^{n} a' x_{\zeta_a(i)} \left(F(P_i) - F(P_{i-1})\right)$. For isolated scenarios, it results from the definition that the P_i, $i = 1, \dots, n$ are locally constant. This shows that the risk measure is locally linear in a. Moreover, we can state:

$$\frac{\partial \rho(a'X)}{\partial a_j} = \sum_{i=1}^{n} x^j_{\zeta_a(i)} \left(F(P_i) - F(P_{i-1})\right). \tag{4.19}$$

This shows that the risk measure sensitivity with respect to portfolio holdings in asset j is a weighted average of asset j returns. Eventually, using the relationship between quantile derivatives and conditional expectations and the decomposition of the risk measure ρ, we can also write:

$$\frac{\partial \rho(a'X)}{\partial a_j} = \sum_{i=1}^{n} E\left[X^j \mid a'X = (a'x)_{i:n}\right] \left(F(P_i) - F(P_{i-1})\right), \tag{4.20}$$

which is another way to state the risk measure derivative as a weighted average of asset j returns.

5 Conclusion

For discrete distributions of asset returns, quantiles and thus VaR risk measures are piecewise linear with respect to portfolio allocation. We can use this result for two purposes. Firstly, it should be easy to check whether for a given joint distribution of asset returns, VaR is sub-additive with respect to portfolio allocation. This results in solving linear programs. Secondly, while optimising under VaR constraints is usually a hard work, our results provide some alternative route. Not surprisingly, spectral risk measures, such as the Expected Shortfall being derived from quantiles, inherit the piecewise linear property with respect to portfolio allocation. Being also convex, optimising under spectral risk measures constraints leads to linear programs as was earlier noticed by Rockafellar & Uryasev [2000], Acerbi & Simonetti [2002]. Eventually, we can provide some invariant decomposition of risk measures for uniform probabilities, which is extremely well suited for most portfolio applications. We also provide some generalization of earlier work on sensitivity analysis of quantiles.

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