Pricing CDOs with state dependent stochastic recovery rates

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Pricing CDOs with state dependent stochastic recovery rates

- Outlook
- Practical context: surge in super senior tranche spreads
- Increase of risk for individual losses leads to increase of risk in aggregate losses
  - *For proper positive dependence*
- Consequences of previous analysis
- Comparing risks for granular portfolios sharing the same large portfolio limit
  - *Stochastic recovery rate versus recovery markdown*
- Numerical issues
**State dependent recovery rates**

- Practical context
  - *Calibration of super senior tranches during the liquidity and credit crisis*
    - Insurance against very large credit losses
    - [30-100] tranche on CDX starts to pay when (approximately) 50% of the 125 major companies in North America are in default
      - Contributed to the collapse of AIG
    - AIG reinsurer of major banks
      - Sold protection through AIG Financial Products (London) and Banque AIG (Paris)
      - Between 440 and 500 billion “CDS” outstanding
      - Issues with accounting, counterparty risk, collateral management and liquidity.
        - Large MTM losses
        - Though no insurance payments were to be made
State dependent recovery rates

- Market tsunami on AAA & AA Asset Backed Securities
  - Increase in spreads induced more damage than actual defaults
  - Prices patterns are quite informative for financial modelling
State dependent recovery rates

- High spreads on super senior tranches
  - Fixed 40% recovery rate assumption used to be market standard
State dependent recovery rates

- High spreads on super senior tranches
  - Could not be calibrated with the standard 40% recovery rate
  - [60-100] tranches traded at positive premiums ...

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Spread (bps)</th>
<th>Base Correlation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-3%</td>
<td>500, Upfront 68.51 points</td>
<td>39.45</td>
</tr>
<tr>
<td>3-7%</td>
<td>773.99</td>
<td>67.12</td>
</tr>
<tr>
<td>7-10%</td>
<td>435.52</td>
<td>72.58</td>
</tr>
<tr>
<td>10-15%</td>
<td>240.05</td>
<td>85.18</td>
</tr>
<tr>
<td>15-30%</td>
<td>126.50</td>
<td>-</td>
</tr>
<tr>
<td>30-100%</td>
<td>69.57</td>
<td>-</td>
</tr>
</tbody>
</table>

Source: Data from March 12, supplied by Markit
State dependent recovery rates

- Practical context
  - Steep “base correlations”
  - Implied dependence as measured by implied Gaussian copula correlation
  - Increases strongly with respect to attachment point
    - Reflecting “fat tails” in aggregate loss distributions
    - A bunch of issues of trading desks
      - Negative or increasing tranchelet prices
      - Delta scattering and weird idiosyncratic gamma
  - These issues are (partly) solved in a stochastic recovery rate approach
  - Main issue since 2008 for investment banks
State dependent recovery rates

- Theoretical context
  - Aggregate loss = sum of individual losses
  - Individual loss = default indicator times loss given default
  - Recovery rate = 1 – loss given default / credit notional
  - Recovery rates are stochastic

- Cross dependencies
  - Amongst default events (copula models, etc.)
  - Between default events and recovery rates
  - Amongst recovery rates

- Dependence through common latent factors
  - For convenience
State dependent recovery rates

When does an increase in individual risk leads to an increase in the risk on the aggregate portfolio (sum of individual risks)?

- (Multivariate) Gaussian risks
  - Individual risks with same expectation
  - Increase in risk = increase in variance
  - Increase in aggregate portfolio risk occurs if and only if pairwise correlations are non negative

- What about the general case?
  - Stochastic orders
    - Univariate case: convex order (close to second order stochastic dominance)
  - Positive dependence between individual risks
State dependent recovery rates

- Positive dependence
  - **MTP2**: *Multivariate Total Positivity of Order 2* (Karlin & Rinott (1980))
    - Log-density is supermodular
  - Conditionally Increasing
    - $X = (X_1, \ldots, X_n)$ is CI if and only if $E \left[ \phi(X_i | (X_j)_{j \in J}) \right]$ is increasing in $(X_j)_{j \in J}$ for increasing $\phi$
  - Positive association (Esary, Proschan & Walkup (1967))
  - PSMD: positive supermodular dependent

- Gaussian copula
  - Positive association = PSMD = positive pairwise correlations
  - MTP2 = CI (Müller & Scarsini (2001))
State dependent recovery rates

Theoretical context

- Non Gaussian framework
  - Individual risks have a probability mass at 0
- Increase of risk of individual risks: convex order
- Theorem (Müller & Scarsini (2001))
  - $X$ and $Y$ random vectors with common conditionally increasing copula
  - $X_i$ smaller than $Y_i$ for all $i$
  - Then $X$ smaller than $Y$ with respect to $d_{cx}$ (directionally convex) order
    - Then $X$ smaller than $Y$ with respect to stop-loss order

- Gaussian copula dependence
  - Conditionally increasing if and only if the inverse of covariance matrix is a $M$-matrix
  - $\Sigma$ non singular, entrywise non negative, $\Sigma^{-1}$ has positive non diagonal entries
State dependent recovery rates

- Dependence in large dimension
- Well known to finance people
- Factor models
  - Arbitrage pricing theory, asymptotic portfolios
    - Chamberlain & Rothschild (1983)
  - Large portfolio approximations (infinite granular portfolios)
    - Conditional law of large numbers
  - Qualitative data with spatial dependence
    - CreditRisk + (Binomial mixtures), CreditMetrics, Basel II (Gaussian copula)
  - Factor models may not be related to a causal view upon dependence
    - De Finetti, exchangeable sequences of Bernoulli variables are Binomial mixtures
    - Mixing random variable latent factor
State dependent recovery rates

- Spatial dependence with qualitative data
  - *Factor models have been used for long in other fields*
    - IQ tests (differential psychology), Bock & Lieberman (1970), Holland (1981)
    - Item Response Models

- Stochastic recovery rates
  - *Modeling of cross dependencies*
State dependent recovery rates

- Stochastic recovery rates
  - Modeling of cross dependencies
    - Individual loss = default indicator times loss given default
    - What is important for the computation of tranche premiums (or risk measures) is the joint distribution of individual losses
    - Direct approach: (discretized) individual loss seen as a polychotomous (or multinomial) variable
      - Multivariate Probit model (Krekel (2008))
      - Dual view of CreditMetrics (default side versus ratings)
  - Sequential models
    - Probit or logit models for default events (dichotomous model)
State dependent recovery rates

- Gaussian copula
  - *When is it conditionally increasing?*
  - *One factor case (positive betas)*
    - Gaussian copula is Conditionally Increasing (proof based on Holland & Rosenbaum (1986))
  - *Multifactor case: more intricate, even if all betas are positive, Gaussian copula may not be Conditionally Increasing*
    - Counterexamples
      - Gaussian copula with positive pairwise correlation
      - Increase of marginal risk (convex order)
      - May lead to a decrease of convex risk measures on aggregate portfolio
      - Constraints on conditional covariance matrix
  - *Hierarchical Gaussian copulas*
    - Conditionally Increasing copula (proof based upon Karlin & Rinott (1980))
State dependent recovery rates

Consequences of previous analysis

- Other examples of Conditionally Increasing copulas
- Archimedean copulas, Müller & Scarsini (2005)
- Dichotomous models with monotone unidimensional representation
  - Default indicators conditionally independent upon scalar $V$
  - Conditional default probabilities are non decreasing in $V$
  - Most known and used models
    - Includes additive factor copula models (Cousin & Laurent (2008)), such as generic one factor Lévy model of Albrecher et al. (2007).

Most portfolio credit risk models associated with CI
State dependent recovery rates

- Consequences of previous analysis
  - Non stochastic recovery rates
  - Analysis of a “recovery markdown”
  - Change recovery rate assumption from 40% to 30% (say)
  - Change marginal default probability so that expected loss unit is unchanged
  - Lemma: increase of marginal risk with respect to convex order

- Then, given a CI copula, increase of risk of the aggregate portfolio with respect to convex order
  - Increase in senior tranche premiums
  - Or CDO senior tranche spreads
State dependent recovery rates

- Consequences of previous analysis
  - *Stochastic recovery rate of Amraoui and Hitier (2008)*
  - *Depends only upon latent factor*
    - As in Altman et al. (JoB 2005)
  - *Specification of recovery rate is such that conditional upon latent factor is the same as in a recovery mark-down case*
  - *Same conditional expected losses*
    - Same large portfolio approximations
    - Same “infinitely granular” portfolios
    - When number of names tends to infinity, strong convergence of aggregate losses to large portfolio limits
- Stochastic recovery rate (AH) versus recovery markdown
  - *Same infinitely granular portfolios*
  - *But finitely granular portfolios behave (slightly) differently*
State dependent recovery rates

- Stochastic recovery rate (AH) vs recovery markdown
  - Main comparison result
  - Aggregate losses are ordered with respect to convex order
  - Smaller risks in stochastic recovery rate specification
  - Smaller spreads on senior tranches
  - Small numerical discrepancies

- Ongoing risk management and theoretical issues
  - Spot recovery versus time to recovery
    - Bennani & Maetz (2009), Li (2000)
  - Risk management for distressed names in a stochastic recovery rate framework
    - Off the run series, bespoke portfolios
State dependent recovery rates

- Numerical issues
  - Computational efficiency
    - Especially important when computing Greeks and risk managing CDOs
  - Needs to be reassessed in case of stochastic recovery models
  - Analytical computations of conditional moments
    - Gram Charlier expansions
    - Same low order approximation than Stein’s method
    - Much quicker than recursions and Monte Carlo