SENSITIVITY ANALYSIS OF VALUES AT RISK

Christian GOURIEROUX

CREST and CEPREMAP

Jean-Paul LAURENT

ISFA, University of Lyon and CREST

Olivier SCAILLET

IAG/Economics Dpt - UCL

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[1] Paper published in the **Journal of Empirical Finance**, Vol.7 (3-4), 225-245, 2000. We will also use some results from *Nonparametric estimation and sensitivity analysis of expected shortfall* by O. Scaillet.

Agenda

1. **Motivation**
    - Risk Management / Portfolio Selection
2. **Main Theoretical Results**
    - Analytical Expressions for 1st and 2nd Derivatives of Value at Risk (VaR) w.r.t. Portfolio Allocation
    - Analytical Expressions of first derivatives of expected shortfall
3. **Some Examples**
    - Gaussian Distribution with/without Unobserved Heterogeneity
4. **Application**
    - VaR Efficient Portfolios
5. **Estimation Procedure**
    - Nonparametric Approach (Kernel)
6. **Empirical Illustration**
    - French Stock Data

1. Motivation

- Framework
  - regulatory environment
  - proprietary Risk Measurement Models
  - VaR = **Synthetic** Measure of Risk Portfolio

- *Risk Management*
  - risk analysis
  - risk control
  - portfolio optimization

- Need to go <u>further</u> than a <u>single</u> number

- What are the *risk drivers* ?
  1. VaR of subportfolios
     - ex-post analysis
     - aggregation issues
     - VaR is neigher additive or subadditive
  2. Statistical analysis of *risk scenarios*
  3. Incremental VaR (**sensitivity analysis**)
     - ex-ante
     - management of risk limits
     - Large Portolios preclude online computations
     - $\Longrightarrow$ **Sensitivity** of VaR w.r.t. portfolio allocation

- Why determine convexity of VaR w.r.t. portfolio allocation ?
- *Portfolio Selection under VaR Constraints*
  - Determination of Optimal Portfolios
  - Need to check that the set of portfolios satisfying VaR constraint is convex
- VaR may not be subadditive (i.e. a *coherent* measure of risk)
  - No bonus for diversification
  - Poor internal risk management
  - subadditivity can be statistically tested ...
  - and depends on financial intermediaries policies ...
    - Credit risk,
    - Out of the money options

2. Main Theoretical Results

- **VaR definition:**
  - n Financial Assets with Prices $p_{i,t}$ at Date t
  - Portfolio Value : $W_t(a) = \sum_{i=1}^{n} a_i p_{i,t} = a' p_t$
  - $P_t \left[W_{t+1}(a) - W_t(a) + VaR_t(a, \alpha) < 0 \right] = \alpha$
  - Upper Quantile $(1 - \alpha)$...
    - since with $y_{t+1} = p_{t+1} - p_t$
    - $P_t \left[-a' y_{t+1} > VaR_t(a, \alpha) \right] = \alpha$

- **First properties**
  - VaR is degree one positively homogeneous: $\lambda \geq 0$
  - $VaR_t(\lambda a, \alpha) = \lambda \times VaR_t(a, \alpha)$
  - $a' \dfrac{\partial VaR_t(a, \alpha)}{\partial a} = VaR_t(a, \alpha)$ (Euler)
  - Risk contribution of i: $a_i \times \dfrac{\partial VaR_t(a, \alpha)}{\partial a_i}$
  - VaR is **not always** subadditive:
  - $VaR_t(a + b, \alpha) \leq VaR_t(a, \alpha) + VaR_t(b, \alpha)$?
  - subadditivity + positive homogeneity $\Rightarrow$ convexity of $VaR(a, \alpha)$ (w.r.t portfolio allocation a)

- **Sensitivities of VaR**

i) 1st derivative of VaR

$$\frac{\partial VaR_t(a,\alpha)}{\partial a} = -E_t[y_{t+1}|a'y_{t+1} = -VaR_t(a,\alpha)]$$

ii) 2nd derivative of VaR

$$\begin{aligned} \frac{\partial^2 VaR_t(a,\alpha)}{\partial a \partial a'} &= \frac{\partial \log g_{a,t}}{\partial z}(-VaR_t(a,\alpha))V_t[y_{t+1}|a'y_{t+1} = -VaR_t(a,\alpha)] \\ &- \left\{\frac{\partial}{\partial z}V_t[y_{t+1}|a'y_{t+1} = -z]\right\}_{z=VaR_t(a,\alpha)}, \end{aligned}$$

with $g_{a,t}$ the conditional p.d.f of $a'y_{t+1}$

- **Expected Shortfall definition**:

  – $m_t(a, \alpha) = E_t[-a'y_{t+1} \mid -a'y_{t+1} > Var_t(a, \alpha)]$

  – Also known as *Mean Excess Loss*, as *TailVaR* or as *Lower Partial Moment.*

  – Another commonly used measure of risk

  – Expected Shortfall is subadditive

i) 1st derivative of Expected Shortfall

$$\frac{\partial m_t(a, \alpha)}{\partial a} = -E_t[y_{t+1}| - a'y_{t+1} > VaR_t(a, \alpha)]$$

Similar result to that obtained on sensitivity of VaR

3. Some Examples

⇒ <u>First example</u>: Gaussian Distribution

- $y_{t+1} \sim N(\mu_t, \Omega_t)$
- $VaR_t(a, \alpha) = -a'\mu_t + (a'\Omega_t a)^{1/2} z_{1-\alpha}$
- $z_{1-\alpha}$ quantile of level $1 - \alpha$ of Gaussian distribution.

Remark:

- $(a'\Omega_t a)^{1/2}$: standard dev. of portfolio absolute returns
- subadditivity of standard deviations:
- $((a+b)'\Omega_t(a+b))^{1/2} \leq (a'\Omega_t a)^{1/2} + (b'\Omega_t b)^{1/2}$
- ⇒ In the Gaussian case, VaR is always subadditive

i) 1st derivative of VaR

$$\begin{aligned}
\frac{\partial VaR_t(a,\alpha)}{\partial a} &= -\mu_t + \frac{\Omega_t a}{(a'\Omega_t a)^{1/2}} z_{1-\alpha} \\
&= -\mu_t + \frac{\Omega_t a}{a'\Omega_t a}(VaR_t(a,\alpha) + a'\mu_t) \\
&= -E_t\left[y_{t+1} | a'y_{t+1} = -VaR_t(a,\alpha)\right]
\end{aligned}$$

ii) 2nd derivative of VaR

$$\frac{\partial^2 VaR_t(a,\alpha)}{\partial a \partial a'} = \frac{z_{1-\alpha}}{(a'\Omega_t a)^{1/2}} \left[\Omega_t - \frac{\Omega_t a a' \Omega_t}{a'\Omega_t a} \right]$$

$$\frac{\partial^2 VaR_t(a,\alpha)}{\partial a \partial a'} = \frac{z_{1-\alpha}}{(a'\Omega_t a)^{1/2}} V_t[y_{t+1} | a'y_{t+1} = -VaR_t(a,\alpha)]$$

Remark :

$$\frac{\partial \log g_{a,t}}{\partial z}(-VaR_t(a,\alpha)) = \frac{VaR_t(a,\alpha) + a'\mu_t}{a'\Omega_t a} = \frac{z_{1-\alpha}}{(a'\Omega_t a)^{1/2}}$$

2nd term = 0 (conditional homoscedasticity)

- Expected Shorfall $m_t(a,\alpha)$
- $m_t(a,\alpha) = -a'\mu_t + (a'\Omega_t a)^{1/2} \dfrac{\varphi(z_{1-\alpha})}{\alpha}$
  - $-\varphi$: Gaussian density

i) 1st derivative of Expected Shortfall

$$\frac{\partial m_t(a,\alpha)}{\partial a} = -\mu_t + \frac{\Omega_t a}{(a'\Omega_t a)^{1/2}} \frac{\varphi(z_{1-\alpha})}{\alpha}$$

$\Rightarrow$ Second example:

Gaussian with *Unobserved Heterogeneity*

- $y_{t+1} \mid u \sim N\left(0, \Omega_t(u)\right)$
- with heterogeneity factor u with distribution Π
- Gaussian Random Walk with Stochastic Volatility

Check of VaR Convexity:

$$\frac{\partial \log g_{a,t}}{\partial z}(-VaR_t(a,\alpha)) = VaR_t(a,\alpha) E_{\tilde{\Pi}}\left[\frac{1}{a'\Omega_t(u)a}\right] > 0.$$

$$\begin{aligned} -\frac{\partial}{\partial z} V_t[y_{t+1}|a'y_{t+1} = -z] &= -\frac{\partial}{\partial z}\left[V_\Pi\left[-z\frac{\Omega_t(u)a}{a'\Omega_t(u)a}\right]\right] \\ &= +2zV_\Pi\left[\frac{\Omega_t(u)a}{a'\Omega_t(u)a}\right], \end{aligned}$$

which is nonnegative for $z = VaR_t(a,\alpha)$!

4. Application : VaR Efficient Portfolios

Budget w allocated among

n Risky Assets and 1 Riskfree Asset (r)

$$\begin{cases} \max_a \; a' E_t y_{t+1} \\ s.t. \quad VaR_t(a;\alpha) \leq VaR_o - w(1+r) = \widetilde{VaR}_o \end{cases}$$

$VaR_o =$ Bound for Authorized Risk (CAD)

First Order Conditions :

$$\begin{cases} E_t y_{t+1} = -\lambda_t^* \dfrac{\partial VaR_t}{\partial a}(a_t^*, \alpha) \\ VaR_t(a_t^*, \alpha) = \widetilde{VaR}_o \end{cases}$$

Proportionality between Global and Local Expectations :

$$E_t y_{t+1} = \lambda_t^* E_t \left[y_{t+1} | a_t^{*\prime} y_{t+1} = -\widetilde{VaR}_o \right]$$

5. Estimation Procedure

Nonparametric approach (kernel) : i.i.d. returns

- Estimation of VaR

$$P[-a'y_{t+1} > VaR(a,\alpha)] = \alpha$$

estimated by Gaussian Kernel

$$\frac{1}{T}\sum_{t=1}^{T}\Phi\left(\frac{-a'y_t - \widehat{VaR}}{h}\right) = \alpha$$

Gauss-Newton Algorithm :

$$var^{(p+1)} = var^{(p)} + \frac{\frac{1}{T}\sum_{t=1}^{T}\Phi\left(\frac{-a'y_t - var^{(p)}}{h}\right) - \alpha}{\frac{1}{Th}\sum_{t=1}^{T}\varphi\left(\frac{a'y_t + var^{(p)}}{h}\right)}$$

Starting values : Gaussian VaR or Empirical Quantile

- Convexity of VaR

  Hessian $\frac{\partial^2 VaR(a,\alpha)}{\partial a \partial a'}$ positive semidefinite

  If, for negative z values

$$\frac{\partial \log g_{a,t}(z)}{\partial z} > 0 \text{ and } \frac{\partial V[y_{t+1}|a'y_{t+1}=z]}{\partial z} \gg 0$$

  Estimator of p.d.f. of portfolio value

$$\hat{g}_a(z) = \frac{1}{Th}\sum_{t=1}^{T} \varphi\left(\frac{a'y_t - z}{h}\right)$$

  Estimator of conditional variance

$$\hat{V}[y_{t+1}|a'y_{t+1}=z] = \frac{\sum_{t=1}^{T} y_t y_t' \varphi\left(\frac{a'y_t - z}{h}\right)}{\sum_{t=1}^{T} \varphi\left(\frac{a'y_t - z}{h}\right)} - \frac{\sum_{t=1}^{T} y_t \varphi\left(\frac{a'y_t - z}{h}\right) \sum_{t=1}^{T} y_t' \varphi\left(\frac{a'y_t - z}{h}\right)}{\left[\sum_{t=1}^{T} \varphi\left(\frac{a'y_t - z}{h}\right)\right]^2}$$

- Estimation of VaR efficient portfolio

Simple forms of 1st and 2nd derivatives of VaR

$\Longrightarrow$ Gauss-Newton Algorithm

$$
\begin{aligned}
a^{(p+1)} &= a^{(p)} - [\frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)}, \alpha)]^{-1} \frac{\partial VaR}{\partial a}(a^{(p)}, \alpha) \\
&+ \left[\frac{2(\widetilde{VaR}_o - VaR(a^{(p)}, \alpha)) + Q(a^{(p)}, \alpha)}{Ey'_{t+1}[\frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)}, \alpha)]^{-1} Ey_{t+1}} \right]^{1/2} \\
&\times [\frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)}, \alpha)]^{-1} Ey_{t+1}
\end{aligned}
$$

with

$$
Q(a^{(p)}, \alpha) = \frac{\partial VaR}{\partial a'}(a^{(p)}, \alpha)[\frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)}, \alpha)]^{-1} \frac{\partial VaR}{\partial a}(a^{(p)}, \alpha)
$$

Theoretical recursion replaced by empirical counterpart

6. Empirical Illustration

French Stock Data from CAC 40

- Thompson-CSF (electronic devices)

- L'Oréal (cosmetics)

Daily Returns : 04/01/1997 to 05/04/1999 (546 obs.)

Empirical Results :

* Standard Normal VaR underestimate (Skew. and Kurt.)

* Smoother patterns for Kernel estimates

* Nonmonotonicity of Sensitivities

* Check for Convexity fails for some allocations

* VaR Symmetry lost

* VaR Efficient Portfolios = Tangency Points of

  $a_1\hat{\mu}_1 + a_2\hat{\mu}_2 = cst$ with IsoVar curve of level $V\widetilde{aR}_o$