SENSITIVITY ANALYSIS
OF VALUES AT RISK

Christian GOURIEROUX
CREST and CEPREMAP

Jean-Paul LAURENT
ISFA, University of Lyon and CREST

Olivier SCAILLET
IAG/Economics Dpt - UCL

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Agenda

1. Motivation
   - Risk Management / Portfolio Selection

2. Main Theoretical Results
   - Analytical Expressions for 1st and 2nd Derivatives of Value at Risk (VaR) w.r.t. Portfolio Allocation
   - Analytical Expressions of first derivatives of expected shortfall

3. Some Examples
   - Gaussian Distribution with/without Unobserved Heterogeneity

4. Application
   - VaR Efficient Portfolios

5. Estimation Procedure
   - Nonparametric Approach (Kernel)

6. Empirical Illustration
   - French Stock Data
1. Motivation
   • Framework
     – regulatory environment
     – proprietary Risk Measurement Models
     – $\text{VaR} = \textbf{Synthetic}$ Measure of Risk Portfolio
   • Risk Management
     – risk analysis
     – risk control
     – portfolio optimization
   • Need to go \textbf{further} than a \underline{single} number
   • What are the \textit{risk drivers}?  
     1. VaR of subportfolios  
        – ex-post analysis
        – aggregation issues
        – VaR is neither additive or subadditive
     2. Statistical analysis of \textit{risk scenarios}
     3. Incremental VaR (\textbf{sensitivity analysis})
        – ex-ante
        – management of risk limits
        – Large Portfolios preclude online computations
        – $\implies \textbf{Sensitivity}$ of VaR w.r.t. portfolio allocation
• Why determine convexity of VaR w.r.t. portfolio allocation?

• Portfolio Selection under VaR Constraints
  – Determination of Optimal Portfolios
  – Need to check that the set of portfolios satisfying VaR constraint is convex

• VaR may not be subadditive (i.e. a coherent measure of risk)
  – No bonus for diversification
  – Poor internal risk management
  – Subadditivity can be statistically tested ...
  – and depends on financial intermediaries policies ...
    * Credit risk,
    * Out of the money options
2. Main Theoretical Results

- **VaR definition:**
  - $n$ Financial Assets with Prices $p_{i,t}$ at Date $t$
  - Portfolio Value: $W_t(a) = \sum_{i=1}^{n} a_i p_{i,t} = a' p_t$
  - $P_t [W_{t+1}(a) - W_t(a) + VaR_t(a, \alpha) < 0] = \alpha$
  - Upper Quantile $(1 - \alpha)$...
    * since with $y_{t+1} = p_{t+1} - p_t$
    * $P_t [-a'y_{t+1} > VaR_t(a, \alpha)] = \alpha$

- **First properties**
  - VaR is degree one positively homogeneous: $\lambda \geq 0$
  - $VaR_t(\lambda a, \alpha) = \lambda \times VaR_t(a, \alpha)$
  - $a' \frac{\partial VaR_t(a, \alpha)}{\partial a} = VaR_t(a, \alpha)$ (Euler)
  - Risk contribution of $i$: $a_i \times \frac{\partial VaR_t(a, \alpha)}{\partial a_i}$
  - VaR is **not always** subadditive:
    - $VaR_t(a + b, \alpha) \leq VaR_t(a, \alpha) + VaR_t(b, \alpha)$?
  - subadditivity + positive homogeneity $\Rightarrow$ convexity of $VaR(a, \alpha)$ (w.r.t portfolio allocation $a$)
• Sensitivities of VaR

i) 1st derivative of VaR

\[
\frac{\partial V a R_t(a, \alpha)}{\partial a} = -E_t[y_{t+1}|a'y_{t+1} = -V a R_t(a, \alpha)]
\]

ii) 2nd derivative of VaR

\[
\frac{\partial^2 V a R_t(a, \alpha)}{\partial a \partial a'} = \frac{\partial \log g_{a,t}}{\partial z}(-V a R_t(a, \alpha))V_t[y_{t+1}|a'y_{t+1} = -V a R_t(a, \alpha)]
\]

\[
- \left\{ \frac{\partial}{\partial z}V_t[y_{t+1}|a'y_{t+1} = -z] \right\}_{z=V a R_t(a, \alpha)},
\]

with \( g_{a,t} \) the conditional p.d.f of \( a'y_{t+1} \)
• **Expected Shortfall definition:**

\[ m_t(a, \alpha) = E_t[-a'y_{t+1} \mid -a'y_{t+1} > Var_t(a, \alpha)] \]

– Also known as *Mean Excess Loss*, as *TailVaR* or as *Lower Partial Moment*.

– Another commonly used measure of risk

– Expected Shortfall is subadditive

i) **1st derivative of Expected Shortfall**

\[ \frac{\partial m_t(a, \alpha)}{\partial a} = -E_t[y_{t+1} \mid -a'y_{t+1} > VaR_t(a, \alpha)] \]

Similar result to that obtained on sensitivity of VaR
3. Some Examples

⇒ **First example**: Gaussian Distribution

- \( y_{t+1} \sim N(\mu_t, \Omega_t) \)
- \( \text{VaR}_t(a, \alpha) = -a'\mu_t + (a'\Omega_t a)^{1/2} z_{1-\alpha} \)
- \( z_{1-\alpha} \) quantile of level \( 1 - \alpha \) of Gaussian distribution.

**Remark:**

- \( (a'\Omega_t a)^{1/2} \): standard dev. of portfolio absolute returns
- subadditivity of standard deviations:
  - \( ((a + b)'\Omega_t(a + b))^{1/2} \leq (a'\Omega_t a)^{1/2} + (b'\Omega_t b)^{1/2} \)
- \( \Rightarrow \) In the Gaussian case, VaR is always subadditive

i) 1st derivative of VaR

\[
\frac{\partial \text{VaR}_t(a, \alpha)}{\partial a} = -\mu_t + \frac{\Omega_t a}{(a'\Omega_t a)^{1/2}} z_{1-\alpha}
= -\mu_t + \frac{\Omega_t a}{a'\Omega_t a} (\text{VaR}_t(a, \alpha) + a'\mu_t)
= -E_t[y_{t+1}|a'y_{t+1} = -\text{VaR}_t(a, \alpha)]
\]
ii) 2nd derivative of VaR

\[
\frac{\partial^2 \text{VaR}_t(a, \alpha)}{\partial a \partial a'} = \frac{z_{1-\alpha}}{(a'\Omega_t a)^{1/2}} \left[ \Omega_t - \frac{\Omega_t a a' \Omega_t}{a' \Omega_t a} \right]
\]

\[
\frac{\partial^2 \text{VaR}_t(a, \alpha)}{\partial a \partial a'} = \frac{z_{1-\alpha}}{(a'\Omega_t a)^{1/2}} V_t[y_{t+1} | a'y_{t+1} = -\text{VaR}_t(a, \alpha)]
\]

Remark:

\[
\frac{\partial \log g_{a,t}}{\partial z} (-\text{VaR}_t(a, \alpha)) = \frac{\text{VaR}_t(a, \alpha) + a' \mu_t}{a' \Omega_t a} = \frac{z_{1-\alpha}}{(a'\Omega_t a)^{1/2}}
\]

2nd term = 0 (conditional homoscedasticity)

• Expected Shorfall \( m_t(a, \alpha) \)

\[
m_t(a, \alpha) = -a' \mu_t + (a'\Omega_t a)^{1/2} \varphi(z_{1-\alpha})
\]

\(-\varphi\): Gaussian density

i) 1st derivative of Expected Shortfall

\[
\frac{\partial m_t(a, \alpha)}{\partial a} = -\mu_t + \frac{\Omega_t a}{(a'\Omega_t a)^{1/2}} \varphi(z_{1-\alpha})
\]
Second example:

Gaussian with *Unobserved Heterogeneity*

- \( y_{t+1} \mid u \sim N(0, \Omega_t(u)) \)
- with heterogeneity factor \( u \) with distribution \( \Pi \)
- Gaussian Random Walk with Stochastic Volatility

**Check of VaR Convexity:**

\[
\frac{\partial \log g_{a,t}}{\partial z} (-VaR_t(a, \alpha)) = VaR_t(a, \alpha)E_{\Pi} \left[ \frac{1}{a'\Omega_t(u)a} \right] > 0.
\]

\[
- \frac{\partial}{\partial z} V_t[y_{t+1} | a'y_{t+1} = -z] = -\frac{\partial}{\partial z} \left[ V_{\Pi} \left[ -z \frac{\Omega_t(u)a}{a'\Omega_t(u)a} \right] \right]
\]

\[
= +2zV_{\Pi} \left[ \frac{\Omega_t(u)a}{a'\Omega_t(u)a} \right],
\]

which is nonnegative for \( z = VaR_t(a, \alpha) \)!
4. Application : VaR Efficient Portfolios

Budget $w$ allocated among

$n$ Risky Assets and 1 Riskfree Asset ($r$)

$$\max_{a} \ a' E_t y_{t+1}$$

$$s.t. \ VaR_t (a; \alpha) \leq VaR_0 - w (1 + r) = V\widehat{a}R_o$$

$VaR_0$ = Bound for Authorized Risk (CAD)

First Order Conditions :

$$E_t y_{t+1} = -\lambda_t^* \frac{\partial VaR_t}{\partial a} (a_t^*, \alpha)$$

$$VaR_t (a_t^*, \alpha) = V\widehat{a}R_o$$

Proportionality between Global and Local Expectations :

$$E_t y_{t+1} = \lambda_t^* E_t \left[y_{t+1} \mid a_t^* y_{t+1} = -V\widehat{a}R_o\right]$$
5. Estimation Procedure

Nonparametric approach (kernel) : i.i.d. returns

- Estimation of VaR

\[ P[-a' y_{t+1} > VaR(a, \alpha)] = \alpha \]

estimated by Gaussian Kernel

\[ \frac{1}{T} \sum_{t=1}^{T} \Phi \left( \frac{-a' y_t - \hat{VaR}}{h} \right) = \alpha \]

Gauss-Newton Algorithm :

\[ var^{(p+1)} = var^{(p)} + \frac{1}{T} \sum_{t=1}^{T} \Phi \left( \frac{-a' y_t - var^{(p)}}{h} \right) - \alpha \]

Starting values : Gaussian VaR or Empirical Quantile
• Convexity of VaR

\[
\text{Hessian } \frac{\partial^2 \text{VaR}(a, \alpha)}{\partial a \partial a'} \text{ positive semidefinite}
\]

If, for negative \(z\) values

\[
\frac{\partial \log g_{a,t}(z)}{\partial z} > 0 \text{ and } \frac{\partial V[y_{t+1}|a'y_{t+1} = z]}{\partial z} \gg 0
\]

Estimator of p.d.f. of portfolio value

\[
\hat{g}_a(z) = \frac{1}{Th} \sum_{t=1}^{T} \varphi \left( \frac{a'y_t - z}{h} \right)
\]

Estimator of conditional variance

\[
\hat{V}[y_{t+1}|a'y_{t+1} = z] = \frac{\sum_{t=1}^{T} y_t y_t' \varphi \left( \frac{a'y_t - z}{h} \right)}{\sum_{t=1}^{T} \varphi \left( \frac{a'y_t - z}{h} \right)} - \frac{\sum_{t=1}^{T} y_t \varphi \left( \frac{a'y_t - z}{h} \right) \sum_{t=1}^{T} y_t' \varphi \left( \frac{a'y_t - z}{h} \right)}{\left[ \sum_{t=1}^{T} \varphi \left( \frac{a'y_t - z}{h} \right) \right]^2}
\]
Estimation of VaR efficient portfolio

Simple forms of 1st and 2nd derivatives of VaR

\[ a^{(p+1)} = a^{(p)} - \left[ \frac{\partial^2 VaR(a^{(p)}, \alpha)}{\partial a \partial a'}(a^{(p)}, \alpha) \right]^{-1} \frac{\partial VaR}{\partial a}(a^{(p)}, \alpha) \]

\[ + \left\{ \frac{2(V\tilde{a}R_o - VaR(a^{(p)}, \alpha)) + Q(a^{(p)}, \alpha)}{Ey_{t+1}[\frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)}, \alpha)]^{-1}Ey_{t+1}} \right\}^{1/2} \]

\[ \times \left[ \frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)}, \alpha) \right]^{-1}Ey_{t+1} \]

with

\[ Q(a^{(p)}, \alpha) = \frac{\partial VaR}{\partial a'}(a^{(p)}, \alpha)[\frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)}, \alpha)]^{-1}\frac{\partial VaR}{\partial a}(a^{(p)}, \alpha) \]

Theoretical recursion replaced by empirical counterpart
6. Empirical Illustration

French Stock Data from CAC 40

- Thompson-CSF (electronic devices)
- L’Oréal (cosmetics)

Daily Returns : 04/01/1997 to 05/04/1999 (546 obs.)

Empirical Results :

* Standard Normal VaR underestimate (Skew. and Kurt.)
* Smoother patterns for Kernel estimates
* Nonmonotonicity of Sensitivities
* Check for Convexity fails for some allocations
* VaR Symmetry lost
* VaR Efficient Portfolios = Tangency Points of

\[ a_1\hat{\mu}_1 + a_2\hat{\mu}_2 = cst \] with IsoVar curve of level \( \widehat{VaR}_o \)