Factor Approaches for Effective Pricing and Risk Management of basket credit derivatives and CDO's

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Slides are also available on my web site
Paper « basket defaults swaps, CDO's and Factor Copulas » available on DefaultRisk.com
« I will survive », technical paper, RISK magazine, june 2003

Outlook

- *Semi-analytical pricing of multiname credit derivatives and CDO's*
- *Use of probability generating functions and conditional independence assumption*
- *Copula approaches including Gaussian, Archimedean, multivariate exponential models*
- *Analytical pricing of multiname credit derivatives in Duffie's affine framework*
- *Effective computation of risk parameters*

What are we looking for ?

- A framework where:
  - *One can easily deal with a large number of names,*
  - *Tackle with different time horizons,*
  - *Compute quickly and accurately:*
    - Basket credit derivatives premiums
    - CDO margins on different tranches
    - Deltas with respect to shifts in credit curves
- Main technical assumption:
  - *Default times are independent conditionnally on a low dimensional factor*

Presentation Overview

- Probabilistic Tools
  - *Survival functions of default times*
  - *Factor copulas*
  - *Number of defaults*
- Basket Credit Derivatives
  - *Valuation of premium leg*
  - *Valuation of default leg: homogeneous baskets*
  - *Valuation of default leg: non homogeneous baskets*
  - *Example: first to default swap*
  - *Risk management of basket credit derivatives*
- Valuation of CDO Tranches
  - *Credit loss distributions*
  - *Valuation of CDO's*
  - *Risk management of CDO tranches*

Probabilistic Tools: Survival Functions

- $i = 1, \ldots, n$ names
- $\tau_1, \ldots, \tau_n$ default times
- Marginal distribution function $F_i(t) = Q(\tau_i \leq t)$
- Marginal survival function $S_i(t) = Q(\tau_i > t)$
  - *Risk-neutral probabilities of default*
    - Obtained from defaultable bond prices or CDS quotes
  - *« Historical » probabilities of default*
    - Obtained from time series of default times

Probabilistic Tools: Survival functions

- Joint survival function:

$$S(t_1, \dots, t_n) = Q(\tau_1 > t_1, \dots, \tau_n > t_n)$$

  - *Needs to be specified given marginals.*

- (Survival) Copula of default times:

$$C(S_1(t_1), \dots, S_n(t_n)) = S(t_1, \dots, t_n)$$

  - *C characterizes the dependence between default times.*

- We need tractable dependence between defaults:
  - *Parsimonious modelling*
  - *Semi-explicit computations for portfolio credit derivatives*

Probabilistic Tools

"This next recipe will involve some calculus"

Probabilistic Tools: Factor Copulas

- Factor approaches to joint distributions:
  - *V low dimensional factor, not observed « latent factor »*
  - *Conditionally on V default times are independent*
  - *Conditional default probabilities*

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right), \quad q_t^{i|V} = Q\left(\tau_i > t \mid V\right).$$

  - *Conditional joint distribution:*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid V) = \prod_{1 \leq i \leq n} p_{t_i}^{i|V}$$

  - *Joint survival function (implies integration wrt V):*

$$Q(\tau_1 > t_1, \ldots, \tau_n > t_n) = E\left[\prod_{i=1}^{n} q_{t_i}^{i|V}\right]$$

Probabilistic Tools: Gaussian Copulas

- One factor Gaussian copula *(Basel 2):*
  - $V, \bar{V}_i,\ i = 1, \dots, n$ *independent Gaussian*

$$V_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
  - *Conditional default probabilities:* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$
- Joint survival function:

$$S(t_1, \dots, t_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\rho_i v - \Phi^{-1}(F_i(t_i))}{\sqrt{1-\rho_i^2}}\right)\right)\varphi(v)dv$$

  - *Copula:*

$$C(u_1, \dots, u_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(u_i) - \rho_i v}{\sqrt{1-\rho_i^2}}\right)\right)\varphi(v)dv$$

Probabilistic Tools : Clayton copula

- *Davis & Lo ; Jarrow & Yu ; Schönbucher & Schubert*

- Conditional default probabilities

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

- *V: Gamma distribution with parameter*

- Joint survival function:

$$S(t_1, \ldots, t_n) = \int \prod_{i=1}^{n} \left(1 - p_{t_i}^{i|V}\right) \frac{1}{\Gamma(1/\theta)} e^{-V} V^{(1-\theta)/\theta} dV$$

- *Copula:*

$$C(u_1, \ldots, u_n) = \left(u_1^{-\theta} + \ldots + u_n^{-\theta} - n + 1\right)^{-1/\theta}$$

Probabilistic Tools: Simultaneous Defaults

- *Duffie & Singleton, Wong*

- Modelling of defaut dates: $\tau_i = \min(\bar{\tau}_i, \tau)$
  - $Q(\tau_i = \tau_j) \geq Q\left(\tau \leq \min(\bar{\tau}_i, \bar{\tau}_j)\right) > 0$ *simultaneous defaults.*
  - *Conditionally on* τ, τ_i *are independent.*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid \tau) = \prod_{1 \leq i \leq n} Q(\tau_i \leq t_i \mid \tau)$$

- Conditional default probabilities:

$$Q(\tau_i \leq t_i \mid \tau) = 1_{\tau > t_i} Q(\bar{\tau}_i \leq t_i) + 1_{\tau \leq t_i}$$

  - *Copula of default times:*

$$C(u_1, \ldots, u_n) = E\left[\prod_{1 \leq i \leq n} Q\left(\tau_i \leq F_i^{-1}(u_i) \mid \tau\right)\right]$$

Probabilistic Tools: Affine Jump Diffusion

- *Duffie, Pan & Singleton ;Duffie & Garleanu.*
- $n+1$ *independent affine jump diffusion processes:*

$$X_1, \ldots, X_n, X_c$$

- Conditional default probabilities:

$$Q(\tau_i > t \mid V) = q_t^{i|V} = V\alpha_i(t)$$

$$V = \exp\left(-\int_0^t X_c(s)ds\right), \quad \alpha_i(t) = E\left[\exp\left(-\int_0^t X_i(s)ds\right)\right].$$

- Survival function:

$$Q(\tau_1 > t, \ldots, \tau_n > t) = E\left[V^n\right] \times \prod_{i=1}^{n} \alpha_i(t).$$

  - *Explicitely known.*

Probabilistic Tools: Conditional Survivals

- Conditional survival functions and factors:
  - *Example: survival functions up to first to default time…;*
  - *Conditional joint survival function easy to compute since:*

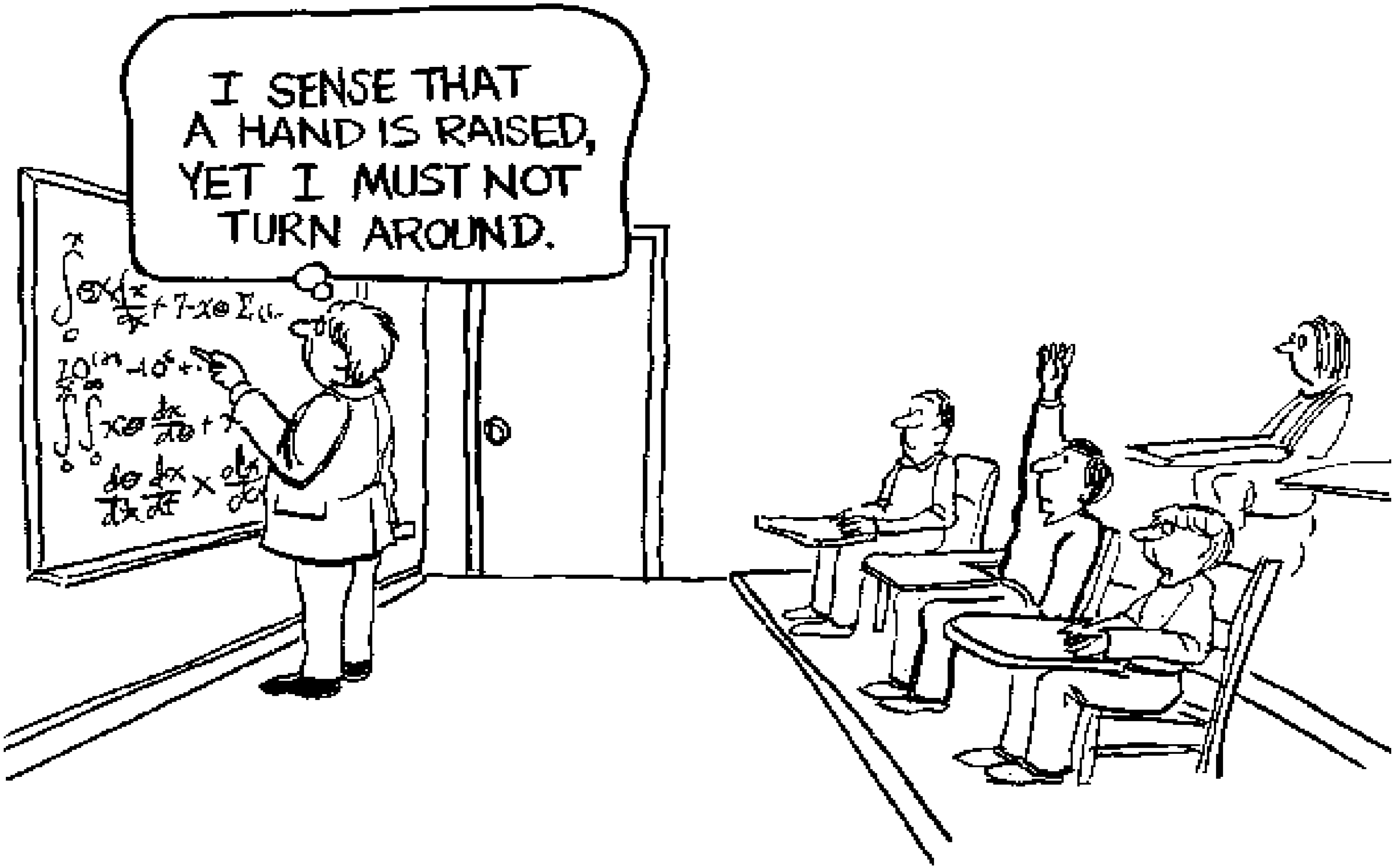
$$Q\left(\tau_1 > t_1, \tau_2 > t_2 \mid \tau_1 > t, \tau_2 > t\right) = \frac{Q\left(\tau_1 > t_1, \tau_2 > t_2\right)}{Q\left(\tau_1 > t, \tau_2 > t\right)}$$

$$Q\left(\tau_1 > t_1, \tau_2 > t_2\right) = E\left[q_{t_1}^{1|V} \times q_{t_2}^{2|V}\right]$$

  - *However be cautious, usually:*

$$Q\left(\tau_1 > t_1, \tau_2 > t_2 \mid \tau_1 \wedge \tau_2 > t\right) \neq E\left[Q\left(\tau_1 > t_1, \tau_2 > t_2 \mid \tau_1 \wedge \tau_2 > t, V\right)\right]$$

Probabilistic Tools



«Counting time is not so important as making time count»

Probabilistic Tools: Number of Defaults

- $N(t) = \sum_{1 \le i \le n} 1_{\{\tau_i \le t\}} = \sum_{1 \le i \le n} N_i(t)$ *Number of defaults at t.*
- τ^k *k^{th} to default time.*
- $S^k(t) = Q(\tau^k > t)$ *Survival function of k^{th} to default.*
- *Remark that:* $\tau^k > t \Longleftrightarrow N(t) < k$
- *Survival function of* τ^k : $S^k(t) = \sum_{l \le k-1} Q(N(t) = l)$
- *Computation of* $Q(N(t) = l)$
- *Use of pgf of N(t):* $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = \sum_{l=0}^{n} Q(N(t) = l)u^l$

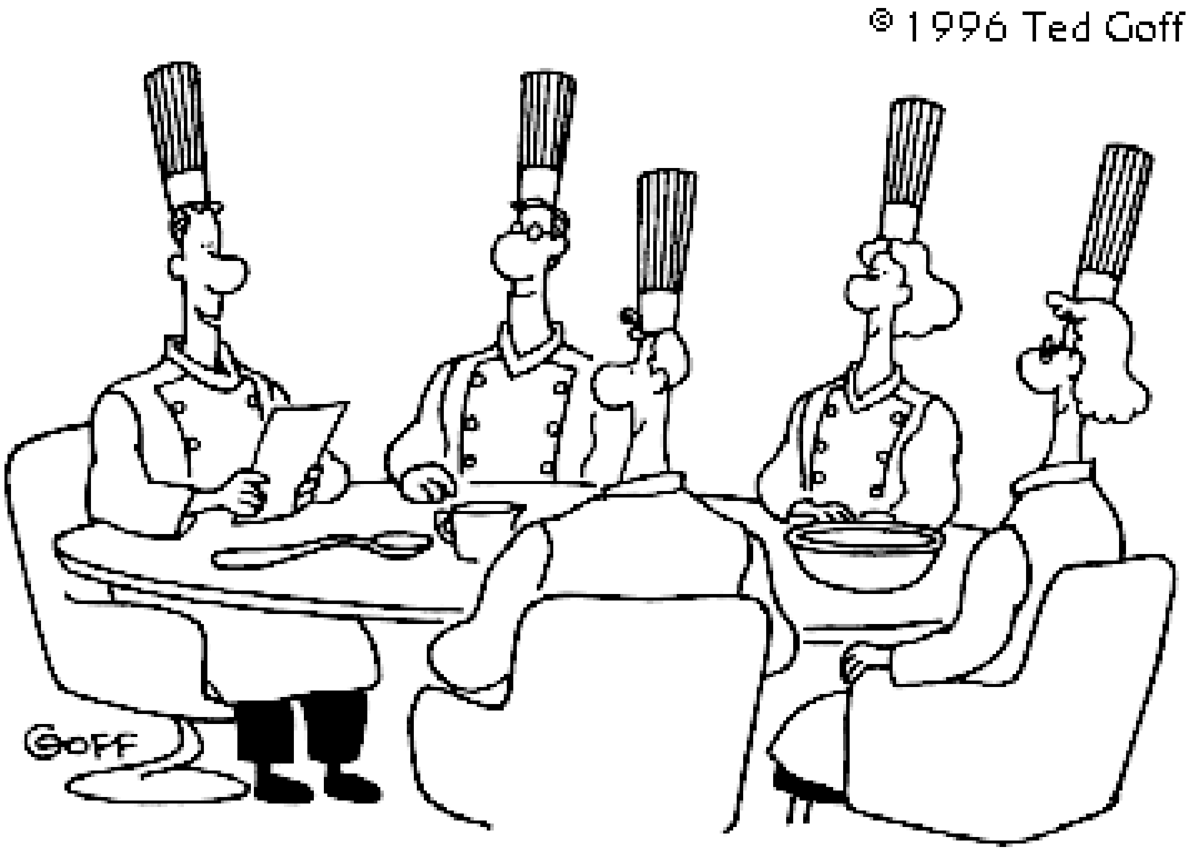
«Counting time is not so important as making time count»

Probabilistic tools: Number of Defaults

- *Probability generating function of* $N(t)$: $\psi_{N(t)} = E\left[u^{N(t)}\right]$
  - $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = E\left[E\left[u^{N(t)} \mid V\right]\right]$ *iterated expectations*
  - $E\left[u^{N(t)} \mid V\right] = \prod_{1 \le i \le n} E\left[u^{N_i(t)} \mid V\right]$ *conditional independence*
  - $E\left[u^{N_i(t)} \mid V\right] = 1 - p_t^{i|V} + p_t^{i|V} \times u$ *binary random variable*
- $\psi_{N(t)}(u) = E\left[\prod_{i=1}^{n}\left(1 - p_t^{i|V} + p_t^{i|V} \times u\right)\right]$ *polynomial in u*
- *One can then compute* $Q(N(t) = k)$
- *Since* $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = \sum_{k=0}^{n} Q(N(t) = k)u^k$

«the whole is simpler than the sum of its parts »

Basket Credit Derivatives Valuation



"Our eggs are all in one basket, no milk has been spilt, and we have plenty of dough."

Valuation of Premium Leg

- k^{th} to default swap, maturity T
  - $t_1, \ldots, t_{l-1}, t_l, \ldots, T$ *premium payment dates*
  - *Periodic premium p is paid until* τ^k
- l^{th} premium payment
  - $\tau^k > t_l$ *payment of p at date* t_l
  - *Present value:* $pB(t_l)S^k(t_l)$
  - $t_{l-1} \leq \tau^k \leq t_l$ *accrued premium of* $(\tau^k - t_{l-1})p$ *at* τ^k
  - *Present value:* $\displaystyle\int_{t_{l-1}}^{t_l} pB(t)(t - t_{l-1})dS^k(t)$
- PV of premium leg given by summation over l

Valuation of Default Leg: Homogeneous Baskets

- $i = 1, \ldots, n$ names
  - *Equal nominal (say 1) and recovery rate (say 0)*
- Payoff : 1 at *k*-th to default time if less than *T*
- Credit curves can be different
  - $S_i(t) = Q(\tau_i > t)$ *given from credit curves*
  - $S^k(t) = Q(\tau^k > t)$ *: survival function of* τ^k
  - $S^k(t)$ *computed from pgf of* $N(t)$

Valuation of Default Leg: Homogeneous Baskets

- Expected discounted payoff

$$E\left[B(\tau^k)1_{\tau^k \leq T}\right] = -\int_0^T B(t)dS^k(t)$$

  - *From transfer theorem*
  - *B(t) discount factor*
- Integrating by parts

$$1 - B(T)S^k(T) + \int_0^T S^k(t)dB(t)$$

  - *Present value of default payment leg*
  - *Involves only known quantities*
  - *Numerical integration is easy*

Valuation of Default Leg: Non Homogeneous Baskets

- $i = 1, \ldots, n$ *names*
- $M_i = (1 - \delta_i) N_i$ *loss given default for i*

- Payment at k^{th} default of M_i if i is in default
  - *No simultaneous defaults*
  - *Otherwise, payoff is not defined*
- i k^{th} default iff k-1 defaults before τ_i
  - $N^{(-i)}(\tau_i)$ *number of defaults (i excluded) at* τ_i
  - *k-1 defaults before* τ_i *iff* $N^{(-i)}(\tau_i) = k - 1$

Valuation of Default Leg: Non Homogeneous Baskets

- *Guido Fubini*

Valuation of Default Leg: Non Homogeneous Baskets

- *(discounted) Payoff* $\sum_{i=1}^{n} M_i B(\tau_i) 1_{\{N^{(-i)}(\tau_i)=k-1\}} 1_{\{\tau_i \leq T\}}$
- *Upfront Premium*
  - *... by iterated expectations theorem*

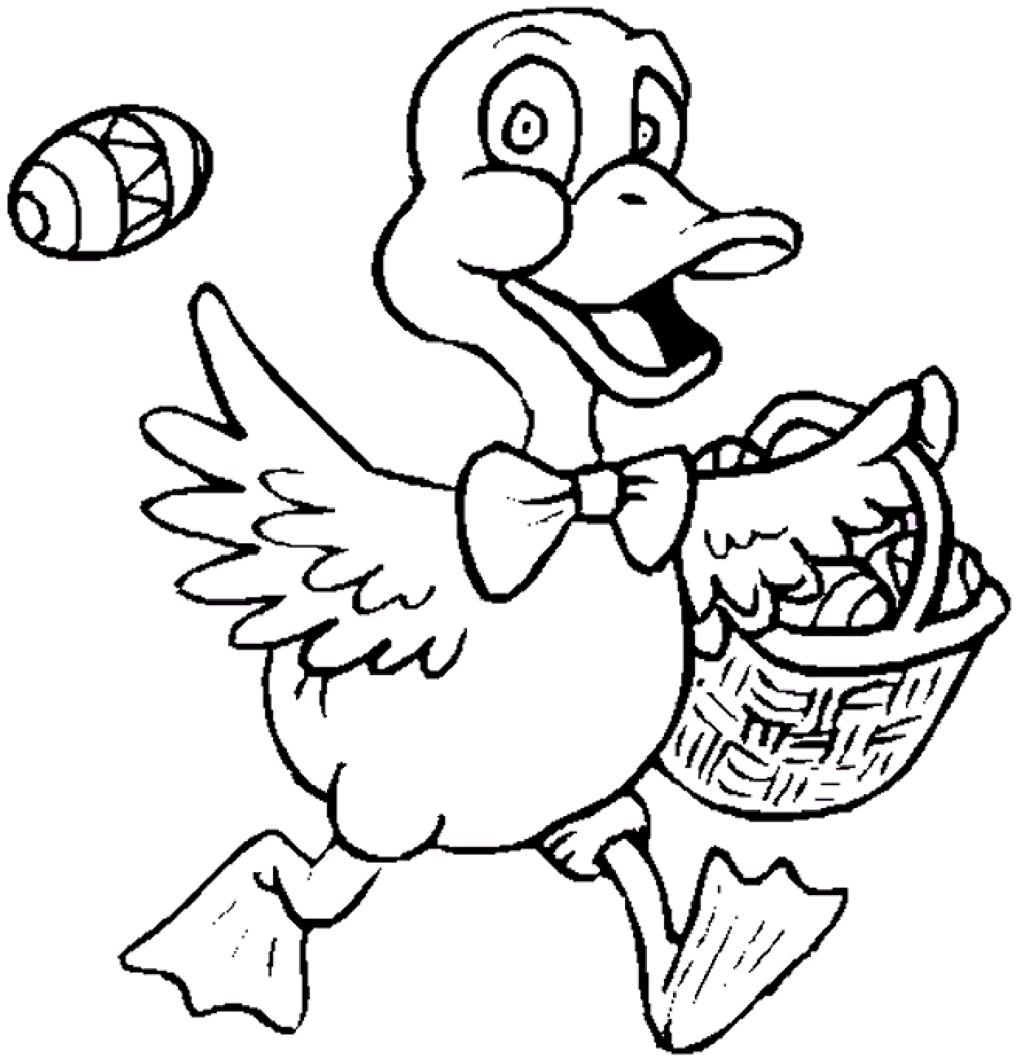
$$\sum_{i=1}^{n} M_i E\left[E\left[B(\tau_i) 1_{\{N^{(-i)}(\tau_i)=k-1\}} 1_{\{\tau_i \leq T\}} \mid V\right]\right]$$

  - *... by Fubini + conditional independence*

$$\int_0^T B(t) Q(N^{(-i)}(t) = k-1 \mid V) dp_t^{i|V}$$

  - *where* $p_t^{i|V} = Q(\tau_i \leq t \mid V)$
  - $Q(N^{(-i)}(t) = k-1 \mid V)$ *: formal expansion of* $\prod_{j \neq i} \left(1 - p_t^{j|V} + p_t^{j|V} u\right)$

Example: First to Default Swap

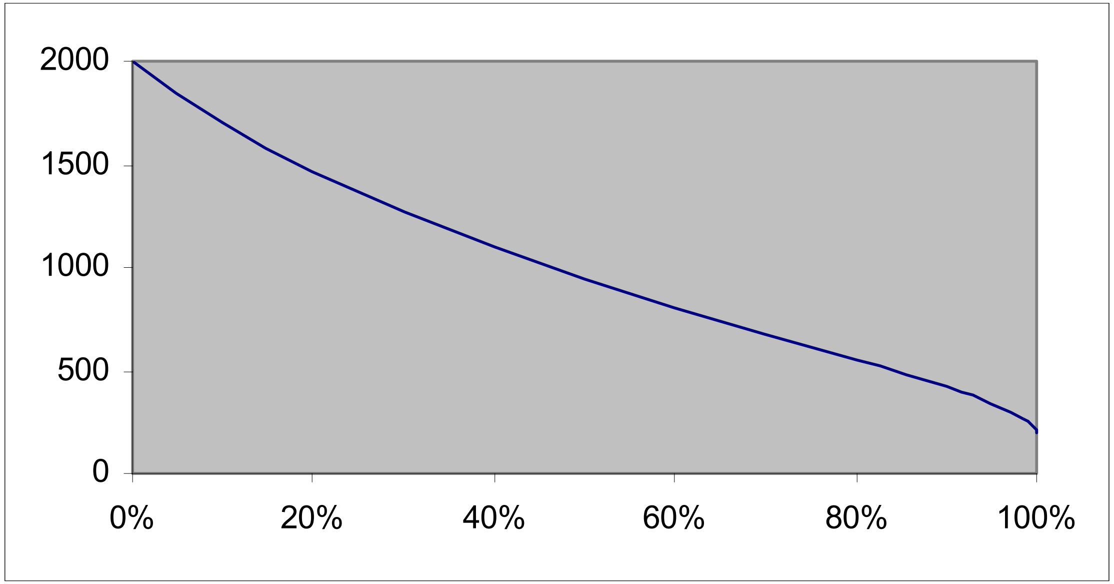


Example: First to Default Swap

- *Case where* $k = 1$
- $Q\left(N^{(-i)}(t) = 0 \mid V\right) = \prod_{j \neq i} \left(1 - p_t^{j|V}\right)$ *no defaults for* $j \neq i$
- *premium* $= \sum_{i=1}^{n} M_i E\left[\int_0^T B(t) \prod_{j \neq i} \left(1 - p_t^{j|V}\right) dp_t^{i|V}\right]$
- $= \int_0^T \sum_{i=1}^{n} M_i B(t) E\left[\prod_{j \neq i} \left(1 - p_t^{j|V}\right) \frac{dp_t^{i|V}}{dt}\right] dt$ *(regular case)*
- *One factor Gaussian* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$
- *Archimedean* $p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$

Example: First to Default Swap

- Dependence upon correlation parameter
  - One factor Gaussian copula
  - 10 names, recovery rate = 40%, maturity = 5 years
  - 5 spreads at 50 bps, 5 spreads at 350 bps



  - x axis: correlation parameter, y axis: annual premium

Risk Management of Basket Credit Derivatives

- Computation of greeks
  - *Changes in credit curves of individual names*
  - *Changes in correlation parameters*
- Greeks can be computed up to an integration over factor distribution
  - *Lenghty but easy to compute formulas*
  - *The technique is applicable to Gaussian and non Gaussian copulas*
  - *See « I will survive », RISK magazine, June 2003, for more about the derivation.*

Risk Management of Basket Credit Derivatives

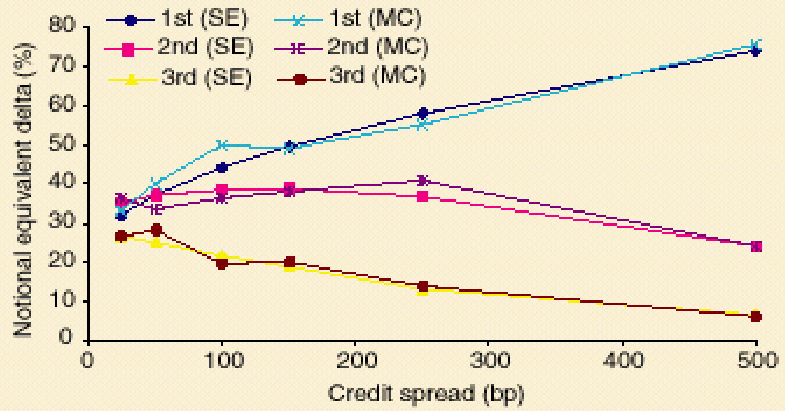
- Example: six names portfolio
- Changes in credit curves of individual names
- Amount of individual CDS to hedge the basket
- Semi-analytical more accurate than 10^5 Monte Carlo simulations.
- Much quicker: about 25 Monte Carlo simulations.

A. Comparison of the semi-explicit formulas with Monte Carlo simulations

	First to default		Second to default		Third to default	
	SE	MC	SE	MC	SE	MC
0%	1,075.1	1,075.9	214.8	214.7	28.2	27.7
20%	927.0	925.9	247.2	247.5	61.4	61.8
30%	859.9	857.9	256.8	257.6	77.6	78.0
40%	796.6	795.2	263.3	264.2	92.7	93.0
60%	679.6	678.0	268.8	268.9	119.5	119.8
80%	573.1	571.7	266.2	266.1	141.0	140.9
100%	500.0	500.0	250.0	250.0	150.0	150.0

Premiums in basis points per annum as a function of correlation for a five-year maturity basket with credit spreads of 25, 50, 100, 150, 250 and 500bp and equal recovery rates of 40%

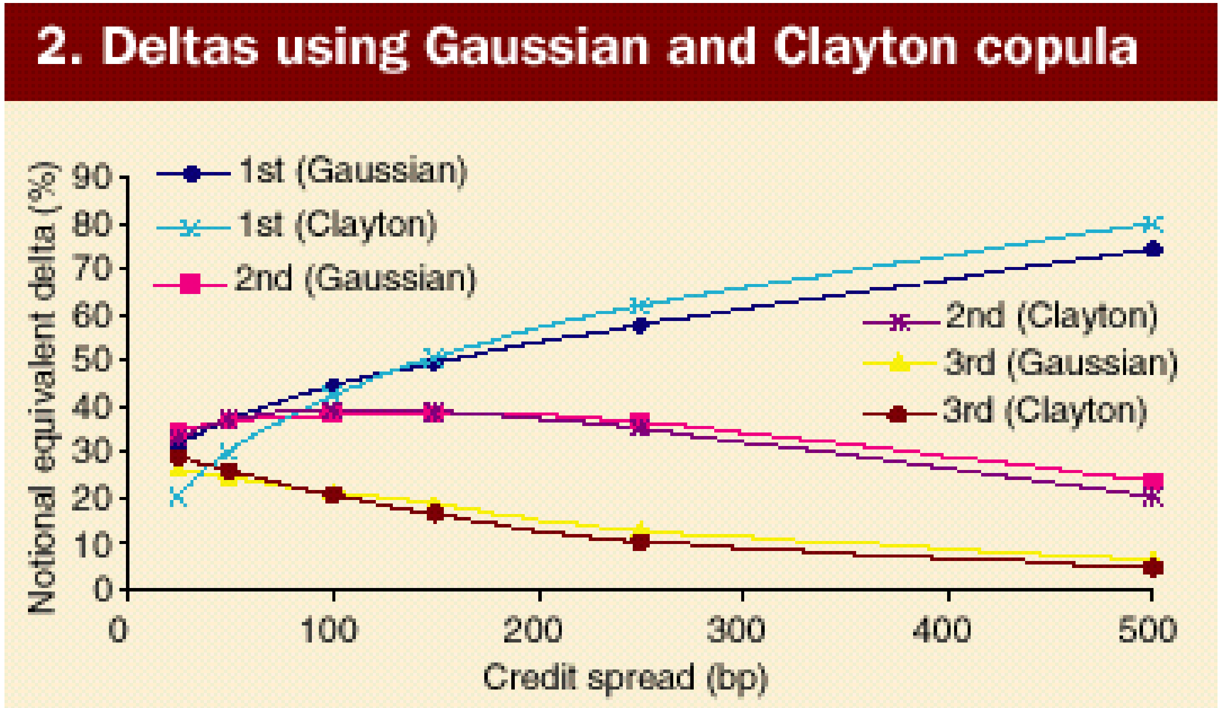
1. Deltas calculated using semi-explicit formulas and Monte Carlo approaches



Comparison of deltas calculated using the analytical formulas and 105 Monte Carlo simulations for the example given in table A. The Monte Carlo deltas are calculated by applying a 10bp parallel shift to each curve

Risk Management of Basket Credit Derivatives

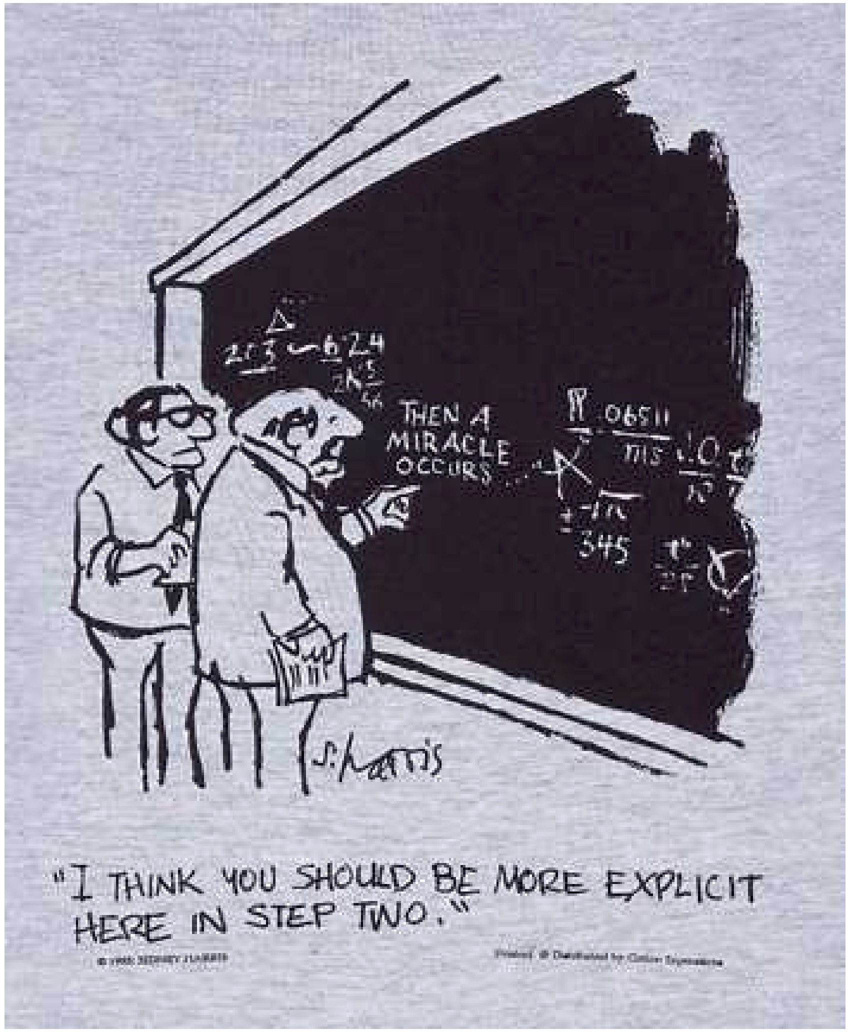
- Changes in credit curves of individual names
  - *Dependence upon the choice of copula for defaults*



Comparison of deltas calculated using Gaussian (30% correlation) and Clayton copulas (θ = 0.27)

CDO Tranches

«Everything should be made as simple as possible, not simpler»



- *Explicit premium computations for tranches*
- *Use of loss distributions over different time horizons*
- *Computation of loss distributions from FFT*
- *Involves integration par parts and Stieltjes integrals*

Credit Loss Distributions

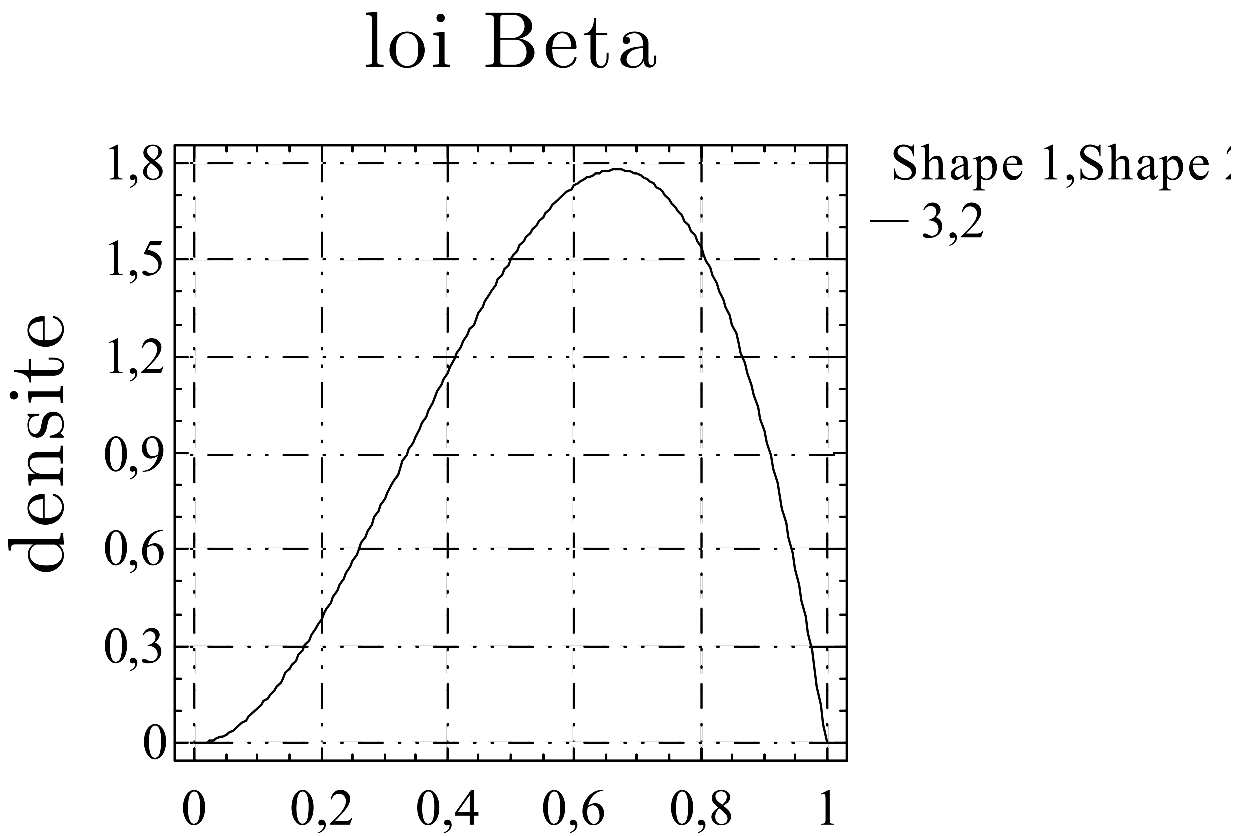
- *Accumulated loss at t:* $L(t) = \sum_{1 \leq i \leq n} N_i(1-\delta_i) N_i(t)$
  - *Where* $N_i(t) = 1_{\tau_i \leq t}$, $N_i(1-\delta_i)$ *loss given default*
- *Characteristic function* $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
- *By conditioning* $\varphi_{L(t)}(u) = E\left[\prod_{1 \leq j \leq n} \left(1 - p_t^{j|V} + p_t^{j|V} \varphi_{1-\delta_j}(uN_j)\right)\right]$
- *If recovery rates follows a beta distribution:*

$$\varphi_{L(t)}(u) = E\left[\prod_{1 \leq j \leq n} \left(1 - p_t^{j|V} + p_t^{j|V} M(a_j, a_j + b_j, iuN_j)\right)\right]$$

  - *where M is a Kummer function,* a_j,b_j *some parameters*
- *Distribution of L(t) is obtained by Fast Fourier Transform*

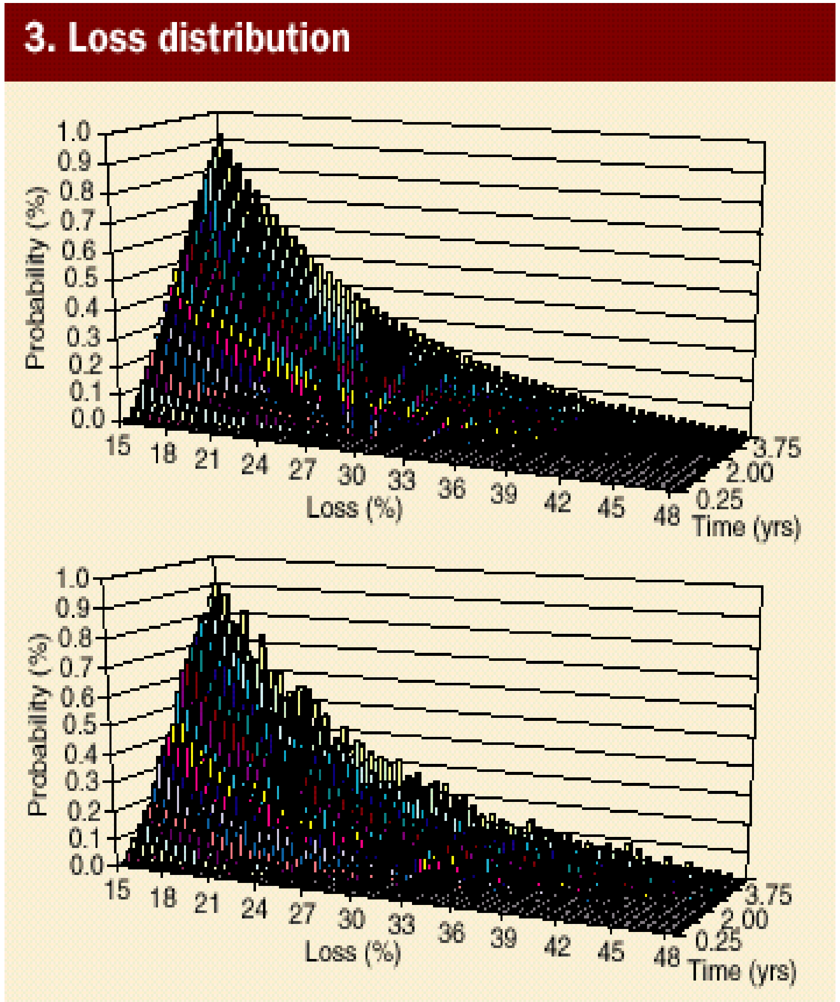
Credit Loss Distributions

- *Beta distribution for recovery rates*



Credit Loss distributions

- One hundred names, same nominal.
- Recovery rates: 40%
- Credit spreads uniformly distributed between 60 and 250 bp.
- Gaussian copula, correlation: 50%
- 10^5 Monte Carlo simulations



Loss distribution over time for the table B example with 50% correlation for the semi-explicit approach (top) and Monte Carlo simulation (bottom)

Valuation of CDO's

- *Tranches with thresholds* $0 \leq A \leq B \leq \sum N_j$
- *Mezzanine: pays whenever losses are between A and B*
- *Cumulated payments at time t: M(t)*

$$M(t) = (L(t) - A))\,1_{[A,B]}(L(t)) + (B - A)1_{]B,\infty[}(L(t))$$

- *Upfront premium:* $E\left[\int_0^T B(t)dM(t)\right]$
  - *B(t) discount factor, T maturity of CDO*
- *Stieltjes integration by parts* $B(T)E[M(T)] + \int_0^T E[M(t)]dB(t)$
- *where* $E[M(t)] = (B - A)Q(L(t) > B) + \int_A^B (x - A)dF_{L(t)}(x)$

Valuation of CDO's

B. Pricing of five-year maturity CDO tranches

	Equity (0–3%)		Mezzanine (3–14%)		Senior (14–100%)	
	SE	MC	SE	MC	SE	MC
0%	8,219.4	8,228.5	816.2	814.3	0.0	0.0
20%	4,321.1	4,325.3	809.4	806.9	13.7	13.7
40%	2,698.8	2,696.7	734.3	731.4	33.4	33.2
60%	1,750.6	1,738.5	641.0	637.8	54.1	53.7
80%	1,077.5	1,067.9	529.5	526.9	77.0	76.6
100%	410.3	406.6	371.2	367.0	110.4	109.6

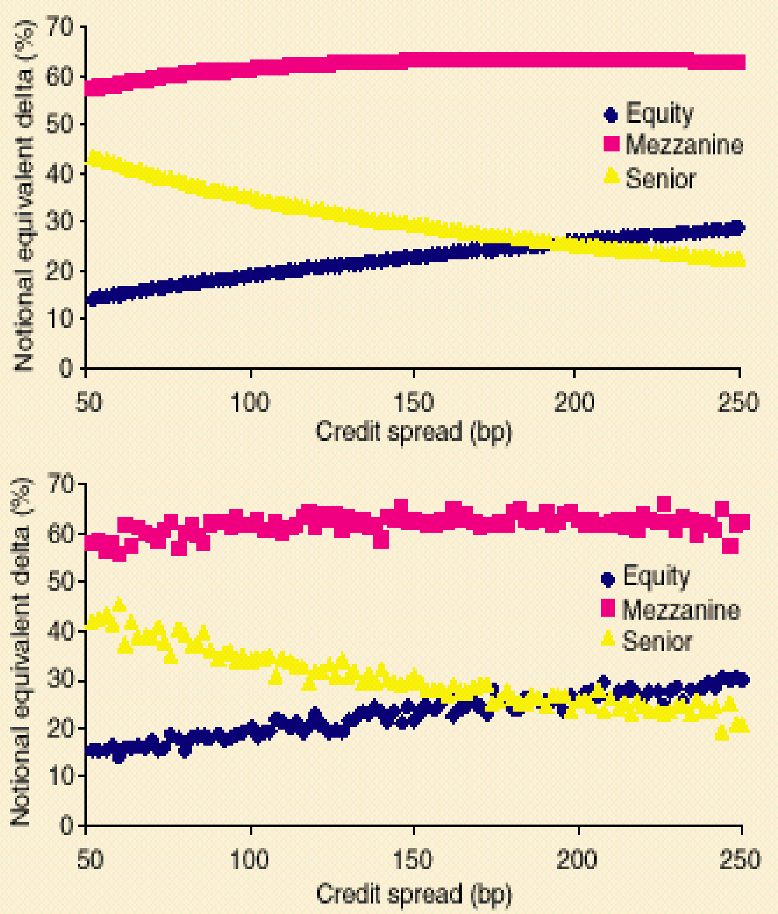
Premiums in basis points per annum as a function of correlation for 5-year maturity CDO tranches on a portfolio with credit spreads uniformly distributed between 60 and 250bp. The recovery rates are 40%

- *One factor Gaussian copula*
- *CDO tranches margins with respect to correlation parameter*

Risk Management of CDO's

- Hedging of CDO tranches with respect to credit curves of individual names
- Amount of individual CDS to hedge the CDO tranche
- Semi-analytic : some seconds
- Monte Carlo more than one hour and still shaky

4. CDO tranche deltas



CDO tranche deltas using the analytical method (top) and Monte Carlo (bottom) for a correlation of 50%

Conclusion

- Factor models of default times:
  - *Very simple computation of basket credit derivatives and CDO's*
  - *One can deal easily with a large range of names and dependence structures*