# *New approaches to the pricing of basket credit derivatives and CDO's*

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*Jean-Paul Laurent*
*Professor, ISFA Actuarial School, University of Lyon*
*& Ecole Polytechnique*
*Scientific consultant, BNP Paribas*

Laurent.jeanpaul@free.fr, http:/laurent.jeanpaul.free.fr

*Paper « basket defaults swaps, CDO's and Factor Copulas » available on DefaultRisk.com*

## *Overview*

- Straightforward approach to baskets and CDO
  - *Direct modelling of default times*
  - *Modelling of dependence through copulas*
- Semi-explicit premiums
  - *Factor copulas : dramatic dimension reduction*
  - *Fast computations for large baskets*
  - *No need of inaccurate Monte Carlo*
- Semi-explicit loss distributions
  - *Computation of VaR, expected shortfall, risk contributions*

# *Overview*

- *Probabilistic tools*
  - Survival functions of default times
  - Factor copulas
  - Moment generating functions
    - *Distribution of k-th to default time*
    - *Loss distributions over different time horizons*
- *Valuation of basket credit derivatives*
  - homogeneous
  - general case
- *Valuation of CDO tranches*
- *How is it related to intensity approaches ?*

# *Probabilistic tools*


by Andrejs Dunkels

# *Probabilistic tools: survival functions*

- $i = 1, \ldots, n$ names
- $\tau_1, \ldots, \tau_n$ default times
- Marginal distribution function $F_i(t) = Q(\tau_i \leq t)$
- Marginal survival function $S_i(t) = Q(\tau_i > t)$
- Joint survival function

$$S(t_1, \ldots, t_n) = Q(\tau_1 > t_1, \ldots, \tau_n > t_n)$$

  - *Needs to be specified given marginals*
- (Survival) Copula of default times

$$C(S_1(t_1), \ldots, S_n(t_n)) = S(t_1, \ldots, t_n)$$

  - *C characterizes the dependence between default times*

# *Probabilistic tools: factor copulas*

- Tractable specification of dependence
  - *Parsimonious modelling*
  - *Suitable for large baskets and CDO's*
  - *Semi-explicit computations*
- Factor approaches
  - *V factor (low dimension)*
  - *Conditionally on V default times are independent*
  - *Conditional default probabilities*

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right)$$

  - *Conditional joint distribution*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid V) = \prod_{i=1}^{n} p_t^{i|V}$$

# *Probabilistic tools: Gaussian copulas*

- One factor Gaussian copula *(Basel 2)*
  - $V, \bar{V}_i,\ i = 1, \ldots, n$ *independent Gaussian*

$$V_i = \rho_i V + \sqrt{1 - \rho_i^2}\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(V_i)$
  - *Conditional default probabilities* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$
  - *Joint survival function*

$$S(t_1, \ldots, t_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\rho_i v - \Phi^{-1}(F_i(t_i))}{\sqrt{1-\rho_i^2}}\right)\right) \varphi(v) dv$$

  - *Copula*

$$C(u_1, \ldots, u_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(u_i) - \rho_i v}{\sqrt{1-\rho_i^2}}\right)\right) \varphi(v) dv$$

# Probabilistic tools : Clayton copula

- *Davis & Lo, Jarrow & Yu, Schönbucher & Schubert*
- *Conditional default probabilities*

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

- *Joint survival function*

$$S(t_1, \ldots, t_n) = \int \prod_{i=1}^{n} \left(1 - p_{t_i}^{i|V}\right) \frac{1}{\Gamma(1/\theta)} e^{-V} V^{(1-\theta)/\theta} dV$$

- *Copula*

$$C(u_1, \ldots, u_n) = \left(u_1^{-\theta} + \ldots + u_n^{-\theta} - n + 1\right)^{-1/\theta}$$

# *Probabilistic tools: simultaneous defaults*

- Modelling of defaut dates $\tau_i = \min(\bar{\tau}_i, \tau)$
  - *Duffie & Singleton, Wong*
  - $Q(\tau_i = \tau_j) \geq Q\left(\tau \leq \min(\bar{\tau}_i, \bar{\tau}_j)\right) > 0$ *simultaneous defaults*
  - *Conditionally on* $\tau$, $\tau_i$ *are independent*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid \tau) = \prod_{1 \leq i \leq n} Q(\tau_i \leq t_i \mid \tau)$$

- Conditional default probabilities
  - $Q(\tau_i \leq t_i \mid \tau) = 1_{\tau > t_i} Q(\bar{\tau}_i \leq t_i) + 1_{\tau \leq t_i}$
- Copula of default times

$$C(u_1, \ldots, u_n) = E\left[\prod_{1 \leq i \leq n} Q\left(\tau_i \leq F_i^{-1}(u_i) \mid \tau\right)\right]$$

# Probabilistic tools: k-th to default time

- $N(t) = \sum_{1 \le i \le n} 1_{\{\tau_i \le t\}} = \sum_{1 \le i \le n} N_i(t)$ *Number of defaults at t*
- $\tau^k$ *k-th to default time*
- $S^k(t) = Q(\tau^k > t)$ *Survival function of k-th to default*
- *Remark that* $\tau^k > t \Longleftrightarrow N(t) < k$
- *Survival function of* $\tau^k$ : $S^k(t) = \sum_{l \le k-1} Q(N(t) = l)$
- *Computation of* $Q(N(t) = l)$
- *Use of pgf of N(t):* $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = \sum_{l=0}^{n} Q(N(t) = l)u^l$

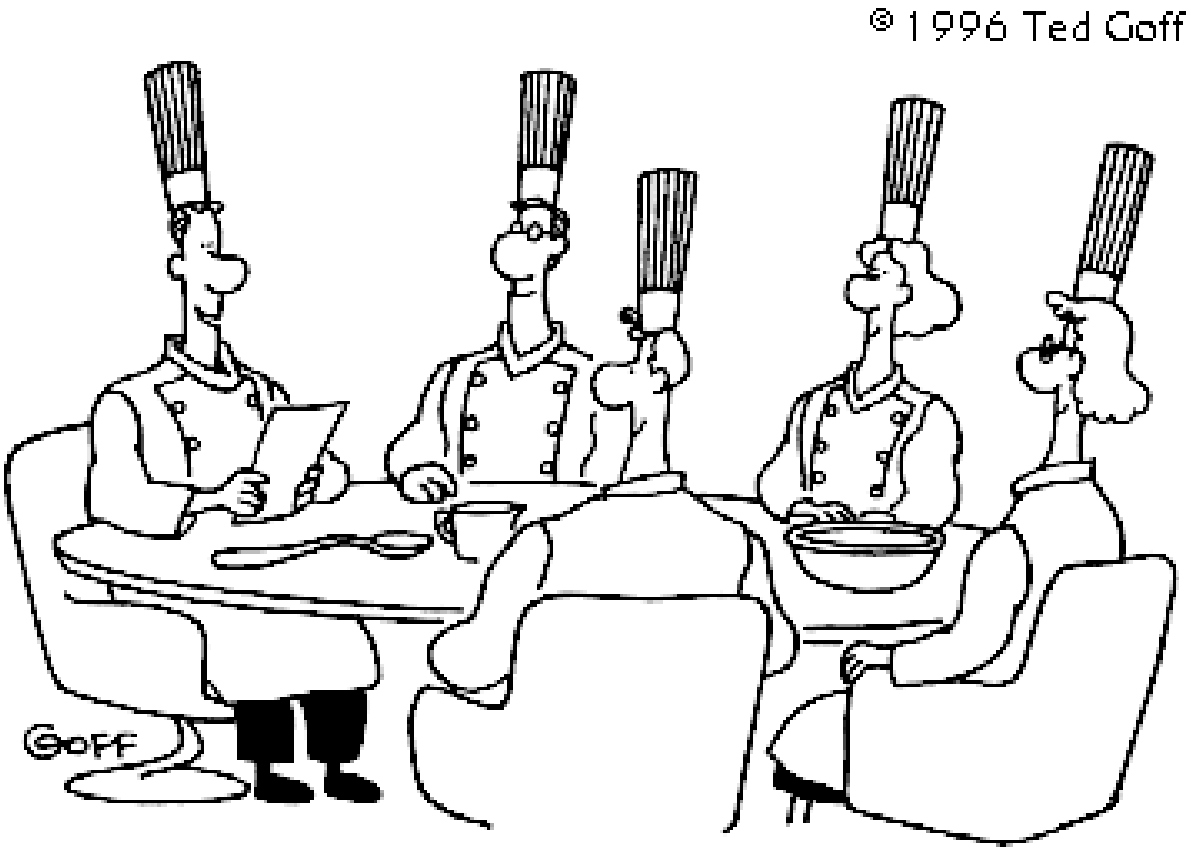
*«Counting time is not so important as making time count»*

# Probabilistic tools: number of defaults

- *Probability generating function of* $N(t): \ \psi_{N(t)} = E\left[u^{N(t)}\right]$
  - $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = E\left[E\left[u^{N(t)} \mid V\right]\right]$ *iterated expectations*
  - $E\left[u^{N(t)} \mid V\right] = \prod_{1 \le i \le n} E\left[u^{N_i(t)} \mid V\right]$ *conditional independence*
  - $E\left[u^{N_i(t)} \mid V\right] = 1 - p_t^{i|V} + p_t^{i|V} \times u$ *binary random variable*
- $\psi_{N(t)}(u) = E\left[\prod_{i=1}^{n}\left(1 - p_t^{i|V} + p_t^{i|V} \times u\right)\right]$ *polynomial in u*
- *One can then compute* $Q(N(t) = k)$
- *Since* $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = \sum_{k=0}^{n} Q(N(t) = k) u^k$

*«the whole is simpler than the sum of its parts »*

# *Basket Valuation*




"Our eggs are all in one basket, no milk has been spilt, and we have plenty of dough."

# *Valuation of homogeneous baskets*

- $i = 1, \ldots, n$ names
  - *Equal nominal (say 1) and recovery rate (say 0)*
- Payoff : 1 at *k*-th to default time if less than *T*
- Credit curves can be different
  - $S_i(t) = Q(\tau_i > t)$ *given from credit curves*
  - $S^k(t) = Q(\tau^k > t)$ *: survival function of* $\tau^k$
  - $S^k(t)$ *computed from pgf of* $N(t)$

# *Valuation of homogeneous baskets*

- Expected discounted payoff

$$E\left[B(\tau^k)1_{\tau^k \leq T}\right] = -\int_0^T B(t)dS^k(t)$$

  - *From transfer theorem*
  - *B(t) discount factor*
- Integrating by parts

$$1 - B(T)S^k(T) + \int_0^T S^k(t)dB(t)$$

  - *Present value of default payment leg*
  - *Involves only known quantities*
  - *Numerical integration is easy*

# *Valuation of premium leg*

- *k*-th to default swap, maturity *T*
  - $t_1, \ldots, t_{l-1}, t_l, \ldots, T$ *premium payment dates*
  - *Periodic premium p is paid until* $\tau^k$
- *l*-th premium payment
  - $\tau^k > t_l$ *payment of p at date* $t_l$
  - *Present value:* $pB(t_l)S^k(t_l)$
  - $t_{l-1} \leq \tau^k \leq t_l$ *accrued premium of* $(\tau^k - t_{l-1})p$ *at* $\tau^k$
  - *Present value:* $\int_{t_{l-1}}^{t_l} pB(t)(t - t_{l-1})dS^k(t)$
- PV of premium leg given by summation over *l*

# *Non homogeneous baskets*

- $i = 1, \ldots, n$ *names*
- $M_i = (1 - \delta_i) N_i$ *loss given default for i*

- Payment at *k*-th default of $M_i$ if *i* is in default
  - *No simultaneous defaults*
  - *Otherwise, payoff is not defined*

- *i k*-th default iff *k*-1 defaults before $\tau_i$
  - $N^{(-i)}(\tau_i)$ *number of defaults (i excluded) at* $\tau_i$
  - *k-1 defaults before* $\tau_i$ *iff* $N^{(-i)}(\tau_i) = k - 1$

# Non homogeneous baskets

- (discounted) Payoff $\sum_{i=1}^{n} M_i B(\tau_i) 1_{\{N^{(-i)}(\tau_i)=k-1\}} 1_{\{\tau_i \leq T\}}$
- Upfront Premium
  - … by iterated expectations theorem

$$\sum_{i=1}^{n} M_i E\left[E\left[B(\tau_i) 1_{\{N^{(-i)}(\tau_i)=k-1\}} 1_{\{\tau_i \leq T\}} \mid V\right]\right]$$

  - … by Fubini + conditional independence

$$\int_0^T B(t) Q(N^{(-i)}(t) = k-1 \mid V) dp_t^{i|V}$$

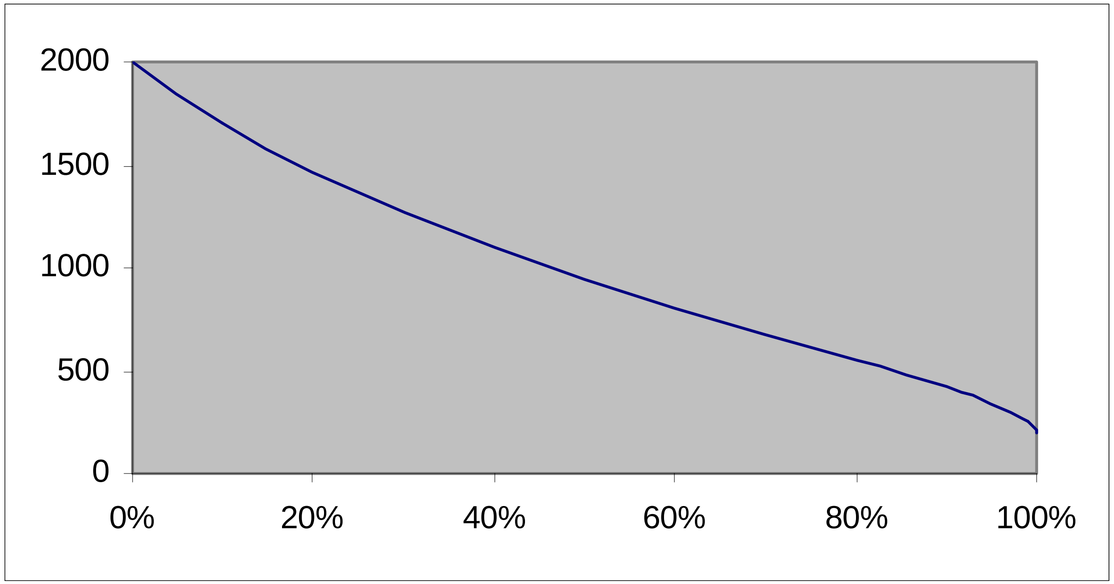
  - where $p_t^{i|V} = Q(\tau_i \leq t \mid V)$
  - $Q(N^{(-i)}(t) = k-1 \mid V)$ : formal expansion of $\prod_{j \neq i} \left(1 - p_t^{j|V} + p_t^{j|V} u\right)$

# *First to default swap*

- *Case where* $k = 1$
- $Q\left(N^{(-i)}(t) = 0 \mid V\right) = \prod_{j \neq i} \left(1 - p_t^{j|V}\right)$ *no defaults for* $j \neq i$
- *premium* = $\sum_{i=1}^{n} M_i E\left[\int_0^T B(t) \prod_{j \neq i} \left(1 - p_t^{j|V}\right) dp_t^{i|V}\right]$
- $= \int_0^T \sum_{i=1}^{n} M_i B(t) E\left[\prod_{j \neq i} \left(1 - p_t^{j|V}\right) \frac{dp_t^{i|V}}{dt}\right] dt$ *(regular case)*
- *One factor Gaussian* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \rho_i^2}}\right)$
- *Archimedean* $p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$

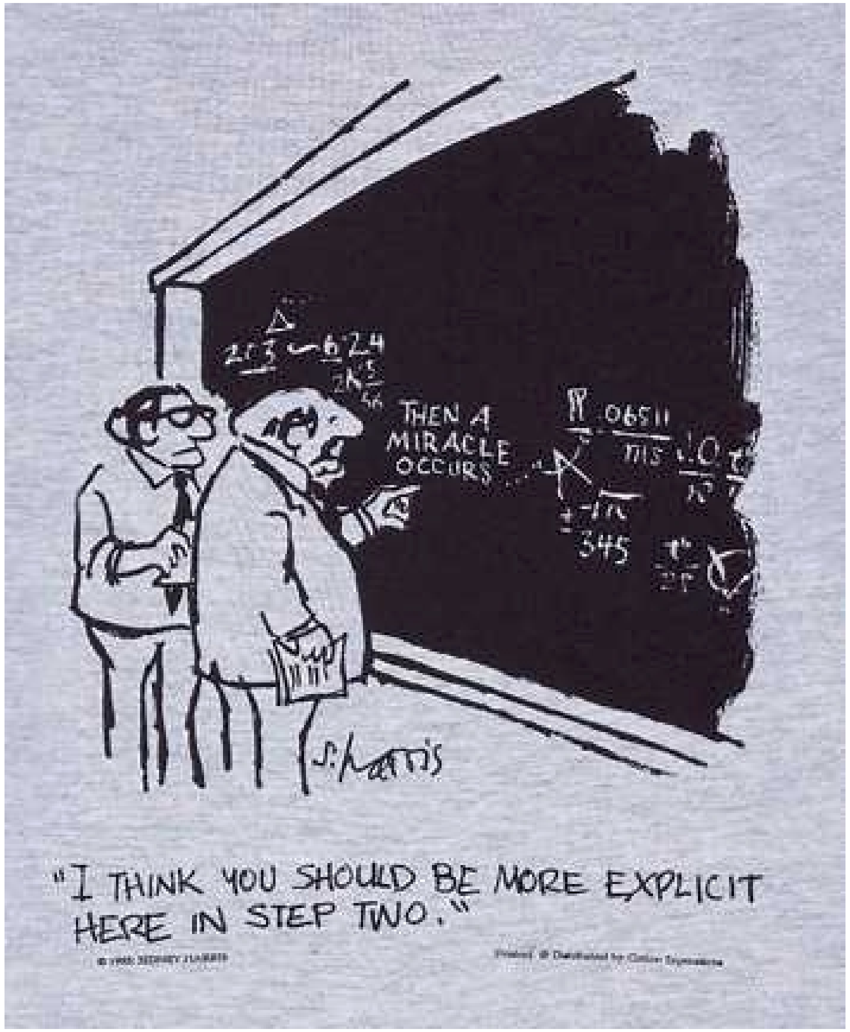
# First to default swap

- One factor Gaussian copula
- n=10 names, recovery rate = 40%
- 5 spreads at 50 bps, 5 spreads at 350 bps
- maturity = 5 years
- x axis: correlation parameter, y axis: annual premium



# *Valuation of CDO's*

*«Everything should be made as simple as possible, not simpler»*




- *Explicit premium computations for tranches*
- *Use of loss distributions over different time horizons*
- *Computation of loss distributions from FFT*
- *Involves integration par parts and Stieltjes integrals*

# Valuation of CDO's

- Loss at t: $L(t) = \sum_{1 \le i \le n} N_i (1-\delta_i) N_i(t)$
  - where $N_i(t) = 1_{\tau_i \le t}$
- Characteristic function $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
- By conditioning $\varphi_{L(t)}(u) = E\left[\prod_{1 \le j \le n} \left(1 - p_t^{j|V} + p_t^{j|V} \varphi_{1-\delta_j}(N_j)\right)\right]$
- If recovery rates follows a beta distribution:

$$\varphi_{L(t)}(u) = E\left[\prod_{1 \le j \le n} \left(1 - p_t^{j|V} + p_t^{j|V} M(a_j, a_j + b_j, iN_j)\right)\right]$$

  - where M is a Kummer function, $a_j$,$b_j$ some parameters
- Distribution of L(t) is obtained by Fast Fourier Transform

# Valuation of CDO's

- Tranches with thresholds $0 \leq A \leq B \leq \sum N_j$
- Mezzanine: pays whenever losses are between A and B
- Cumulated payments at time t: M(t)

$$M(t) = (L(t) - A))\,1_{[A,B]}(L(t)) + (B - A)1_{]B,\infty[}(L(t))$$

- Upfront premium: $E\left[\int_0^T B(t)dM(t)\right]$
  - B(t) discount factor, T maturity of CDO
- Integration by parts $B(T)E[M(T)] + \int_0^T E[M(t)]dB(t)$
- where $E[M(t)] = (B - A)Q(L(t) > B) + \int_A^B (x - A)dF_{L(t)}(x)$

# Valuation of CDO's

- *One factor Gaussian copula*
- *n=50 names, all at 100 bps, recovery = 40%*
- *maturity = 5 years, x axis: correlation parameter*
- *0-4%, junior, 4-15% mezzanine, 15-100% senior*