MULTIMODAL IMPLIED RISK-NEUTRAL DENSITIES

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Abstract

We consider the informational content of a set of USD/DEM forex options with different exercise dates and strikes. We recall how state price densities and european options' prices are related and estimation methods based upon a discrete set of observed option prices. After describing the dataset used, we exhibit some multimodal distributions for future exchange rates at 30, 60, 90, 180, and 270 days' time horizons. High and low probability regions might be explained by non linearities in the local volatility of exchange rates or by the presence of jumps. We look at the steadiness of our results with respect to the chosen methodology.

keywords : state price densities, risk-neutral densities, bayesian estimation, forex options, volatility smile.

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1 Introduction

This paper aims at estimating and analysing the risk-neutral probability measures obtained from the prices of foreign exchange derivative contracts. We shall first remind the various methodologies which permit to recover this probability measure, and investigate the way they are used in practice.

1.1 Options prices and underlying assets

The pricing of derivative contracts is usually based on a specification of the dynamics of the underlying asset. In the BLACK and SCHOLES (1973) and MERTON (1973) framework, markets are typically assumed to be frictionless (i.e. no default risk, no short sale restrictions, no transaction costs) and the time-t price of the underlying asset is supposed to follow a log-normal distribution. In this respect, the probability distributions of the underlying prices are first specified and derivative prices are then derived from these underlying prices. As a consequence, these two prices play different roles. As a matter of fact, this asymmetry was really justified at the beginning, essentially because of the illiquid and potentially inefficient features of these derivative markets. As an example, prices data were generally not available for markets analysts and the Black and Scholes formula played then the role of a convenient benchmark.

Nowadays, options markets have become very liquid (sometimes more liquid than the underlying markets), large option prices data sets are available and the departures from the Black and Scholes formula, though rather weak, cannot be neglected any longer. This fact has motivated a lot of research on how some more complex models could take into account of the actual oberved prices: stochastic volatility models, jump processes... are good examples of these tentatives.

Alternatively, others have tried to infer future prices of some assets from the joint probability distribution of these assets and the (most liquid) corresponding derivative contracts. These approachs are often termed as *reverse engineering* methods of estimation of the risk-neutral probability measure. (see SÖDERLIND and SVENSSON (1997), LAURENT (1998), DEMONSANT (1998)). This is the approach which is investigated in the following sections.

1.2 Observed option prices

The pionnering papers which have first tried to use options prices are now relatively ancient (BREEDEN and LITZENBERGER (1978)). They are in the spirit of ROSS (1976)'s ideas which were building a bridge between (european) options contracts and the Arrow-Debreu contingent prices. For the past few years, the estimation of risk-neutral probability measures[1] has progressed significantly on both empirical and theoretical matters. We now remind the most proeminent results obtained in this approach.

1.2.1 Interpolation

A first approach consists in interpolating the (european) option prices curve $C(t, K)$ with respect to the strike and the maturity. Indeed, we usually observe prices of call or put options for various strikes. BREEDEN and LITZENBERGER (1978) have shown that the state-price density $f(t,e)^2$ can be computed from the price curve $C(t,K)$ through the following formula:

$$\frac{\partial^2 C(t,e)}{\partial K^2} = f(t,e), \tag{1.1}$$

The interpolation of the $C(t,K)$ for all t and all K can be for example achieved by using quadratic functions (SHIMKO (1990)). However such a methodology depends greatly on the interpolation techniques used to perform the computation. DUPIRE (1992, 1994), DERMAN and KANI (1994) have extended the previous result by establishing the link between the local volatility of the underlying price and the partial derivatives of the option prices. More specifically, if we assume that the short rate equals zero and that the price of the underlying asset follows a diffusion process, then we can prove that:

$$\sigma^2(t,K) = 2\,\frac{\dfrac{\partial C(t,K)}{\partial t}}{\dfrac{\partial^2 C(t,K)}{\partial K^2}} \tag{1.2}$$

where $\sigma(t,K)$ is the local volatility of the underlying asset.

DUMAS, FLEMING and WHALEY (1995) investigate the use of the previous formula in the context of options written on stock indices. RUBINSTEIN (1994), DERMAN and KANI (1994) use a binomial tree to compute the local volatility. Finally, LAURENT and LEISEN (1998) extend these results to the case of Markov chains and provide some necessary and sufficient conditions for the risk-neutral probability distribution to be recovered.

1.2.2 Parametric distributions

Another approach consists in searching the unknown distribution in a class of parametric functions. The distribution is such that the distance between theoretical and observed prices is minimized. As an example BAHRA (1997), MELLICK and THOMAS (1997), SÖDERLIND and SVENSSON (1997), CAMPA, CHANG and REIDER (1997), JONDEAU and ROCKINGER (1997) use the class of linear convex combination of two log-normal distributions. Alternatively BATES (1996a, 1996b) studies USD/DEM options in the context of jump processes with possible stochastic volatility. The results show that observed prices are consistent with a jump component of low frequency but large magnitude. Both papers estimate the underlying parameters as well as the skewness and kurtosis parameters but do not investigate the overall shape of the distribution nor the existence of several modes.

MADAN and MILNE (1994), ABKEN, MADAN and RAMAMURTIE (1996) approximate the option with polynomials. Then they are able to compute a polynomial approximation of the risk-neutral density with respect to a gaussian measure.

JARROW and RUDD (1982), CORADO and SU (1996), JONDEAU and ROCKINGER (1997) perform a fourth-order Edgeworth development around the log-normal distribution and apply this technique to the options written on exchange rates and stock indices.

1.2.3 Semi-parametric approach

Eventually, semi-parametric approaches can be implemented for discrete and continuous models. Such approaches do not arbitrarily constrain the shape of risk-neutral densities. Rubinstein (1994), Derman, Kani and Chriss (1996) consider binomial and trinomial trees. Theoretical option prices computed under these trees must be equal to observed option prices. Moreover, the estimated risk-neutral distribution of the underlying asset must be close to a discretised version of the log-normal distribution. Proximity can be measured with a quadratic, entropy or goodness of fit criterion. In the continuous case, Buchen et Kelly (1996), Avellaneda, Friedman, Holmes et Samperi (1996) use the entropy criterion to estimate a risk-neutral distribution from an a priori log-normal distribution. Magnien, Prigent et Trannoy (1996), Laurent et Scaillet (1997) consider the quadratic criterion. Taking into account different information sets, Clément, Gouriéroux et Monfort (1997) build risk-neutral random measures from observed option prices.

1.3 Practical implementation

Several central banks conduct quantitative analysis based on observed option prices. Malz (1995a, 1995b), for the Fed of New-York, Jondeau et Rockinger (1997) for the Bank of France, estimate exchange rate distributions from currency options. Abken, Madan et Ramamurtie (1996) for the Fed of Atlanta, Bahra (1997) for the Bank of England, Coutant, Jondeau et Rockinger (1997) for the Bank of France consider interest rate options. Campa, Chang et Reider (1997) discuss how to use these approaches to assess the credibility of currency bounds within SME and how financial markets react to monetary policy.

Financial institutions trading derivatives also use these methods:

- local volatility estimation and implied trees are commonly used for the pricing and hedging of path-dependent options. This approaches allow to price exotic options consistently with plain vanilla options while being arbitrage free.

- Trading books often include non traded european options, i.e. that do not correspond to standard maturities and exercise prices at mark to market date. The mark to market of these options is usually done by interpolating liquid option prices. However this procedure is not always arbitrage free.

The paper is organised as follows. In Section 2, we present the currency options data that has been used. Section 3 recalls Breeden and Litzenberger (1978) result relating option prices and state price densities. Section 4 presents the Bayesian approach to estimating state price densities. Section 5 presents the estimated risk-neutral densities and discusses some possible explanations. In Section 6, we assess the reliability of our results. Section 7 concludes. Technical proofs are gathered in an Appendix.

2 The data

In the following, the underlying asset is a foreign currency and we thus consider european exchange rate options. We denote by $e(t)$, the exchange rate at date t, i.e. the amount of local currency to get one unit of foreign currency. $B(t,T)$ is the local currency price, at date t, of a zero-coupon bond paying one unit of local currency at T. $\bar{B}(t,T)$ is the foreign currency price, at date t of a zero-coupon bond paying one unit of foreign currency at date T. The forward exchange rate at date T seen from date t is defined by $e(t,T) = e(t)\dfrac{\bar{B}(t,T)}{B(t,T)}$.

At the current date $t_0 = 0$, we observe a set of european calls with different exercise dates $t = t_1, \dots, t_N$ and exercise prices for each exercise date t_i, $K_{i,j}$, $j = 1, \dots, J_i$. The payoff of such an option is equal to $(e(t_i) - K_{i,j})^+$ units of local currency[3]. We denote by $C(t_i, K_{i,j})$ the price at date t_0 of such an option[4].

Several currency exchanges are operating around the world. We can think of:

- The Chicago Mercantile Exchange where currency futures and options on currency futures are traded. Deutschemark and Yen contracts were the most actively traded in 1997, followed by Swiss Franc, Canadian Dollar and Bristish Pound. The traded options are american style and have weekly, monthly and quarterly maturities. The average daily traded volume on USD/DEM options contracts has been 3360 for the first three quarters of 1998[5]. At the end of september 1998, the outstanding volume was of 47 713 contracts. The market is thus rather liquid by not as liquid as the underlying futures market[6]. Quotes on futures and options are available from CME by ftp.

- currency options are also traded on UCOM (*United Currency Options Market*) of the Philadelphia Stock Exchange. These options can have standardised maturities[7] (mid month, quarterly and, long-term), and can be american or european type. For example, une USD/DEM option gives the right to buy at a given date 62500 Deutschemarks at a predetermined price (see table 1). 65 494 currency options contracts were traded in january 1999[8]

Tableau 1 : options contracts on PHLX. Source : PHLX. User's Guide to Currency Options.

local currency	underlying currency	Nominal
U.S. Dollar	Deutsche mark (USD/DEM)	62500 DEM
U.S. Dollar	British Pound (USD/GBP)	31250 GBP
U.S. Dollar	French Franc (USD/FRF)	250 000 FRF
U.S. Dollar	Yen (USD/JPY)	6 250 000 JPY
Deutsche Mark	Yen (DEM/JPY)	62 500 DEM
British Pound	Deutsche Mark (GBP/DEM)	31250 GBP

Currency options traded on exchanges remain less liquid that OTC options. The trading volume on exchanges is about one tenth of volume traded on OTC markets. Several criteria are used to choose quotes. The trading volume should be important, the bid-ask spreads small. Getting synchronous option and underlying prices is also an important issue in order to get unbiased implied volatilities.

For the previous reasons, we have chosen OTC prices. To ease further comparisons of methods, we have used the dataset of Avellaneda, Friedman, Holmes and Samperi (1996). We have prices corresponding to twenty five options categories associated with five exercise dates and five exercise prices. Maturities are one, two, three, six and nine months. The data consist in bid and ask prices and as usual (see e.g.Buchen and Kelly (1996), Avellaneda et al. (1996), Malz (1995a, 1995b)), we have been working with mid-prices.

Tableau 2 : currency options prices. 23 août 1995. Source : Avellaneda et al.

Maturité	Type	Strike	Bid	Offer	Volat.
30 jours	Call	1.5421	0.0064	0.0076	14.9
	Call	1.5310	0.0086	0.0100	14.8
	Call	1.4872	0.0230	0.0238	14.0
	Put	1.4479	0.0085	0.0098	14.2
	Put	1.4371	0.0063	0.0074	14.4
60 jours	Call	1.5621	0.0086	0.0102	14.4
	Call	1.5469	0.0116	0.0135	14.5
	Call	1.4866	0.0313	0.0325	13.8
	Put	1.4312	0.0118	0.0137	14.0
	Put	1.4178	0.0087	0.0113	14.2
90 jours	Call	1.5764	0.0101	0.0122	14.1
	Call	1.5580	0.0137	0.0160	14.1
	Call	1.4856	0.0370	0.0385	13.5
	Put	1.4197	0.0141	0.0164	13.6
	Put	1.4038	0.0104	0.0124	13.6
180 jours	Call	1.6025	0.0129	0.0152	13.1
	Call	1.5779	0.0175	0.0207	13.1
	Call	1.4823	0.0494	0.0515	13.1
	Put	1.3902	0.0200	0.0232	13.7
	Put	1.3682	0.0147	0.0176	13.7
270 jours	Call	1.6297	0.0156	0.0190	13.3
	Call	1.5988	0.0211	0.0250	13.2
	Call	1.4793	0.0586	0.0609	13.0
	Put	1.3710	0.0234	0.0273	13.2
	Put	1.3455	0.0173	0.0206	13.2

Figure 1 plots the implied volatilities corresponding to different exercise prices and dates. We obtain the standard U-shape smiles that tend to flatten for longer maturities.

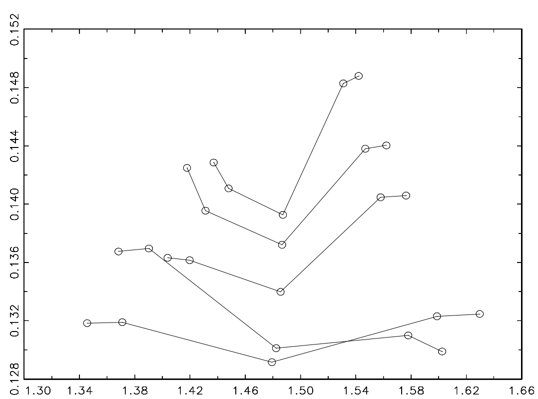

Figure 1 : Smiles de volatilité

3 State price densities and option prices

In this section, we relate state price densities and option prices; in a first step, we proceed heuristically, then we recall Breeden et Litzenberger (1978) result.

3.1 Butterfly premia

Let us consider some option prices, the corresponding payoffs having the same exercise date but different exercise prices.

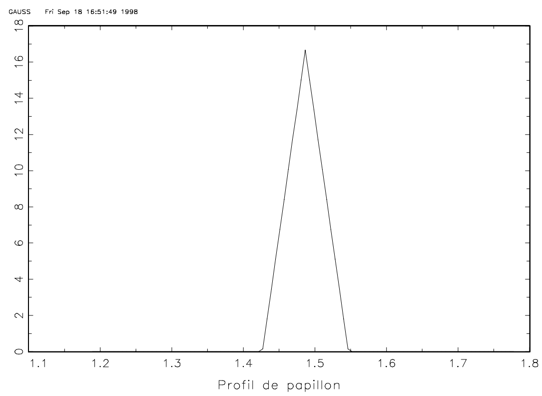

Figure 2 : butterfly profile

Let us assume that we can trade three options, with exercise date t, and exercise price $K-\varepsilon, K, K+\varepsilon$ ($\varepsilon > 0$). A **butterfly** payoff can be synthetised by holding $\frac{1}{\varepsilon^2}$ options with exercise price $K-\varepsilon$, $\frac{-2}{\varepsilon^2}$ options with exercise price K and $\frac{1}{\varepsilon^2}$ options with exercise price $K+\varepsilon$. The butterfly price is then equal to $\dfrac{C(t,K-\varepsilon)-2C(t,K)+C(t,K+\varepsilon)}{\varepsilon^2}$.

The butterfly provides a positive payoff only if the exchange rate at exercise date t is located around K. The price of such a financial product is positively related to the probability for the exchange rate to be close to K at date t is high and to the risk-premia corresponding to that exchange rate level. Let us remark that the butterfly price is close to $\dfrac{\partial^2 C(t,K)}{\partial K^2}$[9]. We will further consider the market price of a payoff contingent on the exchange rate to be around K at date t.

3.2 Asset prices and option premia

At this stage, we assume that we can trade european calls with exercise date t whatever the exercise price K. The corresponding option premia is denoted by $C(t,K)$. We moreover assume that:

i. $C(t,K)$ is positive, decreasing and convex.

ii. $C(t,K)$ is twice differentiable in K, for all $K \geq 0$.

iii. $\dfrac{\partial C}{\partial K}(t,0) = -B(0,t)$.

iv. $\lim_{b\to\infty} C(t,b) - b\dfrac{\partial C}{\partial K}(t,b) = 0.$

Assumption (i) is a consequence of the absence of arbitrage opportunities. Thus, an option price is positive, a call spread option price [10] is positive and, the butterfly prices are positive since the associated payoffs are positive.

Under fixed exchange rate regimes or in discrete state space models such as binomial or trinomial trees, assumption (ii) is not fulfilled. There then exist some exchange rate levels, where the first derivative is discontinuous[11]. These discontinuities correspond to probability masses at given exchange rate levels. The regularity assumption is thus fulfilled for "continuous" models.

To interpret the third assumption, we recall that $-\dfrac{\partial C}{\partial K}(t,0)$ is the price of a digital call with exercise price equal to zero, i.e. a payoff that pays 1 for any strictly positive value of the exchange rate. Assumption (iii) thus means that a contract that pays 1 only if the exchange rate is equal to zero, is traded at a zero price. From a financial point of vue, the exchange rate being equal to zero means that the foreign currency has no value and is closely related to country bankruptcy and the disappearance of this foreign currency.

Similarly, it can be shown (see Appendix) that assumption (iv) means that one can be insured against the disappearance of local currency at a zero price[12].

We can then show (see Appendix):

$$C(t,K) = \int (e-K)^{+} \frac{\partial^2 C}{\partial K^2}(t,e)de \tag{3.3}$$

Option premia thus appear as integrals of the associated option payoffs, $(e-K)^{+}$, w.r.t. the measure $\frac{\partial^2 C}{\partial e^2}(t,e)de$. Breeden and Litzenberger (1978) relate more general payoffs and prices. Let us consider a payoff $g(e)$ at date t, where g is a smooth function of exchange rate e equal to zero above some given exchange rate level. We can then write the price at date t_0 of this payoff as (see Appendix):

$$\int g(e) \frac{\partial^2 C}{\partial K^2}(t,e)de \tag{3.4}$$

We will further denote by $f(t,e) = \dfrac{\partial^2 C(t,e)}{\partial K^2}$, the **state price density** for exercise date t[13]. This state price density is some kind of discounting function conditional on the future exchange rate $e(t)$. Indeed, let us consider a payoff at date t equal to a Dirac measure $\delta(e_0)$ at the level e_0 (this is the limit of the above butterfly). Its price is $\int \delta(e_0) f(t,e)de = f(t,e_0)$. The state price density $f(t,e)$ thus provides the market price of a payoff at time t contingent on the exchange rate being equal to an arbitrary predetermined value.

Thanks to Assumption (iii), on can check that:

$$B(0,t) = \int \frac{\partial^2 C}{\partial K^2}(t,e)de \tag{3.5}$$

$\dfrac{1}{B(0,t)}\dfrac{\partial^2 C}{\partial K^2}(t,e)de$ is thus a probability measure. We denote by $\dfrac{1}{B(0,t)}\dfrac{\partial^2 C}{\partial K^2}(t,e)$, the **risk-neutral density** (with respect to the Lebesgue measure). Let us remark that state price density and risk-neutral density only differ by a multiplicative discounting term.

4 Risk-neutral density estimation

From a practical point of vue, we never start with a continuous record of option prices but only with a discrete set of observed data for liquid options. Since the information set is smaller, it is not possible to fully determine the state price density. A state price density must fulfill:

$$\int (e(t_i) - K_{i,j})^+ f(t_i, e) de = C(t_i, K_{i,j}),$$

$t_i = t_1, \ldots, t_N$, $\forall K_{i,j}$, $\quad j = 1, \ldots, J_i$. Let us remark that we deal with linear constraints on f and that the unknown quantity is a function. We thus deal with a non parametric estimation problem. In the remaining of the paper, we will consider independently the functions $f(t_i, e), f(t_j, e)$, $t_i \neq t_j$ and our approach will be purely static. Without going into technical details, the existence of state price densities is guaranted when markets are frictionless and arbitrage-free.

As a consequence of the estimation of state price densities, we will be able to estimate options prices. Indeed, given a time horizon t_i, and an estimated density $\hat{f}(t_i, e)$ for this horizon, we obtain the following estimate, $\hat{C}(t_i, K)$ for the option price:

$$\hat{C}(t_i, K) = \int (e - K)^+ \hat{f}(t_i, e) de$$

Since state price densities are usually not unique, we need some criterion to decide how to choose one. We rely here upon the Bayesian approach to estimating state price densities introduced by Jackwert and Rubinstein (1995), Buchen and Kelly (1996).

The core idea is to choose the state price density, consistent with observed option prices, the closest as possible, according to some criteria discussed below, to an a priori (or reference) state price density. In the remaining of the paper, the exchange rate will be log-normally distributed under the a priori probability measure. Let us denote by $f_0(t, e)$ the corresponding state price density. $\dfrac{f(t,e)}{f_0(t,e)}$ is the density with respect to the reference probability measure. If the two probability measures are equal, the previous quantity is equal to one. We will more particularly use the quadratic criterion:

$$\frac{1}{2} \int \left(\frac{f(t,e)}{f_0(t,e)} - 1 \right)^2 f_0(t,e) de,$$

the weighting $f_0(t, e)$ means that the differences between f and f_0 have more importance when f_0 is large, that is in regions with large a priori probabilities. Another commonly used criterion is the entropy criterion:

$$\int \frac{f(t,e)}{f_0(t,e)} \log \left(\frac{f(t,e)}{f_0(t,e)} \right) f_0(t,e) de.$$

Once the criterion has been chosen, the estimation problem, for horizon t_i consists in solving the following optimisation problem[14] :

$$\min_f J\left(f(t_i, .), f_0(t_i, .)\right),$$

where J is the proximity criteria between state price densities, under the consistency with observed option prices (with exercise date t_i) constraints, $C(t_i, K_{i,j})$, $j = 1, \dots, J_i$, and consistency constraints with the underlying asset price[15], $\bar{B}(t_0, t_i)e(t_0)$ and with the riskless asset, $B(t_0, t_i)$:

$$\begin{cases} C(t_i, K_{i,j}) & = & \int (e - K_{i,j})^+ f(t_i, e)de, \\ \bar{B}(t_0, t_i)e(t_0) & = & \int ef(t_i, e)de, \\ B(t_0, t_i) & = & \int f(t_i, e)de. \end{cases}$$

The existence of a solution can be easily proved for the quadratic criterion (from the projection theorem) and is more difficult to state for the entropy criterion. The Lagragian $\mathcal{L}(f(t_i, .), \lambda_i)$ can be written as:

$$J - \int \left(\lambda_{i,0} + \sum_{j=1}^{J_i} \lambda_{i,j}(e - K_{i,j})^+ \right) f(t_i, e)de.$$

where $\lambda_i = (\lambda_{i,0}, \lambda_{i,1}, \dots, \lambda_{i,J_i})'$ are the Lagrange multipliers. The first order conditions provide the following expressions, $f_{\lambda_i}(t_i, e)$, respectively for the quadractic and the entropy criteria:

$f_0(t_i, e) \left(1 + \lambda_{i,0} + \sum_{j=1}^{J_i} \lambda_{i,j}(e - K_{i,j})^+ \right),$

$f_0(t_i, e) \exp \left(\lambda_{i,0} - 1 + \sum_{j=1}^{J_i} \lambda_{i,j}(e - K_{i,j})^+ \right).$

The Lagrange multipliers $\lambda_{i,j}$ are determined from the consistency with observed option prices constraints:

$$\int (e - K_{i,j})^+ f_{\lambda_i}(t_i, e)de = C(t_i, K_{i,j}),$$

$\forall j = 1, \dots, J_i$. This is a set of linear equations for the quadratic criterion and non linear in the case of the entropy criterion.

5 Multimodal densities

Figures 3 to 7 represent a priori and estimated risk-neutral densities for 30, 60, 90, 180 and 270 days. We get probability distributions with two or three modes[16]. Thus multimodality may be surprising. We now discuss this point more in detail.

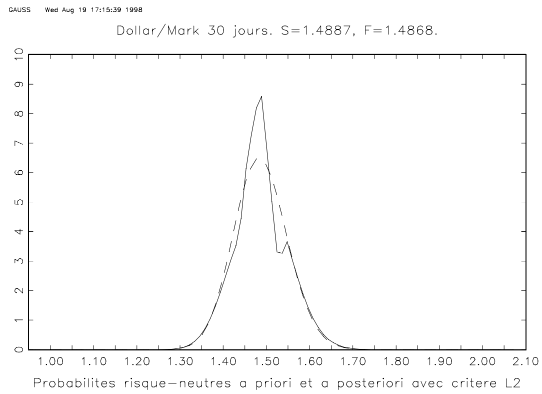

Figure 3 : a priori and estimated densities with L_2 criteria

Let us first notice that in a number of empirical studies, the risk-neutral density is chosen among a set of distributions that are unimodal. This is for example the case with most stochastic volatility models. This precludes a priori the presence of several modes. However, if this implicit constraint is removed, it is not uncommon to get multimodal risk-neutral densities (Jackwert et Rubinstein (1995), Abken, Madan, et Ramamurtie (1996), Jondeau et Rockinger (1997)).

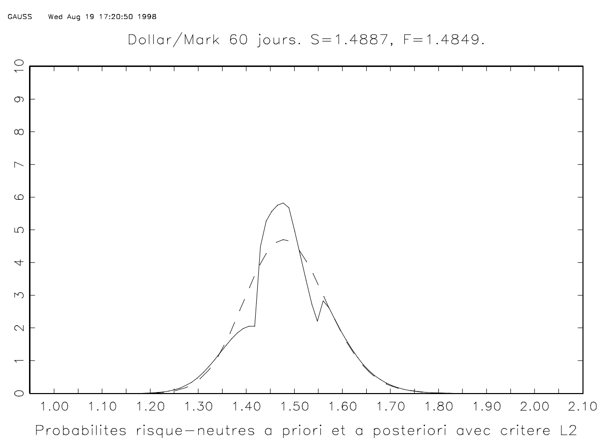

Figure 4 : a priori and estimated densities with L_2 criteria

We may also remark that the presence of several modes is not inconsistent with some standard option pricing models:

- jump-diffusion models of Merton (1976) type are usually associated with multimodal distributions for short term horizons.
- When the exchange rate is modeled by a diffusion process, we may think of non linear dependence of the local volatility coefficient w.r.t. the underlying exchange rate. The exchange rate will then tend to stay during longer periods around levels associated with low volatility. There is thus an inverse relationship between density and local volatility as can be seen from the (1.2) de Derman, Kani, Dupire expression. The square of the local volatility is inversely proportional to the density. Based

upon the same dataset, AVELLANEDA et al. (1996) obtain irregular shapes for local volatiliy, which is consistent with our own results on risk-neutral densities.

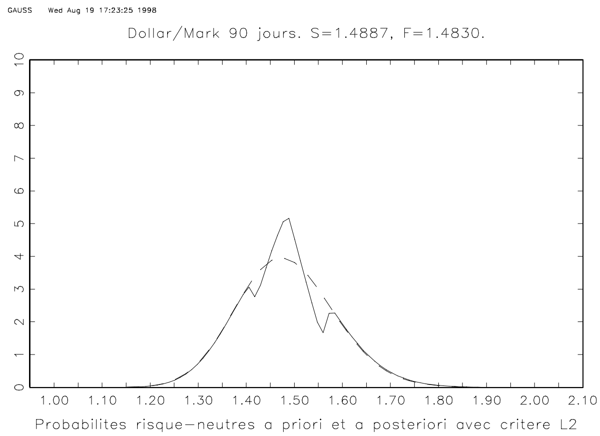

Figure 5 : a priori and estimated densities with L_2 criteria

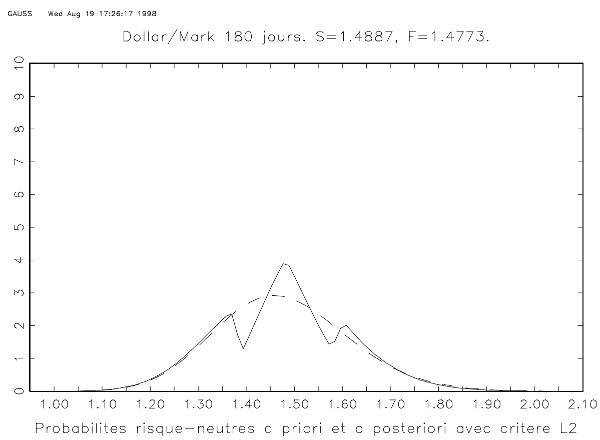

Figure 6 : a priori and estimated densities with L_2 criteria

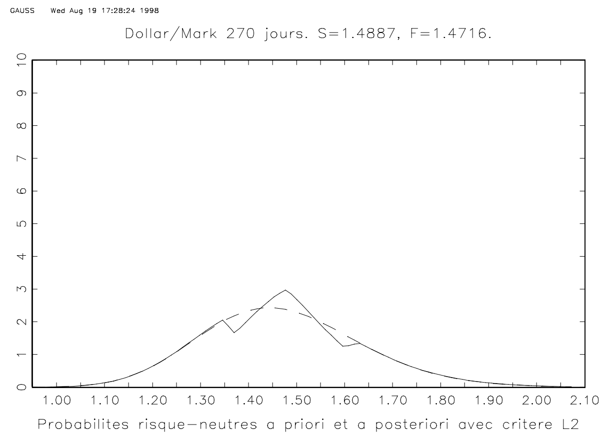

Figure 7 : a priori and estimated densities with L_2 criteria

Skewness and excess kurtosis of asset returns are a standard stylised fact. GARCH models, such as E-GARCH or T-GARCH, and stochastic volatility models (when volatility is negatively correlated with underlying asset) are consistent with these stylised facts. The same applies for the estimated densities with the Bayesian approach. However, excess kurtosis is now a consequence of the presence of two secondary modes (the density is “thinner" around its mean) and is not due to fat tails. Similarly, the skewness is due to the asymetry between these two modes.

Lastly, the modes are located around the same exchange rate levels, whatever the horizon, while the estimations have been conducted independently. Hopefully, this is a preliminary sign that the above results are not purely artificial and will provide some relevant information.

6 Assessing the steadiness of multimodality

We now consider in detail how the previous results depend on our methodology. We study the consequences of observed option prices uncertainty, the choice of a priori probability density function, benchmark numéraire and the proximity criteria between pdf.

6.1 observed option prices and implied probability density functions

In OTC markets, traded option prices may be mid, bid or ask prices and differ among various market-makers. Though short-term currency options are rather liquid, there thus remains uncertainty in the input option prices. In figure 8, we assess the sensitivity of the estimated pdf w.r.t. a change in observed option prices. The put USD/DEM price, with exercise date 1.43 and exercise date 60 days is updated from 1.27% to 1.37%, other option prices remaining unchanged. This has a significant effect on the estimated risk-neutral density: input option prices do bring some information that leads us to modify the a priori log-normal distribution. The effects on the estimated density are located in a neighbourhood of the exercise price of the modified input option. The qualitative features of the estimated density such as multimodality remain unchanged.

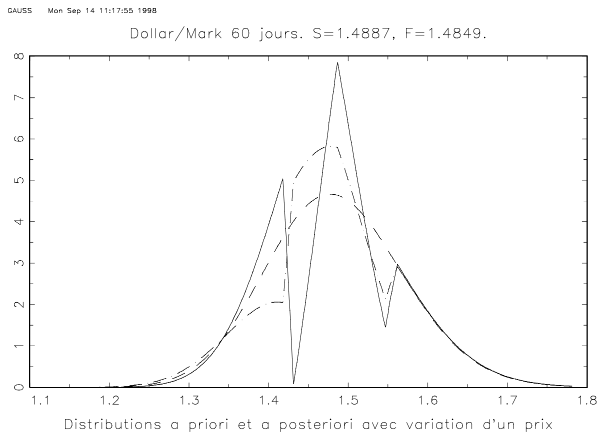

Figure 8 : assessing the sensitivity to input option prices

The quadratic criterion allows a more detailed theoretical analysis of the previous effect. In appendix C, we show that the relative difference between a priori and estimated densities, $\frac{f_\lambda(t,e)}{f_0(t,e)} - 1$ can be written as:

$$(C(t,K) - C_0(t,K))' \mathrm{V}_0[(e(t) - K)^+]^{-1}(e-K)^+,$$

where $C(t,K)$ is the vector of observed option prices with exercise date t, $C_0(t,K)$ the vector of Black and Scholes option prices with exercise date t, $(e-K)^+$ the vector of option payoffs with maturity t and, $\mathrm{V}_0[(e(t) - K)^+]$, the payoff variance-covariance matrix, assuming that $e(t)$ is (log-normally) distributed under the a priori model. The difference between estimated and a priori densities thus depends linearly on the differences between observed option prices and option prices computed under the a priori model.

6.2 Choice of a priori density

Due to the practical relevance of Black and Scholes pricing formula, we have considered that the future exchange rate was log-normally distributed with mean the forward exchange rate. It is indeed the benchmark modeling, and we will deviate from it only if it is strongly required by observed data. Let us remark that empirical studies of more sophisticated option pricing models (stochastic volatility, interest rates, presence of jumps) usually study relative improvements w.r.t. Black and Scholes model. The log-normal prior is the most common one in these Bayesian approaches to the estimation of risk-neutral densities[17].

However, let us remark that the Black and Scholes is parametrised by some volatility coefficient σ. Under the forward (a priori) probability measure, future exchange rate are modelled by:

$$e(t) = e(0,t) \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right),$$

where W_t is centered Gaussian with variance equal to t. The density $f_{0,\sigma}(t,e)$, $\forall e>0$, is now written as:

$$\frac{1}{e\sqrt{2\pi\sigma^2 t}}\exp\left[-\frac{1}{2\sigma^2 t}\left(\frac{\sigma^2 t}{2}+\log\frac{e}{e(0,t)}\right)^2\right].$$

This implies that the estimated state price density $f_{\lambda(\sigma),\sigma}(t,e)$ depends on the volatility parameter in the a priori model, both through $f_{0,\sigma}$ and indirectly through the Lagrange multipliers $\lambda(\sigma)$.

In Figure 9, we show three estimated risk-neutral densities corresponding to $\sigma = 14\%$, $\sigma - 3\%$,$\sigma + 3\%$. Let us remark that, even for large modifications of the volatility parameter, the overall shape of estimated densities remains unchanged.

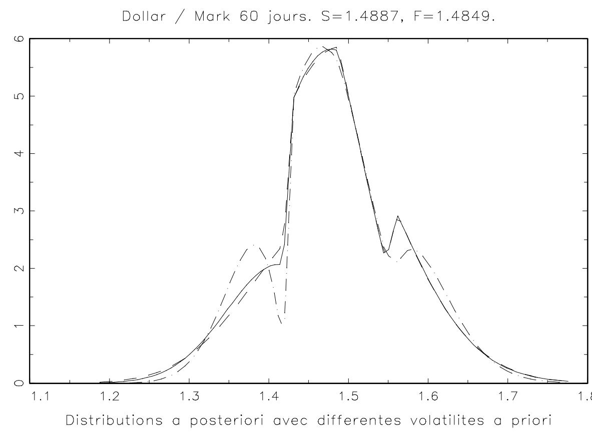

Figure 9 : estimated densities with different volatility parameters

Using "the" implied volatility seems the simplest idea; we see in table two implied volatilities range from 13,8% to 14,5% and depend on exercise prices. We can then consider an optimal implied volatility parameter, which is the solution of:

$$\min_{\sigma}\int\left(\frac{f_{\lambda_i(\sigma),\sigma}(t,e)}{f_{0,\sigma}(t,e)}-1\right)^2 f_{0,\sigma}(t,e)de.$$

Thus, we look for a volatility parameter such that the modification of the a priori model (Black and Scholes) is the smallest possible. It can be shown (see appendix) that the previous optimisation problem is equivalent to:

$[(C-C_{0,\sigma})(t,K)]'\,\mathrm{V}_{0,\sigma}[(e(t)-K)^+]^{-1}\,[(C-C_{0,\sigma})(t,K)]$,

where $C(t,K)$ is the vector of observed option prices for exercise date t, $C_{0,\sigma}(t,K)$ the vector of Black and Scholes option prices computed under volatility σ, $\mathrm{V}_{0,\sigma}[(e(t)-K)^+]$ the variance-covariance matrix of option payoffs, $e(t)$ being log-normally distributed with volatility σ. The optimal implied volatility parameter is in fact a non linear least square estimate and is chosen in order to minimise a the deviations between observed prices and Black and Scholes prices.

6.3 Choice of benchmark numéraire

The previous quantities are expressed in some arbitrary currency or numéraire. If $e(t)$ is the number of USD to be paid to get one DEM, we can equivalently say that $e(t)$ is the price of one DEM in numéraire

DEM. The USD payoff of the call DEM is equal to $(e(t) - K)^+$, while the payoff of the same financial product expressed in DEM is equal to $\left(1 - \frac{K}{e(t)}\right)^+$ (it is a put USD). Similarly, we know that risk-neutral densities are not invariant by change of numéraire (GEMAN, EL KAROUI, et ROCHET (1995)). LAURENT et SCAILLET (1997) show the Bayesian approach provides estimated densities that also depend upon the chosen numéraire. To assess this dependence effect, we have plot the estimated densities taken either USD, either DEM as the benchmark numéraire. Figure 10 shows that, from an empirical point of view, the two densities are almost identical.

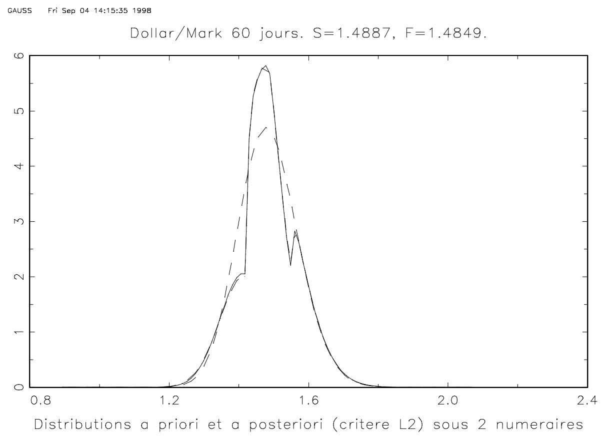

Figure 10 : estimated densities with two different numéraires

6.4 Dependence upon the chosen proximity criterion

Several criteria can be used to measure the proximity between two densities. One may try to choose between quadratic, entropy or other criteria for economic reasons, or from a theoretical point of view, but the final choice remains whatever arbitrary. Figure 11 plots the estimated risk-neutral densities obtained with the quadratic and entropy criteria. We see that these are quite close and from a practical point of view, at least given our dataset, the choice of criterion is not a critical issue.

Figure 11 : estimated densities with L_2 and entropy criteria

This is encouraging since the scarcity of our observed option prices is a rather adverse situation. In the extreme case where one would observe a single option price, the a priori model would allow a perfect calibration thanks to the volatility parameter. The estimated density would then always correspond to Black and Scholes, whatever the proximity criterion. In the opposite case where one would observe option prices for all exercise prices, the estimated density would not either depend upon the chosen criterion. This results from the Breeden and Litzenberger property.

7 Conclusion

We have considered how a posteriori implied probability density functions of USD/DEM exchange rates deviate from log-normal a priori pdf's. In most cases, we obtain multimodal distributions. As a consequence of the presence of several modes, estimated probability distributions exhibit skewness and excess kurtosis. Let us remark that multimodality precludes GARCH or stochastic volatility models in their simplest form. However, our results might be explained by the presence of jumps or by strong non-linearities in the exchange rate diffusion coefficient.

In order to assess the reliability of our results, we study how estimated densities depend on the methodology. We consider both the quadratic and entropy criteria, the effects of the choice of the benchmark numéraire or of the a priori volatility parameter. We find that these have small impact on the estimated densities, which is an indication of good specification. We also consider how estimated risk-neutral densities depend on input observed option prices. The overall shape, including the presence of several modes, is not modified when observed option prices remain within the bid-ask spreads.

However, our results must be handled with caution. We used only observations for one day and one given couple of currencies. It should be interesting to confirm these preliminary results by further empirical studies based upon larger datasets.

Notes

[1]In our framework, a risk-neutral probability measure is equivalent to the historical probability measure and is such that current asset prices are equal to their expected discounted payoffs.

[2]When the state space is continuous, the state price density is the analogue of the prices of Arrow-Debreu state contingent claims in discrete models. Up to a discouting coefficient, state price densities and risk-neutral densities w.r.t. the Lebesgue measure are equal.

[3]Here, we do not take into account lags between exercise and effective payments.

[4]This price also depends on t_0 ; thereafter, t_0 is fixed, say $t_0 = 0$ and we omit the dependence upon t_0 when it is not necessary.

[5]Most of the activity in 1998 was related to the Yen and the volume on DEM options have decreased by half compared the 1997 figures.

[6]Over the tree first quarters of 1998, the average daily volume on DEM futures contracts was 29 251 contracts.

[7]Special maturities can also be traded and the PHLX tries to promote customised contracts.

[8]515 150 currency options were thus traded in 1998, to be compared with the 2 952 700 contrats traded on the CME for three quarters of 1998.

[9]We thereafter assume that this second derivative is well defined.

[10]A call spread with exercise prices K_1 and K_2, $K_1 < K_2$ pays $(e(t) - K_1)^+ - (e(t) - K_2)^+$ at date t.

[11]Since we have assumed convex option prices, there exist left and right limits.

[12]Let us remark that assumption (i) is quite general, while assumptions (ii), (iii), (iv) might be questionned. It is possible to get of these assumptions ; For instance, Jackwerth and Rubinstein (1995) remain in a discrete state space framework where assumption (ii) is not fulfilled.

[13]As for option prices, the state price density depends on the current date t_0 ; to simplify notations, we omit this dependence.

[14]We proceed here heuristically, without detailing the optimisation sets ; see Luenberger (1969) for a presentation of the functional optimisation techniques.

[15]Since the underlying asset can be viewed as a call option payoff with a zero exercise price, we will not further isolate this constraint.

[16]A mode corresponds to a local maximum of the risk-neutral density.

[17]Some authors also use a uniform distribution; this a priori is implicit in the B-spline interpolation approach of Magnien, Prigent et Trannoy (1996)).

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Appendix A : Interpretation of assumption (iv)

- Let us consider a european put option with exercise date t and exercise price K. The payoff of this option is equal to $(K - e(t))^+$ units of local currency. Let us denote by $P(t, K)$ the price of this option at $t_0 = 0$. We recall put-call parity for currency options:

$$C(t, K) - P(t, K) = e(0)\bar{B}(0, t) - KB(0, t).$$

- $(K - e(t))^+$ units of local currency have the same value than $\frac{1}{e(t)}(K - e(t))^+$ of foreign currency, where $\frac{1}{e(t)}$ is the price of one unit of local currency expressed in foreign currency; $K\left(\frac{1}{e(t)} - \frac{1}{K}\right)^+$ now appears as a call payoff on the local currency with exercise date t and exercise price $\frac{1}{K}$. Let us denote by $\bar{C}(t, u)$ the price *in* *foreign currency* of a call option on the local currency with exercise price u. Since a put on foreign currency is equal to a call on local currency, we get:

$$P(t,K) = Ke(0)\bar{C}\left(t, \frac{1}{K}\right).$$

- From the two previous equations and using $\dfrac{\partial \bar{C}(t, u=0)}{\partial u} = -\bar{B}(0,t)$ (which is equivalent to assumtion (iii) when the benchmark numéraire is the foreign currency), we obtain Assumption (iv). This assumption thus means that the local currency will not vanish before date t.

Appendix B : Proof of equation (3.3)

- Since $C(t,u)$ is positive and non decreasing, $C(t,u)$ has a limit when $u \to \infty$.
- As $\dfrac{\partial C(t,u)}{\partial K}$ is negative and non decreasing, it admits a limit when $u \to \infty$. Morevover, since $C(t,u)$ has a limit when $u \to \infty$, the limit of $\dfrac{\partial C(t,u)}{\partial K}$ must be equal to zero: $\lim_{u\to\infty} \dfrac{\partial C(t,u)}{\partial K} = 0$.

Integrating by parts, we get:

$$\int_0^b (e-K)^+ \frac{\partial^2 C}{\partial K^2}(t,e)de = C(t,K) - C(t,b) + (b-K)\frac{\partial C}{\partial K}(t,b),\ \forall b > K.$$

We eventually obtain the stated result by letting b going to infinity, by using the above result about the limit of the derivative of the option price, $\lim_{b\to\infty} \dfrac{\partial C(t,b)}{\partial K} = 0$ and assumption (iv), $\lim_{b\to\infty} C(t,b) - b\dfrac{\partial C}{\partial K}(t,b) = 0$.

Appendix B : Proof of equation (3.4)

Integrating by parts, we can express the payoff as:

$$g(e) = g(0) + eg'(0) + \int_0^{+\infty} g''(u)(e-u)^+ du \tag{7.6}$$

The first term $g(0)$ corresponds to a constant payoff(a zero-coupon bond with maturity t and nominal value $g(0)$), the second term corresponds to a holding of $g'(0)$ units of foreign currency at t. The last term can be seen as an infinite sum of call payoffs with exercise price u, $(e-u)^+$, and holding amounts $g''(u)du$. From the law of one price, the prices of these three payoffs are respectively, $g(0)B(t_0,t)$, $g'(0)e(t_0)\bar{B}(t_0,t)$ and $\displaystyle\int g''(u)C(t,u)du$. Thanks to a repeated integration by parts, we can write the previous integral as $g(0)\dfrac{\partial C}{\partial K}(t,0) - g'(0)C(t,0) + \displaystyle\int g(u)\frac{\partial^2 C}{\partial K^2}(t,u)du$. Since a call option with zero exercise price is always exercised, we get $C(t,0) = e(t_0)\bar{B}(t_0,t)$. The stated result is now obtained thanks to assumption (iii).

Appendix C : Lagrance Multipliers

Let us consider the quadratic criterion; the Lagrange multipliers are the solution of a set of linear equations:

$$\begin{cases} \displaystyle\int f_\lambda(t,e)de = \int f_0(t,e)de, \\ \displaystyle\int (e-K_l)^+ f_\lambda(t,e)de = C(t,K_l), \quad \forall l = 1\ldots J. \end{cases}$$

with

$$f_\lambda(t,e) = f_0(t,e)\left(1+\lambda_0+\sum_{j=1}^{J}\lambda_j(e-K_j)^+\right).$$

Let us assume that the a priori model is consistent with the observed underlying prices; the first equation can then be written as:

$$\lambda_0 B(0,t)+\sum_{j=1}^{J}\lambda_j C_0(t,K_j)=0.$$

The remaining equations then provide:

$$(1+\lambda_0)C_0(t,K_l)+\sum_{j=1}^{J}\lambda_j\int(e-K_l)^+(e-K_j)^+f_0(t,e)de = C(t,K_l),\quad \forall l=1..J$$

Substituting λ_0 gives:

$$C(t,K_l)-C_0(t,K_l)=\sum_{j=1}^{J}\lambda_j\left[\int(e-K_l)^+(e-K_j)^+f_0(t,e)de-\frac{C_0(t,K_l)C_0(t,K_j)}{B(0,t}\right]$$

The running term between brackets is equal to $B(0,t)\mathrm{Cov}\ _0[(e(t)-K_j)^+,(e(t)-K_l)^+]$. This provides :

$$B(0,t)\lambda = \mathrm{Var}_{0,\sigma}[(e(t)-K)^+]^{-1}[C(t,K)-C_0(t,K)]$$

The final result is obtained by using f_λ expression.

Appendix D : Expression of $f_{\lambda(\sigma),\sigma}(t,e)$

From the expression of $f_{\lambda(\sigma),\sigma}$, we can write $\int\left(\frac{f_{\lambda(\sigma),\sigma}(t,e)}{f_{0,\sigma}(t,e)}-1\right)^2 f_{0,\sigma}(t,e)de$ as:

$$\int\left(\lambda_0(\sigma)+\sum_{j=1}^{J}\lambda_j(\sigma)(e-K_j)^+\right)\left(\frac{f_{\lambda(\sigma),\sigma}(t,e)}{f_{0,\sigma}(t,e)}-1\right)f_{0,\sigma}(t,e)de$$

Moreover, from the consistency of $f_{\lambda(\sigma),\sigma}$ with observed option prices, we can write the criterion as:

$$\sum_{j=1}^{J}\lambda_j(\sigma)(C(t,K_j)-C_{0,\sigma}(t,K_j))$$

Substituting the expression of Lagrange multipliers provides the stated result.