Hedging Interest Rate Margins on Demand Deposits

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This paper deals with risk mitigation of interest rate margins related to a bank's demand deposits. We assume the demand deposits to be both related to interest rates and business risk which cannot be fully hedged on financial markets. The dynamics of forward Libor rates follows a standard market model and takes into account some risk premium associated with investing in longer term assets. The deposit rates are related to the market rates in linear or non linear ways. We take the viewpoint of an asset and liability manager focusing on the bank's net operating income at a given quarter according to standard accounting rules, faced with market incompleteness and dealing with interest rate derivatives. We distinguish two types of dynamic hedging strategies, one involving the information related to interest rates only and the other one also including the current amount of demand deposits. In the first case, the bank treasurer wears blinders while the latter corresponds to an integrated asset and liability management. We show that the optimal hedging strategy in the former case involves a replication of European type Libor payoffs, thus revealing the hidden optionality in demand deposits. We also derive optimal strategies based on the full information set in feedback form. We compare the hedging performance of the two approaches with respect to the correlation parameter between demand deposits and market rates, which can be seen as a proxy for market incompleteness. We also discuss the trade-off between the alleviation of interest rate risk and the excess return when investing in longer term assets. Eventually, we study the robustness of optimal hedging strategies with respect to the choice of risk criterion (quadratic, VaR and Expected Shortfall). The main result of the paper is the writing of analytical and easy to implement dynamical strategies that deal with key features of the hedging problem.

JEL Classification: D21, G11, G13, G21, G32

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1. Introduction

Bank demand deposits are a major component of their liabilities. Under IFRS – the current international accounting standards – banks account demand deposits at amortized cost as opposed to fair value computations[1]. The European Commission endorsed in November 2007 the IAS 39 Fair Value Option Amendment and two carve outs, allowing hedging strategies that lead to a smooth income associated with demand deposits (Carved-Out Fair Value Hedge)[2]. The IASB and the European Banking Federation (EBF) thoughts in order to replace the latter carve outs by a new kind of hedging strategy, the Interest Margin Hedge (IMH) (see e.g. Adam (2007)), aim at assessing the volatility of demand deposits' interest rate margins rather than the volatility of their fair value. In the US, there is still some uncertainty regarding the accounting treatment of assets and liabilities within the banking books. It is likely that the outstanding approach based upon interest paid and received will hold for a while.

On the regulatory side, the US Securities and Exchange Commission asks American banks to report in annual (10-K) and quarterly (10-Q) documents, indicators concerning interest rate margins and their sensitivities to interest rate shocks. The Basel Committee on Banking Supervision's Third Pillar also recommends qualitative and quantitative disclosures for the interest rate risk in the banking book (IRRBB)[3]. As for demand deposits, quantitative disclosures include the "increase (decline) in earnings or economic value (or relevant measure used by management) for upward and downward rate shocks according to management's method for measuring IRRBB, broken down by currency (as relevant)" (see Part 4 – Section II – Table 13).

In their internal interest rate risk management process, some banking establishments compute the fair value of their assets and liabilities. As for demand deposits, this corresponds to the approach developed by Hutchison and Pennacchi (1996), Jarrow and van Deventer (1998) O'Brein (2000). However, as stated in Jarrow and van Deventer (1998), the demand deposit amount is not contingent to interest rates: it carries some business risk independent of market risk. Ho and Saunders (1981) and later Wong (1997) and Saunders and Schumacher (2000),

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[1] The IFRS actually ask banking establishments to account demand deposits at a Fair Value equal to their nominal value, which is equivalent. See e.g. IAS 39 – Measurement – Subsequent Measurement of Financial Liabilities – Official IASB Website http://www.iasb.org/

[2] See European Commission's (EC) Reference Document IP/04/1385 – Official EC Website http://ec.europa.eu

[3] See e.g. International Convergence of Capital Measurement and Capital Standards – A Revised Framework – Official Bank for International Settlements Website http://www.bis.org/

Kalkbrener and Willing (2004) show that not only interest rates on financial markets but also the regulatory framework, the bank's market structure and its credit risk exposure may influence the demand deposit amount and thus its hypothetical fair value. Indeed, the IASB and the FASB mention[4] that the fair value of demand deposits "involves consideration of non financial components" and subsequently propose to postpone the recognition of those liabilities at fair value. In particular, Basel II's Third Pillar refers to the disclosure of qualitative "assumptions regarding […] the behavior of non-maturing deposits", which are part of those non financial components.

In their valuation approach, Jarrow and van Deventer (1998) do not deal with the residual terms that appear when relating demand deposits and market rates. This dramatically eases the computation of the fair value of demand deposits which are subsequently seen as interest rate contingent claims: there is no need in that simplified framework to deal with risk premia associated with the volatility of deposit amounts and historical and pricing measures are mixed-up. Of course, whether this makes sense is difficult to assess.

Consequently banks, as well as financial analysts and banking managers, pay more attention to the demand deposit income at amortized cost. Indeed, a worldwide study of the Bank for International Settlements (English (2002)) shows that risk mitigation in interest rate margins has been a significant concern for banks during the past twenty years. In this paper, according to outstanding accounting rules, market practice and standard banking theory as evidenced by Ho and Saunders (1981), we propose to assess the interest rate risk on demand deposits from the interest rate margins rather than some hypothetical and heavily model dependent fair value of such demand deposits.

Fortunately enough, when considering interest rate margins, the stringent assumptions involved in the fair value approach are not required to compute optimal risk mitigation strategies based on interest rate derivatives. Thanks to a number of theoretical finance results, explicit derivations can be achieved in a dynamical mean-variance framework. Moreover, the optimal hedging strategies prove to be additive with respect to the choice of the balance sheet item, paving the way for managing the whole balance sheet. We check in this paper the robustness issues associated with the choice of the quadratic risk criterion. We also assess the magnitude of risk mitigation that can be achieved when taking into account market incompleteness and the departures from the complete market case, as far as dealing with hedging instruments such as interest rate swaps or FRA's is concerned.

We consider two different types of hedging strategies. The first one gathers strategies where the amounts of interest rate derivatives are dynamically managed according to the information set related to market rates only – the *financial market information set*. This illustrates the situation where the risk management of interest rate margins is decentralized into some treasury unit where the market information is observable, but not the fluctuations of the deposit amount. Optimal strategies are then obtained by *projecting* the interest rate margin on the set attainable payoffs under the financial market information set. This typically leads to replicating interest rate derivatives. In our framework, these turn out to be quite simple European type options on the terminal Libor rate.

[4] See Financial Accounting Standards Advisory Council's document on Fair Value Option (March 2006) – Official FASB Website http://www.fasb.org/

Regarding interest rate margins, it is worth noting that a path of future quarterly figures is usually involved in the risk assessment process. Thus, the risks to be dealt with have two components. The first one involves the variability of given quarterly results, stated independently of other quarters (say intra quarter volatility). The second one involves the smoothness of the path and involves the magnitude of changes from one quarter to another (say inter quarter volatility). In this paper, we focus of the first kind of risks which, fortunately, leads to a huge simplification of the optimization problem. We do not feel that this is a practical issue: whenever the expected amount of demand deposits does not change in a hectic way, which seems the most common and sensible case, risk reduction over each quarter mechanically leads to a smooth pattern of quarterly margins. Furthermore, as is further detailed in the paper, the minimization of risk over the different quarters breaks down from a multidimensional to a set of one dimensional optimization problems to be solved independently. This dramatic dimensionality reduction leads the way to analytical or easy-to-implement dynamic hedging strategies, a goal which would be out of reach without properly setting-up the risk management criterion.

With the second type of strategies, the dynamic investment processes involve the *full information set*, related to the observation of both the market rates and the deposits' specific risk. In the latter case, Duffie and Richardson (1991) derive explicit dynamic hedging strategies in a framework where both the underlying asset and the asset to hedge follow geometric Brownian processes. Therefore, we can rely on these techniques when deposit rates are linear functions of market rates. To a certain extent, this is the case when restricting Jarrow and van Deventer's (1998) model on deposit rates, to its long term effects. However, when dealing with the more general case where customer rates are not linearly related to market rates, we need to use results of Gouriéroux, Laurent and Pham (1998) or Pham, Rheinländer and Schweizer (1998), which extend Duffie and Richardson's (1991) approach. In our framework, this allows to derive explicit dynamic strategies to hedge the interest rate margin for a given quarter.

In both cases, the trade-off between the alleviation of interest rate risk and the excess return when investing in longer term assets, an important issue in bank management is being studied. Actually, the main result of the paper is the writing of analytical and easy to implement dynamical strategies that deal with key features of the interest rate hedging problem.

This paper is organized as follows. In Section 2 we set the modeling framework and we show how interest rate margins have become a major point of concern for banking establishments today. In Section 3 we present the two types of hedging approaches and derive the optimal strategies to hedge the interest rate margin for a given quarter. In Section 4 we exhibit some features for these strategies and also study their performance under alternative risk-return criteria. We show that the optimal strategy based upon the full information set better accounts for effects due to demand deposits' specific risk.

2. Modeling Framework

2.1. Market Rates

We consider some year quarter $\left[T;T+\frac{1}{4}\right]$ such that we deal with the corresponding quarterly interest rate margin. Besides, we assume that the forward Libor rate at date T for the time period of the interest rate margin – a quarter – follows a Libor Market Model, as defined in Brace, Gatarek and Musiela (1997) and Miltersen, Sandmann and Sondermann (1997):

$$dL_t = L_t\left(\mu_L dt + \sigma_L dW_L(t)\right), \tag{1}$$

where we denote $L_t = L\left(t,T,T+\frac{1}{4}\right)$ for the forward Libor rate. This dynamics are defined under some historical probability measure $\mathbf{P}$.

For model simplicity, μ_L and σ_L are assumed to be constant and we denote the related interest rate risk premium by $\lambda = \frac{\mu_L}{\sigma_L}$. We will thereafter be able to account for greater average returns when investing in long term bonds than in short-term assets (see e.g. Chapter 11 in Campbell, Lo and MacKinlay (1997)).

The framework can readily be extended to the case of deterministic parameters and with extra – but reasonable – computation when both μ_L and σ_L depend upon the forward Libor rate.

2.2 Demand Deposit Amount

We assume that the demand deposit amount follows:

$$dK_t = K_t\left(\mu_K dt + \sigma_K d\overline{W_K}(t)\right), \tag{2}$$

where $\overline{W_K}$ is a standard Brownian process. For simplicity, the trend μ_K and the volatility σ_K are assumed to be constant, though the results readily extend to the deterministic case, for instance to deal with seasonal effects. Clearly, the trend and volatility terms depend upon the liabilities to be considered.

We also propose to assess the possibility of a massive bank run within demand deposits, in the future. Thus, to deal with such a severe liquidity crash, we add some Poisson process to the deposit amount. We choose a Poisson process $N = (N_t)_{0\le t\le T}$ of intensity γ, independent of $\overline{W_K}$ and W_L:

$$dK_t = K_t\left(\mu_K dt + \sigma_K d\overline{W_K}(t) - dN_t\right). \tag{2b}$$

γ cannot be unambiguously estimated upon historical data and one may also rely on expert advice and a Bayesian approach as for operational risk **(**see e.g. Chavez-Demoulin, Embrechts and Neslehova (2005)**)**. Nonetheless, (2b) constitutes a natural and tractable extension of the Brownian case of equation (2).

Let us remark that bank runs may cause bankruptcy. After such an event, one could wonder why keeping on managing demand deposit margins. The answer is two-fold. First, because regulators would pay careful attention to the related hedging portfolio – which still exists after the bank run occurred. More generally, even a bank that does not manage non maturing deposits can still invest in long term assets. Second, if the massive bank run happens in a subsidiary within a holding, then liquidity injections from the holding company may maintain the entity alive and the management of the hedging portfolio should not be given up.

The demand deposit model might involve several jump processes with distinct intensities, to better cope with the demand deposit amount specificities. For simplicity, we do not go further into that direction, but deriving optimal hedging strategies in this latter case is very similar to the case of one total bank run as in Equation (2b).

In both models, at a given time, the outstanding nominal amount results from cash inflows and withdrawals from existing clients, including account cancellations. This point of view is usually chosen by auditors and accountants for the sake of caution. On the other hand, one could include the net cash-flows resulting from the opening of new accounts, which could come either from endogenous growth or external development. This can be associated with the "appraisal value" in the insurance terminology and should rather be the point of view of stockholders. Our approach applies to both cases though the parameters obviously need to be changed.

As seen from the above discussion, the changes in K_t can be associated either with liquidity risk, coming from concerns about the credit worthiness of the managing bank or due to customer arbitrage between demand deposits and other asset classes, either with business risk, for instance if a given bank loses some market share due to poor management of deposit accounts.

The following tables provide maximum likelihood estimations of μ_K and σ_K in a number of cases, at an aggregate level. We considered monthly amounts (seasonally adjusted) issued from the American, European (Euro Zone) and Japanese markets between January 1999 and September 2007. We collected:

- amounts of each market's M2 monetary aggregates – excluding currency in circulation (M0) – endowing overnight deposits, check accounts, savings and certificates of deposit of agreed maturity up to 2 years, as defined by central banks;
- amounts of each market's M1 monetary aggregate, excluding currency in circulation ; this aggregate endows only overnight deposits and check accounts.

Table 2.2 contains estimations for two submarkets in the Euro Zone – France and Germany – showing very little transfer effects from a submarket to another (overall and submarket's volatilities being close) but a more significant growth (9.24%) in the overall market due to the inclusion of new countries in the Euro Zone during the estimation period. This phenomenon can be compared to a bank's establishment external growth policy. Moreover, Table 2.3 contains parameter estimations for two examples of emerging markets – Turkey and Ukraine – showing the tremendous growth of such markets during the last decade.

We also notice that aggregates containing both savings and sight deposits (M2 and assimilated) feature greater stability than those containing only demand deposits (M1). Indeed, there exist money transfers between the different types of accounts, generating volatility on the M1 aggregate while the aggregate including saving accounts remains stable. Of course, this observation strongly depends on the various types of deposits that banks propose to their customers, on each marketplace. For example, in the US, clients often own several types of accounts (MMDA, NOW, checkable accounts, etc.) in addition to asset management services, which feature significant transaction costs or heavy tax conditions, thus not as convenient as usual deposits.

Market	Monetary aggregate	μ_K	σ_K
US	Demand Deposits	-2.29%	8.24%
US	Demand and Checkable Deposits	-0.31%	5.16%
US	M2 - M0	5.99%	1.30%
Euro Zone	Demand Deposits	9.24%	6.08%
Euro Zone	M2-M0	6.27%	2.33%
Japan	M2-M0	2.83%	2.26%

Table 2.1. Estimation of Demand Deposit Parameters for US, Euro Zone and Japan's Monetary Aggregates.
Estimation Period: January 1999 to September 2007. Sources: Federal Reserve Bank of St. Louis (http://www.stlouisfed.org), European Central Bank (http://www.ecb.int/) and Bank of Japan (http://www.boj.or.jp/en/). The estimations are all given on a yearly basis and the input data are seasonally adjusted.

Market	Monetary aggregate	μ_K	σ_K
Euro Zone	Demand Deposits	9.24%	6.08%
France	Demand Deposits	5.93%	5.77%
Germany	Demand Deposits	8.47%	6.19%
Euro Zone	M2-M0	6.27%	2.33%
Germany	M2-M0	3.21%	1.63%

Table 2.2. Estimation of Demand Deposit Parameters for Euro Zone and Submarkets (France, Germany).
Source: European Central Bank (http://www.ecb.int/), Banque de France (http://www.banque-france.fr/) and Deutsche Bundesbank (http://www.bundesbank.de/). The estimations are all given on a yearly basis and the input data are seasonally adjusted.

Market	Monetary aggregate	μ_K	σ_K
Turkey	M1 - M0	37.93%	35.97%
Turkey	M2 - M0	33.63%	11.00%
Ukraine	M1 - M0	33.41%	13.45%
Ukraine	M2 - M0	36.68%	9.12%

Table 2.3. Estimation of Demand Deposit Parameters for Some Emerging Markets (Turkey, Ukraine).
Sources: Central Bank of Republic of Turkey[5] (http://www.tcmb.gov.tr/yeni/eng/) and National Bank of Ukraine (http://www.bank.gov.ua/ENGL/). The estimations are all given on a yearly basis and the input data are seasonally adjusted.

Table 2.4 contains estimations of μ_K and σ_K for the eight largest US banks by deposits. We notice very high values of μ_K for Bank of America, JP Morgan Chase, Regions Bank, etc., which may be related to external growth.

[5] Data are available on the Internet at http://www.tcmb.gov.tr/yeni/eng/

Financial Institution	**Total Deposits** June 30th, 2008 - $ thousands	μ_K	σ_K
Bank of America National Association	642 252 215	12%	17%
JP Morgan Chase Bank National Association	461 008 000	19%	26%
Wachovia Bank National Association	397 759 000	13%	10%
Wells Fargo Bank, National Association	276 306 000	2%	12%
Citibank, National Association	224 325 823	15%	16%
SunTrust Bank	114 276 117	9%	15%
U.S. Bank National Association	127 819 352	3%	7%
Regions Bank	86 225 760	26%	31%

Table 2.4. Estimation of Demand Deposit Parameters for Some US Banks.
Source: Federal Deposit Insurance Corporation (FDIC) (http://www.fdic.gov/). Estimation period: quarterly data from June 30, 2004 to June 30, 2008. The estimations are given on a yearly basis. These figures are indicative of the global growth of the deposit amount within the eight largest US banking establishments by deposits (according to the FDIC on June 30th 2008).

Let us point out that the further results can be adapted with the assumption of a stochastic or time-dependent volatility in the demand deposit amount process. This would enable for example to deal with broader issues concerning liquidity risk.

2.3 Linking Deposit Amount and Interest Rates

We assume the dynamics of the demand deposit amount to be correlated with interest rates:

$$d\overline{W}_K(t) = \rho dW_L(t) + \sqrt{1-\rho^2}\, dW_K(t), \tag{3}$$

where W_K is a Brownian process independent of W_L, and ρ some constant correlation parameter. Let us emphasize that, like in Fraundorfer and Schurle (2003) and in Kalkbrener and Willing (2004), the demand deposits may feature other sources of risk that the one related to interest rates. Then W_K can be considered as some component independent of interest rates movements.

The correlation between the variations of demand deposit amount and that of interest rates can be related to money transfers between deposit accounts and other types of deposits. Janosi, Jarrow and Zullo (1999) refer to this phenomenon as *disintermediation*. They estimate the correlation parameter using bank data coming from the Federal Reserve Bulletin for various types of accounts – namely Negotiable Orders of Withdrawal (NOW), passbooks, statement and demand deposit accounts. Their study exhibits negative values for all account types, causing demand deposit amounts fall when short rates rise. Going further, the deposit amount may exhibit the contrary movement with respect to interest rates because of the “liquidity trap” introduced for example by Hicks (1937). Indeed, like in Japan in the 90’s, the deposit amount may rise while interest rates are too low. This happens because there is no advantage for customers to invest money on assets like bonds or on savings accounts, since such resources are not enough rewarding to compensate the related commissions and fees (see e.g. Krugman (1998)).

We refer to the Engle and Granger method detailed in Ericsson and MacKinnon (1999) to estimate the correlation parameter between the deposit amount and the market rate. Janosi et al. (1999) use a very similar method, although they also pay attention to autocorrelation and short term effects. We show our results in Tables 2.5 and 2.6.

Market	Monetary Aggregate	Related Market Rate	σ_L	σ_K	ρ
US	Demand Deposits	USD 3M Libor	21.80%	8.24%	0%
US	Demand and Checkable Deposits	USD 3M Libor	21.80%	5.16%	-11.28%
Euro	Demand Deposits	3M Euribor	15.42%	6.08%	-70.85%

Table 2.5. Estimation of Correlation Parameter for US and Euro Zone's Demand Deposits.
The estimations of the volatility parameters are given on a yearly basis.

Market	Monetary Aggregate	Related Market Rate	σ_L	σ_K	ρ
Euro	Demand Deposits	3M Euribor	15.42%	6.08%	-70.85%
France	Demand Deposits	3M Euribor	15.42%	5.77%	-46.70%
Germany	Demand Deposits	3M Euribor	15.42%	6.19%	-84.65%

Table 2.6. Estimation of Correlation Parameter for Euro Zone's and Some Submarkets' Demand Deposits.
The estimations of the volatility parameters are given on a yearly basis.

We did not show any results for M2-type monetary aggregates, since they do not exhibit significant correlation with interest rates. Indeed the transfers between saving accounts and more elaborate investment schemes may not be driven by interest rates only. There are likely additional factors like transaction costs and taxes, that drive customers' arbitrages between M2-type deposits and time deposits or asset management investment opportunities.

The preceding estimators of the correlation parameter do not integrate lag and short term effects either but these effects may be analyzed though. We notice that they vary significantly from one aggregate to another and when switching from a marketplace to its submarkets. This may also occur among individual banking establishments and their subsidiaries, and when modifying the perimeter among demand deposits. In the US, demand deposit volatility is higher (8.24% vs. 5.16%) but almost not correlated with interest rate variations, comparing to demand and checkable deposits. Conversely, in the Euro Zone, most of the demand deposit amount volatility (-70.85%) seems to be due to interest rate variations.

2.4. Deposit Rate Modeling

Depending on the local business model, deposit accounts may bear interests for clients. As suggested by Hutchison (1995), Hutchison and Pennacchi (1996) or Jarrow and van Deventer (1998), the deposit rate may exhibit some dependence with respect to market rates.

Hutchison and Pennacchi (1996) assume the deposit rate to fulfill some affine relation with the market rate and the residuals to be linked with the deposit amount's elasticity, thanks to some equilibrium model developed in Hutchison (1995). Indeed, for example, when we perform a linear regression of the US M2 own rate upon the 3-month Libor rate, the residuals feature a correlation of -10% with M2's growth[6]. This is possible to derive optimal hedging strategies in the case where the deposit rate is a linear function of the Libor rate which

[6] This estimation is significant at 1% confidence level, according to the Fisher's zero correlation test (see e.g. Campbell et al. (1997)).

features a residual term correlated with the deposit amount's growth $\overline{W_K}$. Indeed, the optimal hedging strategies we derive in section 4 are linear with respect to the interest rate margin. However, from now on, we assume the deposit rate to be a deterministic function g of the market rate $L_T = L(T, T, T + \delta T)$.

The graphs below (see Figure 2.7) confirm the intuition of some linear dependence between the deposit rate and the market rate.

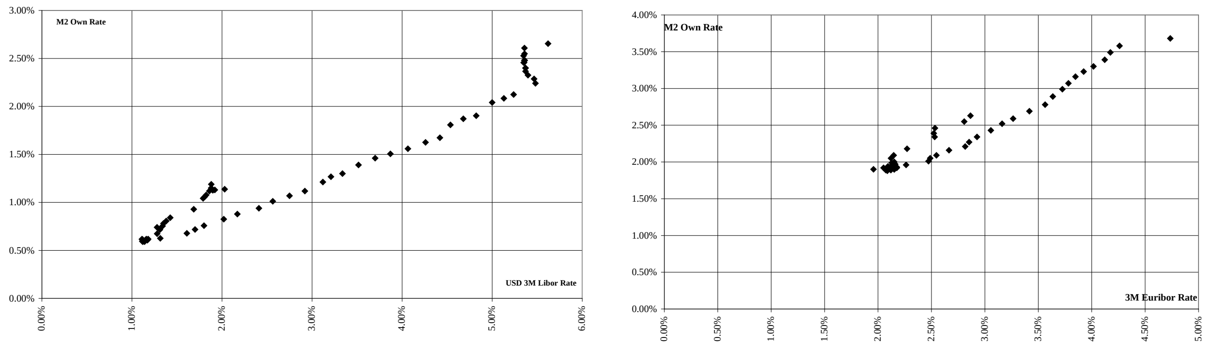

Figure 2.7. Deposit rate and market rate in US (left) and Euro Zone (right).
The scatters represent the deposit rate on the Y axis and the market rate on the X axis. Data sample period ranges from January 2002 to September 2007.

We computed the estimations of α and β corresponding to the linear regression $g(L_T) = \alpha + \beta L_T + \varepsilon_T$. To achieve that, we focus on long term effects only thanks to Engle and Granger's method (see e.g. Ericsson and MacKinnon (1999)), though Jarrow et al. (1999) and Hutchison and Pennacchi (1996) deal more thoroughly with lag and short term effects. Using data on individual bank retail deposit interest rates, Hutchison and Pennacchi (1996) find a β coefficient equal to 0.40 for NOW and 0.83 for MMDA. As Table 2.8 shows, our estimations are located in this range.

Market	Monetary Aggregate	Market Rate	α (Intercept)	β (Market Rate)	R^2
US	M2 - M0	USD 3M Libor	-0.41%	0.66	86%
Euro Zone	M2 - Enterprises and Households	3M Euribor	0.46%	0.69	96%

Table 2.8. Estimating relationship between deposit rate and market rate in the US and in the Euro Zone.
The estimation period for the US is Nov. 1986 – Sept. 2007 and for the Euro Zone, Jan. 2003 – Sept. 2007. The alpha parameter is given on a yearly basis.

The case of the Japanese market differs from the US and the Euro Zone. Indeed, between June 2001 and February 2006, market rates were very low (below 0.08% on a yearly basis), compelling banks with shrinking dramatically deposit rates in order to keep positive margins. Indeed deposit rates were very close to zero during this period. We propose a focus on these facts in Table 2.9.

Explanatory Variables		Coefficient	Standard Error	T-statistic	P-value
When Libor Rate < 0.08%	**Intercept**	*0.004%*	0.027%	0.17	87%
	3M Libor Rate	*0.546*	0.444	1.23	22%
When Libor Rate > 0.08%	**Intercept**	0.045%	0.006%	7.89	<<1%
	3M Libor Rate	0.387	0.015	25.24	<<1%

Table 2.9. Japanese Deposit Rate Modeling.
We gathered deposit and market rates from April 1999 to August 2007 at a monthly frequency. We estimate the following model: $CR_t = \mathbf{1}_{L_t < 0.08\%}(\alpha_1 + \beta_1 L_t) + \mathbf{1}_{L_t > 0.08\%}(\alpha_2 + \beta_2 L_t) + \varepsilon_t$ where CR_t (resp. L_t) is the deposit (resp. market) rate at date t. Italic values (coefficients when Libor Rate <0.08%) are non significant at 5% probability level. (Source: Bank of Japan). Estimations are given on a yearly basis.

Hence, we will further focus on the two following sub-cases as for the modeling of deposit rates:

$$g(L_T) = \alpha + \beta L_T \text{ (US and Euro Zone case),} \quad (5a)$$

and

$$g(L_T) = (\alpha + \beta L_T)\mathbf{1}_{L_T \geq R} \text{ (Japanese case).} \quad (5b)$$

Let us concede that our framework does not account for non Markovian features, either in the client rate determination, either in the link between the deposit amount and market rates, for example. Such features are particularly important in a short-term framework such as demand deposit day-to-day management. However, in our framework, we focus on the middle and long-term management of demand deposits, on the time horizon of a quarter, which somehow alleviates short-term non Markovian effects. Given that, we propose to rely upon middle-term equilibrium relations as for client rates (see (5a) and (5b) above) and demand deposit amounts (see (3) above) to determine our results in the following.

2.5. Interest rate margins within banking regulation – Case of the SEC

In the 1990's, a number of studies focused on the fair value of demand deposits within a bank. For example, this approach was recommended and detailed in the Office of Thrift Supervision's official publication about the *Net Portfolio Value Model* (1994)[7]. However, since the demand deposits' fair value is set as the discounted sum of future cash flows related to demand deposits, its computation may involve an assessment of future interest rate margins. This is what we observe in Selvaggio (1996) or Hutchison and Pennacchi (1996). In a study for the BIS, English (2002) showed that banks have been avoiding significant exposures of the interest rate margin to market interest rates although we still notice slight sensitivities towards the yield curve slope in some European countries (Germany, Norway, Switzerland and Sweden).

Since the adoption of the IFRS in 2005, regulators have been paying increasing attention to interest rate margins and ask banks detailed information about them. Indeed, banks' quarterly (10-Q) and annual (10-K) reports to the SEC[8] contain specific sections about their net interest incomes and the related sensitivities within one year horizon, towards standardized interest

[7] See OTS Official Website – http://www.ots.treas.gov/

[8] See e.g. Item 7 'Management's Discussion and Analysis of Financial Condition and Results of Operations' in 10-K reports, and the corresponding Item 2 in 10-Q reports.

rate shocks. These shocks are usually +/- 200bps interest rate gradual shocks during the upcoming year[9].

From a broader view, the *net interest margin* is defined as the spread between the average interest rate received on assets minus the average interest rate paid on liabilities (see e.g. Maudos and Fernández de Guevara (2004)). Thus, consistently with the SEC approach, we define the *interest rate margin* as the interest income received on assets minus the interest expenses paid on liabilities.

As for the whole balance sheet, the interest rate margin gathers the interest rate revenues generated upon assets minus the expenses paid on liabilities. Computed on a given year quarter $\left[T, T+\frac{1}{4}\right]$, this can be written as follows:

$$IRM^{BS}\left(T, T+\frac{1}{4}\right) = \frac{1}{4}\sum_{i\in BS_A} K_T^i r_T^i - \frac{1}{4}\sum_{j\in BS_L} K_T^j r_T^j \ ,$$

where BS_A and BS_A stand respectively for the Balance Sheet Asset items and Liability items, K_T^i, the amount of balance sheet item i and r_T^i, the interest rate paid (resp. earned) on liabilities (resp. assets). We can simplify this notation by setting the convention of negative amounts for liabilities (positive amounts for assets) and then we get:

$$IRM^{BS}\left(T, T+\frac{1}{4}\right) = \frac{1}{4}\sum_{i\in BS} K_T^i r_T^i$$

on the whole balance sheet.

Considering the specific case of demand deposits, we define the interest rate margin for demand deposits as the income generated by the investment of demand deposits on interest rate markets, net of the interest paid to customers. This corresponds to some simplified bank, deposit-only oriented, which would invest the deposit amount on short term inter-bank contracts, thus generating the interest rate revenue on demand deposits.

In a way, we show in Appendix A that the optimal hedging strategy on the whole balance sheet is the sum of optimal strategies on stand-alone balance sheet items. Thus taking apart demand deposits as we latter proposed does not alleviate generality in any manner.

More formally, the choice of the investment rule before hedge does not influence optimal hedging strategies for any set of strategies we define further[10]. Thus, we choose to assume that, as a first approach, the demand deposits will be invested at the short market rate of the corresponding quarter. This somehow corresponds to investing demand deposits on the very short term, as initially recommended by the IASB for example[11]. Thus, in our viewpoint, the interest rate margin for a given quarter measures the income related to the spread between market and deposit rates, leveraged by the demand deposit amount.

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[9] See Item 7A ‘Quantitative and Qualitative Disclosures About Market Risk’ in 10-K reports and the corresponding Item 3 in 10-Q reports.

[10] This statement holds in cases where the investment rate is a measurable function of the Libor rate L_T, or more generally, any stochastic integral of the Libor rate $\int_0^T \theta_t dL_t$ with $\theta \in \Theta^L$ (for the definition of Θ^L, see subsection 3.1, further).

[11] See e.g. http://www.iasb.org – minutes from the January and February 2004 discussion meetings.

Definition 2.1. (Interest Rate Margin) The Interest Rate Margin at date T stands for the cash-flow generated upon the quarter $\left[T, T+\frac{1}{4}\right]$ *by the investment of the amount of demand deposits on the short term Libor rate* L_T *minus the interests* $g(L_T)$ *paid to customers. In our framework, we express it as follows:*

$$IRM_g(K_T, L_T) = \frac{1}{4} \cdot K_T(L_T - g(L_T)). \qquad (6)$$

The investment of the deposit amount on the Libor rate is an investment practice which can be questioned today. Indeed, the Libor rate is influenced by inter-bank transactions and does not represent the actual refinancing rate of a given bank. Moreover, we focus on swaplet-based hedging strategies and interest rate-based investment of demand deposits because, from an accounting viewpoint, an equity-based hedge or any other kind of non interest rate-based hedge would be separated from the interest rate margin and marked-to-market.

Usually, the time-length of a quarter corresponds to the time interval at which the net interest income is measured in SEC reports[12]. This also corresponds to the time interval between two consecutive ALM committees in most of banking establishments. However, in the following sections, we will consider a time period of a year, for convenience, but rescaling to a time period of a quarter is straightforward.

We have been collecting data concerning the net interest incomes and their sensitivities for 20 US banking establishments[13] of almost the same asset size and featuring a similar involvement in retail banking. As for the latter point, we took the number of branches within the United States as an indicator of the involvement in retail banking activities: thus, each of these establishments feature between 179 and 1711 branches within the United States. Moreover, they feature similar net interest income / asset size and number of agencies / asset size ratios (cf. Appendix A).

During year 2005 Libor rates have – almost gradually – increased by nearly 200 basis points, closely reproducing interest rate scenarios recommended by the SEC, as stated above. Thus we have been measuring the explanatory power of the computed sensitivities for the upcoming year[14]. To achieve that, we examine how the *ex-post* variations of the net interest incomes with respect to their previsions in some central interest rate scenario differ from the sensitivities displayed *ex-ante* in SEC reports. The coefficient of 1.37 in the table below shows that the sensitivity computed *ex-ante* follows but slightly underestimates reality. Moreover, because of the weakness of the R-square and the F-statistic being far beyond its critical value, the explanative power of the *ex-ante* sensitivity seems pretty limited.

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[12] Actually, by measuring the interest rate margin upon a quarter, we choose to ignore the issue related to the intra-quarter variation of the demand deposit amount and the related revenues. This generates volatility in the quarter's interest margin when deposits are invested on overnight basis, for instance.
[13] See Appendix B: List of US Banks used in interest rate margins analysis.
[14] Let us remark that the related time period (2005-2006) is located way before the 2007 subprime crisis.

	Intercept (Standard Deviation)	Ex-ante Sensitivity (StDev)	R²	F-statistic (Critic Value)
Net Interest Income Variation with respect to central IR scenario	8.32% (2.43%)	1.37 (1.05)	29%	1.72 (0.21)

Table 2.10. Net Interest Income Variation: Ex-Ante vs. Ex-Post.
The Net Interest Income Variation with respect to central interest rate scenario stands for the relative difference between the net interest income observed during year 2005 (*ex-post*) and the income that could be expected, according to some central interest rate scenario. It is regressed upon the *ex-ante* sensitivity which is the same variation, computed *ex-ante* by banking establishments at the beginning of year 2005, using internal models. This is the variation in the net interest income subsequent to a gradual linear rise of 200 basis points in interest rates during the upcoming year, with respect to what it would be in some central interest rate scenario.

Actually, this fact is not surprising, since banks' disclaimers in SEC reports already warn us about other factors that could damage the explanatory power of the computed sensitivities.

3. Hedging Strategies for Interest Rate Margins

In this article, we aim at reducing the latter interest rate margin's variance, at some term T, upon sets of hedging strategies that will be defined in this section. Indeed, in subsection 3.1, we define several sets of payoffs corresponding to hedging strategies on the interest rate margin for the quarter $\left[T;T+\frac{1}{4}\right]$. Thus, for such a set of payoffs H, we consider the problem:

$$\min_{S\in H}\mathbf{Var}^{\mathbf{P}}\left[IRM_g\left(K_T,L_T\right)-S\right]. \tag{7}$$

Here we assume that the bank may develop hedging strategies that will impact the interest rate margin at historical cost, from the accounting viewpoint. However the recognition of hedging strategies at historical cost is far from being obvious. Yet, most banking practices tend to design hedging strategies on interest rate securities to alleviate the volatility of the net interest income at historical cost. Thus, from an accounting viewpoint, banks tend to invest on securities recognized as available-for-sale, thus impacting the income statement at historical cost[15]. In the European case, the IFRS-EU Carved-Out Fair Value Hedge[16] allows banks to design swap-based hedging strategies for deposit accounts by date of origination. Thus, each new deposit generation is hedged individually, given the hedging strategies designed for past generations.

The problem (7) deals with risk minimization within the interest rate margin as the only objective, such that there is no minimal return constraint on the final income $IRM_g\left(K_T,L_T\right)-S$. Indeed, we also propose to add some return constraint by replacing the set H by the subset of payoffs which ensure some level of return $m\in\mathbf{R}$ to the bank. We thus

[15] Let us notice that, according to IFRS-IAS standards, the mark-to-market variations of AFS securities must impact the equity. Yet, these variations are actually very difficult to identify within equity fluctuations. Consequently, we can almost consider that the influence of AFS securities is only noticeable in the income statement at historical cost.
[16] See e.g. See European Commission's (EC) Reference Document IP/04/1385 – Official EC Website http://ec.europa.eu.

define $H(m)=\{S \in H \mid \mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - S] = m\}$ and the optimization problem above becomes:

$$\min_{S \in H(m)} \mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - S]^2 . \tag{8}$$

Under this latter form, the problem is equivalent to some orthogonal projection of the interest rate margin upon the set of payoffs $H(m)$.

We determine optimal hedging strategies for both problems – constrained and non constrained – in subsection 3.2, for the sets of payoffs defined in subsection 3.1.

3.1. Hedging instruments, hedging strategies and information sets

The hedging of interest rate margins can be performed using interest rate swaps or forward rate agreements, thus reducing the sensitivity of future incomes to upcoming market rate variations. In this section we define several sets of payoffs each corresponding to a hedging strategy based on forward rate agreements.

From now on, we propose to consider a set of dynamic FRA-based investment strategies:

$$H_{Market} = \left\{ \int_0^T \theta_t dL_t \middle| \theta \in \Theta^L \right\}, \tag{9}$$

where Θ^L is the set of admissible[17] strategies adapted tot the filtration generated by the forward Libor rate. In particular, H_{Market} contains the European-type options on the terminal Libor rate priced at zero. Besides, the hedging strategies in H_{Market} rely upon the only financial market information – carried by the Libor rate. This situation corresponds to a bank where the risk management of interest rate margins would be transferred to a decentralized treasury entity for which the specific information carried by the demand deposit amount is not observable.

In H_{Market}, the dynamic strategy is myopic regarding the amount of demand deposit at each intermediary date, though such information is easily available to any asset and liability manager. Therefore we propose to extend the set H_{Market} above to a larger set of dynamic self-financed investment strategies, enabling the manager to adapt his FRA-based hedging strategy to the evolution of the demand deposit amount. We thus define the following set:

$$H_{ALM} = \left\{ V_T(\theta) := \int_0^T \theta_t dL_t , \theta \in \Theta \right\}, \tag{10}$$

where Θ is the set of admissible strategies[18], adapted to the filtration generated by W_L (Libor rate process) and W_K (deposit amount process). By construction we have $H_{Market} \subset H_{ALM}$, both being closed subspaces of $L^2(\mathbf{P})$ (see e.g. Delbaen et al. (1997)).

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[17] From now on, we refer to the notion of admissibility as defined in Gouriéroux, Laurent and Pham (1998) (Definition 2.1) and Pham, Rheinländer and Schweizer (1998) (Section 1.). This definition ensures the closedness of the set H_{ALM}, which allows us to refer to the projection theorem to find optimal hedging strategies.

[18] See footnote n°10.

3.2. Variance Minimal Measure and **Hedging Numéraire**

The literature provides numerous mathematical tools to deal with quadratic dynamic hedging in incomplete markets, which is our framework. Moreover, the related results propose quasi-analytical expressions for the hedging strategy. As suggested in Föllmer and Schweizer (1990), we consider the martingale minimal measure defined as follows.

Definition 3.1. Consistently with Föllmer and Schweizer (1990), we define the usual minimal martingale measure $\overline{\mathbf{P}}$ *with respect to the 'historical' forward probability measure* $\mathbf{P}$*, thanks to the following pricing kernel:*

$$\frac{d\overline{\mathbf{P}}}{d\mathbf{P}} = \exp\left(-\frac{1}{2}\lambda^2 T - \lambda W_L(T)\right). \tag{11}$$

We recall that $\lambda = \frac{\mu_L}{\sigma_L}$ is the interest rate risk premium involved in investing in long term bonds financed by short term liabilities. In particular, when $\lambda = 0$, there is no difference between the martingale minimal and the historical measure. In our framework, the pricing kernel above is a function of L_T given by $\frac{d\overline{\mathbf{P}}}{d\mathbf{P}} = \left(\frac{L_T}{L_0}\right)^{\frac{-\lambda}{\sigma_L}} \exp\left(\frac{1}{2}\lambda(\lambda + \sigma_L)T\right)$.

The minimal martingale measure is actually the relevant pricing measure for interest rate derivatives.

Moreover, in our framework, the minimal martingale measure $\overline{\mathbf{P}}$ coincides with the variance minimal measure as defined in Pham, Rheinländer and Schweizer (1998) or Delbaen et al. (1997). This is more generally the case for "almost complete models" (see Pages (1987)), including models where the market rate's diffusion coefficients μ_L and σ_L are random processes adapted to the filtration related to W_L. This particularly accounts for models featuring time dependent diffusion coefficients for the Libor rate or local volatility models.

Definition 3.2. Hedging Numéraire. *The* hedging numéraire *is the value process of the dynamic portfolio corresponding to the investment strategy* $\theta^{Num} = \left(\theta_t^{Num}\right)_{0\le t\le T}$ *solving*

$$\min_{\theta\in\Theta} \mathbf{E}^{\mathbf{P}}\left[1 + \int_0^T \theta_t dL_t\right]^2. \tag{12}$$

In our framework, the *hedging numéraire* is a power function of the forward Libor rate and verifies $\frac{dNum_t}{Num_t} = -\frac{\lambda}{\sigma_L L_t} dL_t$ at each date *t*. This portfolio represents the way a manager should invest on interest rate derivatives so as to aim at some fixed return of -1 with minimal variance.

Let us point out that the latter diffusion equation for the *hedging numéraire* is still valid when the modeling of the Libor rate involves deterministic diffusion coefficients or coefficients adapted to the Libor rate process. For example, this holds for local volatility interest rate models, like the CEV model (see e.g. Andersen and Andreasen (2000)).

3.3. Optimal Investment Strategy – Market Information Set

We show in Appendix B that we can restrict the set $H_{Market} = \left\{ \int_0^T \theta_t dL_t \middle| \theta \in \Theta^L \right\}$ to $H^*_{Market} = \left\{ \varphi(L_T) \middle| \varphi : \mathbf{R} \to \mathbf{R},\ \mathbf{E}^{\overline{\mathbf{P}}}[\varphi(L_T)] = 0 \right\}$ in the optimization problem. Then, due to the use of the quadratic criterion, we prove that the optimal strategy in H_{Market} is a European-type option on the Libor rate L_T. This idea is quite similar to the same developments made by Cox and Huang (1989) earlier.

Moreover, we show that the optimal risk profile in H_{Market} - for the non constrained problem (7) – is given by $\varphi^{Market}(L_T)$ with:

$$\varphi^{Market}(L_T) = \mathbf{E}^{\mathbf{P}}\left[IRM_g(K_T, L_T) \middle| L_T\right] - \mathbf{E}^{\overline{\mathbf{P}}}\left[IRM_g(K_T, L_T)\right]. \tag{13}$$

As an example, when the deposit rate is equal to zero, the interest rate margin is equal to $K_T L_T$ and we explicitly derive $\varphi^{Market}(L_T)$ by:

$$\varphi^{Market}(L_T) = \mathbf{E}^{\overline{\mathbf{P}}}[K_T L_T] \times \left\{ \exp\left[-\frac{\rho\sigma_K\sigma_L}{2}\left(1 + \rho\frac{\sigma_K}{\sigma_L}\right)\right]\left(\frac{L_T}{L_0}\right)^{\left(1+\rho\frac{\sigma_K}{\sigma_L}\right)} - 1 \right\}, \tag{14}$$

where $\mathbf{E}^{\overline{\mathbf{P}}}[K_T L_T] = K_0 L_0 \exp[(\mu_K + \rho\sigma_K\sigma_L - \rho\sigma_K\lambda)T]$.

From (14), we notice that the convexity of the optional profile with respect to L_T depends on the correlation parameter $\rho < 0$ being greater or not than $-\frac{\sigma_L}{\sigma_K}$. Indeed we have:

	Convexity of Optional Profile
$\lvert\rho\sigma_K\rvert < \sigma_L$	Convex
$\lvert\rho\sigma_K\rvert > \sigma_L$	Concave

More generally, the risk profile $\varphi^{Market}(L_T)$ involves a non linear regression of the interest rate margin with respect to the Libor rate at date *T*. Let us emphasize that on practical grounds, the optimal risk profile can be achieved either through a dynamic replication[19] of $\varphi^{Market}(L_T)$ or through a buy-and-hold investment in a European-type option on the interest rate derivatives market.

Let us point out that the replication price of $\varphi^{Market}(L_T)$ is equal to zero[20]. However, although all payoffs in H_{Market} have a price equal to zero, this does not imply anything as for their

[19] This is feasible thanks to the almost complete market assumption.

[20] According to the martingale minimal measure $\overline{\mathbf{P}}$ (see Definition 3.1). Indeed, we have:
$\mathbf{E}^{\overline{\mathbf{P}}}[\varphi(L_T)] = \mathbf{E}^{\mathbf{P}}\left[\frac{d\overline{\mathbf{P}}}{d\mathbf{P}}\mathbf{E}^{\mathbf{P}}\left[IRM_g(K_T, L_T) \middle| L_T\right]\right] - \mathbf{E}^{\overline{\mathbf{P}}}\left[IRM_g(K_T, L_T)\right] = 0$ since $\frac{d\overline{\mathbf{P}}}{d\mathbf{P}}$ is a function of L_T.

expected return from a 'historical' viewpoint – under $\mathbf{P}$. Thus, dealing with a return constraint is not a trivial problem and in Appendix B, we also give the solution for the constrained optimization – in each set $H_{Market}(m)$.

3.4. Optimal Investment Strategy – Full Information Set

We firstly deal with the non constrained version of the problem:

$$\min_{S \in H_{ALM}} \mathbf{Var}^{\mathbf{P}}\left[IRM_g(K_T, L_T) - S\right], \quad (14)$$

where we recall that $H_{ALM} = \left\{ V_T(\theta) = \int_0^T \theta_t dL_t, \theta \in \Theta \right\}$ and that Θ is the set of admissible strategies that involve the information contained in both the interest rate process and the deposit amount process. Due to the quadratic nature of this problem, its solution fulfills moment conditions which are also detailed in the following theorem.

T*heorem 3.3. Optimal dynamic strategy. The solution* θ^{**} *to* $\min_{\theta \in \Theta} \mathbf{Var}^{\mathbf{P}}\left[IRM_g(K_T, L_T) - V_T(\theta)\right]$ *verifies the following moment conditions:*

$$\forall \theta \in \Theta, \mathbf{E}^{\mathbf{P}}\left[\left(IRM_g(K_T, L_T) - x^{**} - V_T(\theta^{**})\right) \cdot V_T(\theta)\right] = 0 \,. \quad (15)$$

And it is determined by:

$$\theta_t^{**} = \frac{\partial}{\partial L_t} \mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g(K_T, L_T)\right] + \frac{\lambda}{\sigma_L L_t}\left[\mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g(K_T, L_T)\right] - x^{**} - V_t(\theta^{**})\right], \quad (16)$$

with $x^{**} = \mathbf{E}^{\overline{\mathbf{P}}}\left[IRM_g(K_T, L_T)\right]$.

We give a proof of this theorem in Appendix C.

In this theorem, we split the optimal investment strategy into two parts:

- the delta of the interest rate margin under the variance minimal measure, which acts here as a pricing measure: $\theta_{\Delta,t}^{**} = \frac{\partial}{\partial L_t} \mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g(K_T, L_T)\right]$;
- some feedback corrective term: $\theta_{F,t}^{**} = \mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g(K_T, L_T)\right] - x^{**} - V_t(\theta^{**})$ invested in the *hedging numéraire* introduced in subsection 2.2 above[21].

We also notice that the optimal strategy is linear with respect to the interest rate margin. Thus, in theory, we obtain optimal dynamic strategies for the global interest rate risk within the bank, by summing the optimal hedging strategies for each balance sheet item. This widens the perspective of our study, initially dedicated to demand deposits.

When modeling the probability of bank runs (see equation (2b)), we actually notice very little change for the optimal strategy. Let us remark that the variance minimal measure and the

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[21] The full filtration stands for the filtration related to both W_L and W_K. Let us point out that, from a mathematical viewpoint, the conditional expectation which appears in the delta term $\theta_{\Delta,t}^{**}$ stands for the optional projection of the interest rate margin upon the full filtration at date *t*, as defined in Protter (2003).

hedging numéraire remain the same in this new framework. Therefore, the optimal investment strategy can be derived along the same lines as in Theorem 3.3 above[22]:

$$\theta_t^{**} = e^{-\gamma(T-t)} \cdot \frac{\partial}{\partial L_t} \mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g\left(K_T, L_T\right)\right] + \frac{\lambda}{\sigma_L L_t}\left[e^{-\gamma(T-t)} \cdot \mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g\left(K_T, L_T\right)\right] - x^{**} - V_t\left(\theta^{**}\right)\right] \quad (17),$$

where $\mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g\left(K_T, L_T\right)\right]$ corresponds to the same term as computed in the previous section, with a deposit amount diffusion as assumed in Equation (2). We notice that the expression of the optimal strategy is quite the same as in Theorem 3.3 except an additional amortizing factor $e^{-\gamma(T-t)}$ on the deposit amount, corresponding to the probability of bank run. Let us also notice that when the bank run occurs at some date t, the investment strategy only consists in managing the hedging portfolio's value according to the hedging numéraire. After that, the deposit amount is equal to zero.

Finally, the following Corollary addresses the constrained problem. It shows that adding a return constraint on the strategy implies some slight modification to the feedback corrective term in the optimal investment strategy.

Corollary 3.4. Optimal dynamic strategy in mean-variance framework. The solution $\theta^*(m)$ *to* $\min_{\theta \in \Theta} \mathbf{Var}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - V_T(\theta)\right]$ *u.c.* $\mathbf{E}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - V_T(\theta)\right] = m$ *is recursively determined by:*

$$\theta_t^*(m) = \theta_{\Delta,t}^{**} + \frac{\lambda}{\sigma_L L_t}\left[\mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g\left(K_T, L_T\right)\right] - x(m) - V_t\left(\theta^*(m)\right)\right], \qquad (18)$$

with $x(m) = x^{**} + \dfrac{m - x^{**}}{1 - e^{-\lambda^2 T}}$.

We give a proof of this Corollary in Appendix C.

3.5. Optimal full-information strategy with linear deposit rate.

As an example, in this subsection we assume the deposit rate to be a linear function of the market rate, according to equation (5a): $g\left(L_T\right) = \alpha + \beta L_T$.

The optimization in this particular case can be solved using Duffie and Richardson's (1991) results. Indeed, due to the linear form of the deposit rate, the interest rate margin is the sum two lognormal random variables. Then, the optimal dynamic strategy is the sum of the optimal dynamic strategies corresponding to each term: more precisely, assuming $\alpha = 0$, then assuming $\beta = 0$, yields the two components of the optimal strategy. These can be determined using Duffie and Richardson's (1991) results. Hopefully, our approach and theirs lead to the same analytical formulas.

Moreover, in our framework, the strategy can be explicitly derived. In the $\alpha = 0$ case, Theorem 3.3's feedback term can be deduced from:

$$\mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g\left(K_T, L_T\right)\right] = (1-\beta) K_t L_t \exp\left[(T-t)\left(\mu_K - \rho\sigma_K \lambda + \rho \sigma_K \sigma_L\right)\right], \qquad (19)$$

[22] The proof of this result is available upon request from the authors.

and the delta term can be explicitly derived as follows:

$$\theta^{**}_{\Delta,t} = \frac{\partial \mathbf{E}^{\overline{\mathbf{P}}}_t\left[IRM_g\left(K_T, L_T\right)\right]}{\partial L_t} = (1-\beta)\left(1+\frac{\rho\sigma_K}{\sigma_L}\right)K_t \exp\left[(T-t)\left(\mu_K - \rho\sigma_K\lambda + \rho\sigma_K\sigma_L\right)\right]. \quad (20)$$

4. Empirical Comparison of Optimal Strategies

In the following subsections, as an example, we consider parameters corresponding to the Euro Zone as for the deposit amount and the interest rates, that is $\mu_K = 9.24\%$, $\sigma_K = 6.08\%$, $\mu_L = 5.15\%$, $\sigma_L = 15.42\%$, $\rho = -70.85\%$. We also deal with a horizon of 2 years ($T = 2$). We set $K_0 = 100$ as the initial amount of deposits. As stated earlier, for convenience, we report returns and volatilities for a time interval of a year ($\delta T = 1$); rescaling is straightforward.

4.1. Checking Optimality of Dynamic Strategies

In this subsection, we check the independence of the dynamically hedged margin (in the case of H_{ALM} (see subsection 3.1)), towards self-financed strategies on the Libor rate process, at minimum variance point[23]. Indeed, Theorem 3.3 states that the strategy θ^{**} minimizing the variance fulfills the usual first order conditions, that is $\mathbf{Cov}^{\mathbf{P}}\left(IRM_g\left(K_T, L_T\right) - V_T\left(\theta^{**}\right), V_T(\theta)\right) = 0$ for any admissible strategy $\theta \in \Theta$. This reminds us about the condition fulfilled by the GMM estimator, thus the related tests of specification may be applied here, using simulations (cf. Appendix A.2 in Campbell, Lo and McKinlay (1997) for example). This can be viewed as a numerical check of the optimality of hedging strategies.

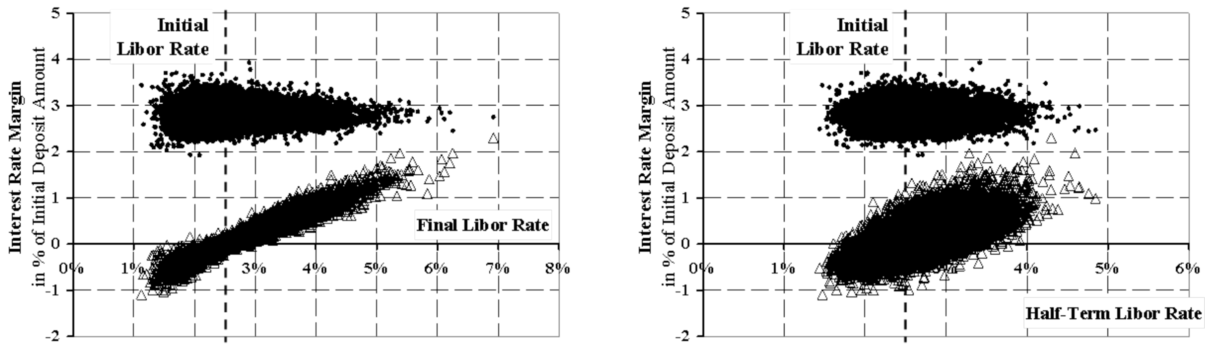
For example we used the Fisher test for zero correlation, in two cases[24]:

- between $IRM_g\left(K_T, L_T\right) - V_T\left(\theta^{**}\right)$ and $V_T(\theta) = L_T - L_0$ (case A);
- between $IRM_g\left(K_T, L_T\right) - V_T\left(\theta^{**}\right)$ and $V_T(\theta) = L_{T/2} - L_0$ (case B),

using 20 000 simulations and the correlations satisfy the Fisher test at 1% and 5% confidence levels. Then, hopefully, the optimal strategy fulfills the first order condition in the latter two cases. The graphs in Figure 4.1 summarize the independence of the hedged margin with respect to the final Libor rate and the half-term Libor rate.

[23] We recall that by "minimum variance point", we mean the optimum in the non constrained problem $\min_{S \in H_{ALM}} \mathbf{Var}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - S\right]$ (see also equation (7)).

[24] The first case corresponds to $\theta_t = 1$ for any $0 \le t \le T$; the second corresponds to $\theta_t = 1$ for $0 \le t \le T/2$ and $\theta_t = 0$ for $T/2 \le t \le T$.

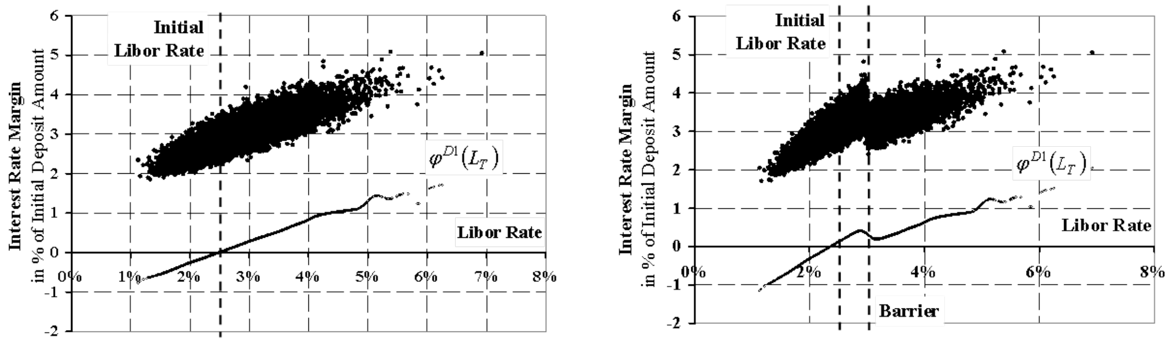


(Case A) (Case B)

Figure 4.1. Hedged Interest Rate Margin and Optimal Hedging Portfolio with respect to Final Libor Rate (left) and Half-Time Libor Rate (right).
We represented the optimally hedged interest rate margin (in dots) and the associated hedging portfolio (in empty triangles) in function of the final Libor rate (Case A – left) and the half-term Libor rate (Case B – right).

4.2. Influence of the Deposit Rate's Model Specification

To study the influence of the deposit rate specification, we consider two different profiles, corresponding to its modeling in the US (linear: $g(L_T) = \alpha + \beta L_T$, case (a)) and in Japan (featuring a barrier: $g(L_T) = \mathbf{1}_{L_T \geq R}(\alpha + \beta L_T)$, case (b)). We set, in our example, $\alpha = -0.50\%$, $\beta = 30\%$ and $R = 3.00\%$ with $L_0 = 2.50\%$. Due to business risk and deposit amount uncertainty, there are several possible levels for the interest rate margin, for a given rate L_T. In Figure 4.2 below, we plotted 20 000 points using Monte Carlo simulations on both the deposit process and the interest rate process. We also plotted φ^{Market} in these two cases. We recall that $\varphi^{Market}(L_T)$ is the risk profile which best fits the interest rate margin in a quadratic manner (see section 3.3). Actually, the optimal profile φ^{Market} may be computed analytically in some cases; in our case, we determined it thanks to Monte Carlo simulations and non parametric estimation. Indeed, we estimated the conditional expectation within φ^{Market} using Gaussian kernel smoothing on 1 000 simulations of the interest rate margin and L_T and then simulating 20 000 scenarios for L_T [25].



Case (a) Case (b)

Figure 4.2. Interest Rate Margin and Exotic Option Hedge.
Case (a): linear deposit rate (US case); Case (b): non linear deposit rate (Japanese case). We mentioned the

[25] We use the Gaussian kernel smoothing method described in the Chapter 3 of Pagan and Ullah (1999).

level of L_0 and the barrier in the Japanese case. In both graphs, the upper scatter represents the interest rate margin without hedge (*y* axis) with respect to L_T (*x* axis) and the lower scatter represents the optimal exotic option $\varphi^{Market}(L_T)$ in H_{Market}. Let us notice that in Case (b), $\varphi^{Market}(L_T)$ can be closely reproduced using caps and floors on the rate L_T. The shift between the hedging profile $\varphi^{Market}(L_T)$ and the interest rate margin's scatter is due to the price of the hedging strategy related to $\varphi^{Market}(L_T)$, which is equal to zero. Thus, the final income including hedge still carries some non zero expected return, which is due to the risk premium on interest rates.

Let us notice that the optimal hedging payoff is nearly linear in case (a), corresponding to a linear deposit rate. This somehow legitimates the investment of the deposit amount on plain vanilla interest rate swaps, following the liquidity amortizing schedule of demand deposits. Indeed, thanks to behavioral modeling as recommended by the Basel II's Pillar 3, banks can forecast the decay of their current demand deposits' amount and compute the correlation parameter ρ with respect to interest rates. Then, the best practice for them is to manage the interest rate margin for each quarter $\left[T, T+\frac{1}{4}\right]$ of the amortizing schedule. This may be performed by settling amortizing interest rate swaps which deal with several quarters simultaneously or FRA-based dynamic portfolios for the margin at each quarter.

In Figure 4.3, we plotted the initial hedge amount (θ_0^{**} in Theorem 3.3) in function of the quarter's time horizon T, with the same values for model parameters. As discussed above, θ_0^{**} is the starting point of the dynamic strategy which the bank has to settle for the management of the interest rate margin at each time horizon. As discussed above, we represent the demand deposit amount's decay by choosing $\mu_K = -9.24\%$. Thus, as discussed in section 2.2, we adopt the viewpoint of the outstanding amount's amortization, which we also plotted in the graph below.

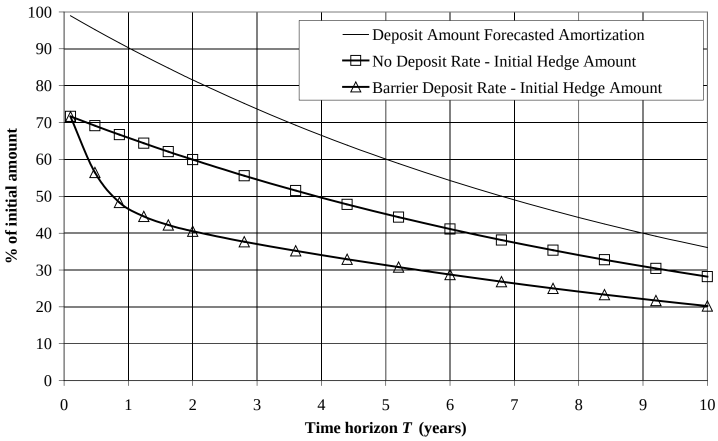


Figure 4.3. Outstanding deposit amount's amortization and related initial amount of hedge.
Same parameters as mentioned at the beginning of section 4, except $\mu_K = -9.24\%$. We represent the initial hedge amount θ_0^{**} (see Theorem 3.3) as a function of the time horizon T, with no deposit rate and a barrier deposit rate. We compare it to the forecasted deposit amount's amortization, which equals $K_0 \exp(\mu_K T)$ in our framework.

Let us remark that, even when there is no deposit rate, the optimal investment for the short-term interest rate margin is equal to 70% the initial deposit amount, instead of 100% as we could anticipate. This is chiefly due to the interest rate option embedded in the deposit amount through the correlation parameter with market rates.

The previous study and the scatters in Figure 4.2 show that the embedded optionality in deposit accounts is influenced by the deposit rate profile, rather than by the business risk. The graphs suggest that the business risk is responsible for some dispersion of the interest rate margin at each level of the terminal Libor rate but has limited influence on the shape of the payoff $\varphi^{Market}(L_T)$, at least given the chosen parameters.

A further issue is the amount of risk reduction involved by choosing the dynamic strategy with full information set rather than the only market information set. As detailed below, the deposit rate's model specification is critical.

Table 4.4 shows some results about risk reduction within the final interest rate margin, when relying upon the full information set and the market information set, for two deposit rate profiles – linear and barrier. We thus measure how important the hedging of interest rate margins is, since the risk reductions reaches almost half the risk carried by the margin.

Moreover, the table below shows that the gap between the full information set and the market information set increases as the deposit rate's profile becomes more exotic. Indeed, the complete market hedging strategy performs nearly as well as $\varphi^{Market}(L_T)$ when the deposit rate is linear, but we notice a slight risk shift between them when the deposit rate features a barrier. Then, unless the deposit rate's model specification is accurate enough[26], proceeding with the full information set provides more stable results.

	Linear Deposit Rate			Barrier Deposit Rate		
	Income's Standard Deviation	Risk Reduction (*)	Expected Return	Income's Standard Deviation	Risk Reduction (*)	Expected Return
Interest Rate Margin without hedge	0.395		2.961	0.390		3.158
Variance-Minimal Point with full-information set	0.194	**-0.201**	2.811	0.222	**-0.167**	3.017
Variance Minimization with Market Information set	0.209	-0.186	2.809	0.230	-0.160	3.007

(*) compared to no hedging at all.

Table 4.4. Risk Reduction for Several Hedging Strategies. (left: linear deposit rate (US case) – right: non linear deposit rate (Japanese case). The levels (standard deviations, risk reductions, expected returns) are given in % of the initial amount. Here we compare the risk reduction within the interest rate margin, when experiencing the two kinds of hedging strategies. In both cases, the risk reduction cuts almost half of the risk carried by the margin. However, the strategy based on the full information set is slightly more efficient, and this is all the more significant as the deposit rate's profile gets more complex (here, from linear to barrier).

Then, the latter result shows some robustness of the dynamic strategy based on full information set, with respect to the deposit rate's specification. This is an noticeable feature, because banks experience tough difficulty identifying the interest rate option within demand deposit interest rate margins.

4.3. Deposit Specific Risk Mitigation

[26] Moreover, Hutchison (1995) and later Hutchison and Pennacchi (1996) show that the deposit rate carries various determinants, not only market rates.

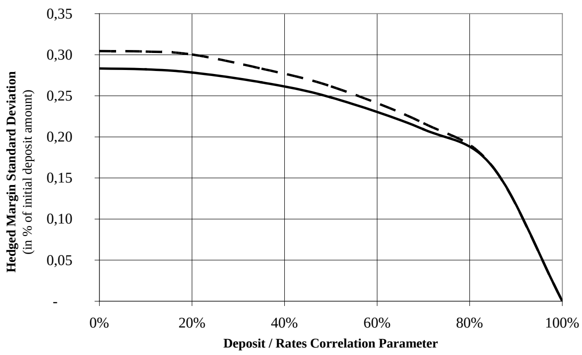
Comparing to dynamic strategies based only on market information, using dynamic hedging strategies with the full information set should imply a better alleviation of the specific risk embedded in demand deposit amount.

Indeed, in Table 4.5 below, we notice that the shift between the minimal standard deviation for each kind of strategy increases from 8.6% to 11.4% as the correlation between the deposit amount and interest rates gets smaller. In other words, when the relative weight of the business risk increases, the strategy related to the full information set increasingly over-performs the strategy related to the market information set. Therefore, as expected, the risk profile $\varphi^{Market}\left(L_T\right)$ features imperfections as for the assessment of the specific risk carried in deposits.

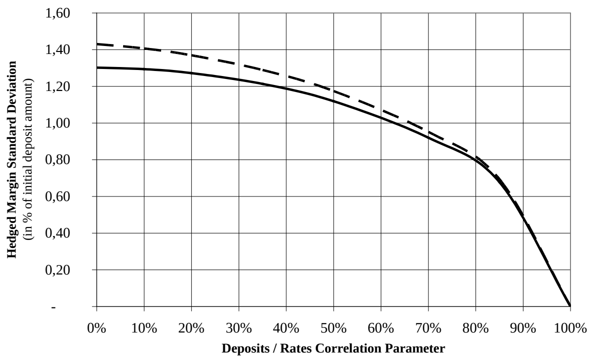
Correlation parameter ρ	**-100%**	**-90%**	**-65%**	**-30%**	**-10%**	**0%**
Optimal hedge with full information set Assessing jointly the information contained in the deposit amount and the market rate	0.000	0.124	0.216	0.272	0.285	0.287
Optimal hedge with market information set Hedging with the optimal risk profile on the terminal Libor rate	0.000	0.134	0.236	0.301	0.317	0.320
Relative Difference	*N/A*	*8.6%*	*9.4%*	*10.5%*	*11.1%*	*11.4%*

Table 4.5. Minimal Standard Deviation for Dynamic Hedging Strategies for Various Levels of the Correlation Parameter (Barrier Customer Rate).
Standard deviations are given in % of the initial deposit amount. The first line the standard deviation at the left endpoint of the efficient frontier (see above), when varying the level of ρ. The first line refers to the case of H_{ALM} while the second line corresponds to H_{Market}. The 'Relative Difference' assesses the relative shift of standard deviations in line 2 over line 1.

When $\rho = -100\%$, there is no specific deposit risk to assess and no difference between the two hedging strategies. However, when $|\rho|$ decreases, the market becomes incomplete and being myopic towards the deposit amount process yields differences in the risk assessment within interest rate margins (from 8.4% to 11.6%). The results are summarized in Figure 4.6 below.



(Case A)



(Case B)

Figure 4.6. Minimal Standard Deviation with respect to Correlation Parameter $|\rho|$ (left: $\sigma_K = 6.08\%$, right: $\sigma_K = 30\%$).
We represented the standard deviation of the hedged margin at minimal variance point in function of the correlation

parameter ρ , for a barrier customer rate, and for two levels of deposit volatility. The dotted line corresponds to the use of the risk profile $\varphi^{Market}(L_T)$ in H_{Market} and the continuous line, to the use of the optimal hedging strategy in H_{ALM} with full information set.

4.4. Robustness towards Risk Criterion

Banks' internal risk measurement procedures often involve Value-at-Risk and Expected Shortfall computations (see definitions in Acerbi and Tasche (2002) for example). In Tables 4.7a and 4.7b, we propose to compare the performances of the various hedging strategies developed up to now, for these criteria. We choose a 99.95% threshold for the VaR and 99.5% for the Expected Shortfall.

Barrier Deposit Rate	**Expected Return**	**Standard Deviation**		**ES** (99.5%)		**VaR** (99.95%)	
		Level	*Risk Reduction*	Level	*Risk Reduction*	Level	*Risk Reduction*
Margin without hedge	3.16	0.39		-2.02		-1.90	
Market Information Set	3.01	0.23	*-0.16*	-2.26	*-0.24*	-2.04	*-0.14*
Complete Market Framework	3.01	0.24	*-0.15*	-2.35	*-0.33*	-2.25	*-0.35*
Full Information Set	3.01	0.22	***-0.17***	-2.38	***-0.36***	-2.29	***-0.39***

Table 4.7a. Risk Measurement for Various Hedging Methods – Barrier Deposit Rate.
In the upper line, we present the risk metrics of the margin without hedge. In the lines below, we display the same risk metrics and the corresponding difference with respect to the upper line. The 'Market Information Set' line stands for the fitting of the interest rate margin using a function of the terminal Libor rate (case of H_{Market}). The 'Complete Market Framework' line corresponds to the framework described in equation (4) and at the end of subsection 3.2, where the deposit amount is contingent only to interest rates. The 'Full Information Set' Line corresponds to the performances of the optimal strategy with full information set, in H_{ALM} .

The negative values for the Expected Shortfall and the VaR in the table above are due to the fact that the margin at final date is mostly positive and to the convention we use for VaR and Expected Shortfall computation[27]. Thus, the VaR at 99.95% of the interest rate margin is (-1.90) for an initial deposit amount of 100. We see, in Table 4.7a above, that using the full information set makes the risk decrease by 0.39 to (-2.29), thus constituting a better risk reduction than other strategies, in a VaR framework. The same actually holds for the Expected Shortfall and the standard deviation.

No Deposit Rate	**Expected Return**	**Standard Deviation**		**ES** (99.5%)		**VaR** (99.95%)	
		Level	*Risk Reduction*	Level	*Risk Reduction*	Level	*Risk Reduction*
Margin without hedge	3.35	0.58		-2.02		-1.90	
Market Information Set	3.11	0.24	*-0.34*	-2.35	*-0.33*	-2.17	*-0.27*
Complete Market Framework	3.11	0.24	*-0.34*	-2.35	*-0.33*	-2.25	*-0.35*
Full Information Set	3.11	0.22	***-0.36***	-2.47	***-0.45***	-2.36	***-0.46***

Table 4.7b. Risk Measurement for Various Hedging Methods – No Deposit Rate.
Same caption as above.

[27] Here, consistently with Acerbi and Tasche (2002), we define the VaR at level 99.95% as the opposite of the 0.05% quantile of the distribution. We use the same convention for the Expected Shortfall.

Generally speaking, the risk reduction implied by the strategy related to the full information set is almost always better: even when there is no customer rate (Table 4.7b), the risk reduction reaches 0.45 in Expected Shortfall framework and 0.46 in VaR framework. This shows some robustness of the optimal dynamic strategy also with respect to the choice of the risk criterion. Moreover, this somehow makes us confident with the mean-variance optimization framework, more tractable than some dynamic mean-VaR or mean-Expected Shortfall framework.

5. Conclusions

In this article we dealt with the mitigation of the risk contained in interest rate margins of demand deposits. We assume the demand deposit amount to carry some source of risk called 'business risk', independent of market risk. Thus we deal with mean-variance hedging of the margins in an incomplete market framework. Thanks to Duffie and Richardson's (1991) results and the theory of the *hedging numéraire*, we derive explicit dynamic hedging strategies. There are various ways to model the demand deposit amount and rates, but the method we developed in this article can cope with a wide range of them. Indeed we detail the case of some non linear behavior of the customer rate with respect to market rates and the possibility of bank runs.

We compared these optimal dynamic strategies based upon the full information set with some strategies that involve only forward Libor rates. We show that identifying the interest rate-related optionality in interest rate margins with market information-based strategies is a quite satisfactory alternative to hedging strategies based on the full information set. Moreover, both methods lead to quite robust results, with respect to model specification of the deposit rate.

However, using the full information set implies better mitigation of the specific risk carried by the deposit amount. Moreover, we show that the related strategies exhibit some robustness with respect to other risk criteria like the Expected Shortfall and the Value-at-Risk. This is a positive conclusion for the use of mean-variance optimization and the related dynamic hedging strategies, since they display good results with respect to other risk measures.

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Appendix A: List of US Banks used in interest rate margins analysis[28]

Bank Name	City and State	Number of Branches	Asset Size *(in millions of dollars)*	Net income (year2005) *(in millions of dollars)*
Webster Bank, National Association	WATERBURY, CT	179	16 622	468
Associated Bank, National Association	GREEN BAY, WI	310	20 312	553
Colonial Bank, National Association	MONTGOMERY, AL	321	23 325	567
Compass Bank	BIRMINGHAM, AL	415	36 914	780
TD BankNorth, National Association	PORTLAND, ME	587	42 368	927
Fifth Third Bank	CINCINNATI, OH	408	53 249	3 048
M&I Marshall and Ilsley Bank	MILWAUKEE, WI	290	49 334	1 132
Union Bank of California, National Association	SAN FRANCISCO, CA	320	52 743	1 624
The Huntington National Bank	COLUMBUS, OH	710	54 186	911
Manufacturers and Traders Trust Company	BUFFALO, NY	637	56 713	1 735
Bank of the West	SAN FRANCISCO, CA	663	56 963	1 352
Comerica Bank	DETROIT, MI	394	58 088	1 810
Capital One, National Association	MCLEAN, VA	695	95 204	3 003
PNC Bank, National Association	PITTSBURGH, PA	1 055	117 232	1 969
Branch Banking and Trust Company	WINSTON SALEM, NC	1 473	120 906	3 348
National City Bank	CLEVELAND, OH	1 363	135 636	4 433
HSBC Bank USA, National Association	WILMINGTON, DE	453	166 101	7 802
SunTrust Bank	ATLANTA, GA	1 711	174 962	3 685
Commerce Bank, National Association	PHILADELPHIA, PA	405	43 053	497
First Tennessee Bank, National Association	MEMPHIS, TN	262	38 178	856

Appendix B: Optimal Strategy in H_{D1} and $H_{D1}(m)$.

Restricting the Set $H_{D1}(m)$.

We consider the set:

$$H^*_{D1}(m) = \{\varphi(L_T) | \varphi : \mathbf{R} \to \mathbf{R}, \mathbf{E}^{\overline{\mathbf{P}}}[\varphi(L_T)] = 0, \mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - \varphi(L_T)] = m\},$$

where $\overline{\mathbf{P}}$ is the variance minimal measure (see Definition 3.1) and $\mathbf{P}$ is the historical measure. By construction, any payoff in $H^*_{D1}(m)$ can be dynamically replicated on interest rate markets using some $\theta \in \Theta^L$, so $H^*_{D1}(m) \subset H_{D1}(m)$. Then, as a first step, let us prove that $\min_{S \in H_{D1}(m)} \mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - S]^2 = \min_{S \in H^*_{D1}(m)} \mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - S]^2$.

Given any $S \in H_{D1}$ we have:

$$\mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - S]^2 = \mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - \mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) | F^L]]^2 + \mathbf{E}^{\mathbf{P}}[\mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) | F^L] - S]^2$$

since S is F^L-measurable.

In our framework, $\mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) | F^L]$ can be set under the form $\varphi(L_T)$ for some $\varphi : \mathbf{R} \to \mathbf{R}$. Therefore, the minimum in $\min_{S \in H_{D1}(m)} \mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - S]^2$ is attained in $H^*_{D1}(m)$.

Optimal Strategy in $H^*_{D1}(m)$.

Here, we look for the solution φ^{D1}_m to the constrained problem:

$$\min_{\varphi \in \Phi} \mathbf{Var}^{\mathbf{P}}[IRM_g(K_T, L_T) - \varphi(L_T)]$$
$$\text{u.c. } \mathbf{E}^{\overline{\mathbf{P}}}[\varphi(L_T)] = 0 \text{ and } \mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - \varphi(L_T)] = m. \qquad \text{(B1)}$$

[28] Source FFIEC (www.ffiec.org) and SEC (www.secinfo.com).

We recall that Φ is the set of functions $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi(L_T)$ is square integrable with respect to $\mathbf{P}$.

Due to its convexity, the constrained problem (B1) is equivalent to:

$$\min_{f,\lambda,\mu} \Gamma(f,\lambda,\mu) = \mathbf{E}^{\mathbf{P}}\left[IRM_g(K_T, L_T) - f(L_T)\right]^2 + 2\lambda \mathbf{E}^{\mathbf{P}}\left[\frac{d\overline{\mathbf{P}}}{d\mathbf{P}} f(L_T)\right]$$
$$+ 2\mu \mathbf{E}^{\mathbf{P}}\left[IRM_g(K_T, L_T) - f(L_T) - m\right]$$

which yields:

$$\varphi_m^{D1}(L_T) = \mathbf{E}^{\mathbf{P}}\left[IRM_g(K_T, L_T) \middle| L_T\right] - \lambda \frac{d\overline{\mathbf{P}}}{d\mathbf{P}} + \mu \text{ with } \begin{cases} -\lambda \mathbf{E}^{\mathbf{P}}\left[\dfrac{d\overline{\mathbf{P}}}{d\mathbf{P}}\right]^2 + \mu = -\mathbf{E}^{\overline{\mathbf{P}}}\left[IRM_g(K_T, L_T)\right] \\ -\lambda + \mu = -m \end{cases},$$

which gives the expression of $\varphi_m^{D1}(L_T)$, the optimal risk profile in $H_{D1}(m)$.

Optimal Strategy in H_{D1}**.**
Then we deal with:

$$\min_m \mathbf{Var}^{\mathbf{P}}\left[IRM_g(K_T, L_T) - \varphi_m^*(L_T)\right]. \tag{B2}$$

We readily obtain the optimal m^* in (B2) by:

$$m^* = \mathbf{E}^{\overline{\mathbf{P}}}\left[IRM_g(K_T, L_T)\right],$$

yielding $\varphi_{S2}^*(L_T) = \mathbf{E}^{\mathbf{P}}\left[IRM_g(K_T, L_T) \middle| L_T\right] - \mathbf{E}^{\overline{\mathbf{P}}}\left[IRM_g(K_T, L_T)\right]$ as expected.

Appendix C: Proof of Theorem 3.3 and Corollary 3.4.

We provide here a proof for Theorem 3.3 and Corollary 3.4.

Using Lemma 3 in Duffie and Richardson (1991), we readily establish that the problem $\min_{\theta\in\Theta} \mathbf{Var}^{\mathbf{P}}\left[IRM_g(K_T, L_T) - V_T(0,\theta)\right]$ yields the same optimal strategy θ^{**} as the problem $\min_{\substack{x\in\mathbf{R}\\ \theta\in\Theta}} \mathbf{E}^{\mathbf{P}}\left[IRM_g(K_T, L_T) - x - V_T(\theta)\right]^2$.

Then, Propositions 3.2 and 5.1 in Gouriéroux, Laurent and Pham (1998) show that solving the problem $\min_{\theta\in\Theta} \mathbf{E}^{\mathbf{P}}\left[IRM_g(K_T, L_T) - (x + V_T(\theta))\right]^2$ for some $x \in \mathbf{R}$ yields the following solution:

$$\theta_t^*(x) = \left(\frac{\partial \overline{L}_t}{\partial L_t}\right)^{-1}\left(\frac{\partial}{\partial L_t} \mathbf{E}_t^{\overline{\mathbf{P}}_{HN}}\left[\frac{IRM_g(K_T, L_T)}{N_T}\right]\right)\left(1 + \frac{\lambda}{\sigma_L}\right) - \frac{\lambda}{\sigma_L L_t}\left(x + V_t(\theta^*)\right), \tag{C1}$$

where $N = (N_t)_{0\le t\le T}$ is the value process of the *hedging numéraire* (see our Definition 3.2) and $\overline{\mathbf{P}}_{HN}$, the probability measure equivalent to the variance minimal measure $\overline{\mathbf{P}}$ (see our Definition 3.1) and defined by $\dfrac{d\overline{\mathbf{P}}(a^{HN})}{d\overline{\mathbf{P}}} = N_T$ (see Proposition 3.1 in Gouriéroux et al. (1998)). Finally, $\overline{L} = \dfrac{L}{N}$ corresponds to the forward Libor process in *hedging numéraire* units. Then, the optimal strategy becomes:

$$\theta_t^*(x) = \frac{\partial}{\partial L_t} \mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g\left(K_T, L_T\right)\right] - \frac{\lambda}{\sigma_L L_t}\left[\mathbf{E}_t^{\overline{\mathbf{P}}}\left[IRM_g\left(K_T, L_T\right)\right] - x - V_t\left(\theta^*\right)\right].$$

Then the optimal x in problem $\min_{\substack{x \in \mathbf{R} \\ \theta \in \Theta}} \mathbf{E}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - x - V_T(\theta)\right]^2$ is given by $x^{**} = \mathbf{E}^{\overline{\mathbf{P}}}\left[IRM_g\left(K_T, L_T\right)\right]$ (see Gouriéroux et al.'s (1998) Theorem 5.2)

The justification for our Theorem 3.3's moment conditions is exactly the same as in Duffie and Richardson's (1991) Lemma 1 and Pham, Rheinländer and Schweizer's (1998) Theorem 7.

This achieves the proof of Theorem 3.3.

Remark: Let us note that Gouriéroux, Laurent and Pham (1998)'s Theorem 5.1 holds if we extend our framework to non Markovian effects in the client rate modelling or deposit amount correlation with interest rates. However, in such a case, the optimal strategy cannot be expressed in such a convenient way as in our Theorem 3.3.

Proof of Corollary 3.4

We recall that Propositions 3.2 and 5.1 in Gouriéroux et al. (1998) state that solving the problem $\min_{\theta \in \Theta} \mathbf{E}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - x - V_T(\theta)\right]^2$ for some $x \in \mathbf{R}$ yields:

$$\theta_t^*(x) = \left(\frac{\partial \overline{L}_t}{\partial L_t}\right)^{-1}\left(\frac{\partial}{\partial L_t} \mathbf{E}_t^{\overline{\mathbf{P}}_{HN}}\left[\frac{IRM_g\left(K_T, L_T\right)}{N_T}\right]\right)\left(1 + \frac{\lambda}{\sigma_L}\right) - \frac{\lambda}{\sigma_L L_t}\left(x + V_t\left(\theta^*\right)\right),$$

with the same notations as above.

Then we have:

$$\mathbf{E}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - V_T\left(\theta^*(x)\right)\right] - \mathbf{E}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - V_T\left(\theta^{**}\right)\right] = \left(1 - \mathbf{E}^{\mathbf{P}}\left[N_T\right]\right)\left(x - x^{**}\right), \quad \text{(C2)}$$

with $\mathbf{E}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - V_T\left(\theta^{**}\right)\right] = x^{**}$ and $\mathbf{E}^{\mathbf{P}}\left[N_T\right] = e^{-\lambda^2 T}$.

Thanks to Lemma 4.3 in Duffie and Richardson (1991), we state that for some $x \in \mathbf{R}$, $\theta_t^*(x)$ solves $\min_{\theta \in \Theta(m)} \mathbf{E}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - V_T(\theta)\right]^2$ where $m = \mathbf{E}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - V_T\left(\theta^*(x)\right)\right]$. In other words, according to our notations, for any $x \in \mathbf{R}$, $\theta^*(x)$ coincides with $\theta^*(m)$ for $m = \mathbf{E}^{\mathbf{P}}\left[IRM_g\left(K_T, L_T\right) - V_T\left(\theta^*(x)\right)\right]$. Then, from Theorem 5.1 in Gouriéroux et al. (1998), we have $\frac{\partial V_t\left(\theta_t^*(x)\right)}{\partial x} = N_t$ and this yields:

$$\frac{\partial \theta_t^*(x)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial V_t\left(\theta_t^*(x)\right)}{\partial L_t}\right) = \frac{\partial}{\partial L_t}\left(\frac{\partial V_t\left(\theta_t^*(x)\right)}{\partial x}\right) = \frac{\partial N_t}{\partial L_t} = -\frac{\lambda}{\sigma_L L_t} N_t.$$

Integrating the latter equation between x^{**} and x and using (C2), we obtain:

$$\theta_t^*(m) = \theta_t^{**} - \frac{\lambda}{\sigma_L L_t} N_t\left(x - x^{**}\right) = \theta_t^{**} - \frac{\lambda}{\sigma_L L_t} \frac{N_t}{\left(1 - \mathbf{E}^{\mathbf{P}}\left[N_T\right]\right)}\left(m - x^{**}\right).$$

This achieves the proof of Corollary 3.4.