Basket Default Swaps, CDO's and Factor Copulas

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Slides available on my web site
Paper « basket defaults swaps, CDO's and Factor Copulas » available on DefaultRisk.com

French Finance Association Conference

- June 23-25 in Lyon
- Special session on mathematical finance chaired by Monique Jeanblanc
- Submission to: affi2003@isfa.univ-lyon1.fr
- with mention of « mathematical finance session »
- Web site: http://affi.asso.fr/lyonus.html

Changes in our environment

- What are we looking for ?
  - *Not really a new model for defaults*
    - Already a variety of modelling approaches to default
  - *Rather a framework where:*
    - one can easily deal with a large number of names,
    - Tackle with different time horizons,
    - Compute loss distributions, measures of risk (VaR, ES) …
    - And credit risk insurance premiums (pricing of credit derivatives).
- Straightforward approach:
  - *Direct modelling of default times*
  - *Modelling of dependence through copulas*
  - *Default times are independent conditionnally on factors*

Overview

- *Probabilistic tools*
  - Survival functions of default times
  - Factor copulas
- *Valuation of basket credit derivatives*
  - Moment generating functions
    - *Distribution of k-th to default time*
    - *Loss distributions over different time horizons*
  - k^{th} to default swaps
- *Valuation of CDO tranches*

Probabilistic tools: survival functions

- $i = 1, \ldots, n$ names
- $\tau_1, \ldots, \tau_n$ default times
- Marginal distribution function $F_i(t) = Q(\tau_i \leq t)$
- Marginal survival function $S_i(t) = Q(\tau_i > t)$
  - *Risk-neutral probabilities of default*
    - Obtained from defaultable bond prices or CDS quotes
  - *« Historical » probabilities of default*
    - Obtained from time series of default times

Probabilistic tools: survival functions

- Joint survival function

$$S(t_1, \ldots, t_n) = Q(\tau_1 > t_1, \ldots, \tau_n > t_n)$$

  - *Needs to be specified given marginals*

- (Survival) Copula of default times

$$C(S_1(t_1), \ldots, S_n(t_n)) = S(t_1, \ldots, t_n)$$

  - *C characterizes the dependence between default times*

- We need tractable dependence between defaults
  - *Parsimonious modelling*
  - *Semi-explicit computations for portfolio credit derivatives*

Probabilistic tools

by Andrejs Dunkels

Probabilistic tools: factor copulas

- Factor approaches to joint distributions
  - *V low dimensional factor, not observed « latent factor »*
  - *Conditionally on V default times are independent*
  - *Conditional default probabilities*

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right), \quad q_t^{i|V} = Q\left(\tau_i > t \mid V\right).$$

  - *Conditional joint distribution*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid V) = \prod_{1 \leq i \leq n} p_{t_i}^{i|V}$$

  - *Joint survival function (implies integration wrt V)*

$$Q(\tau_1 > t_1, \ldots, \tau_n > t_n) = E\left[\prod_{i=1}^{n} q_{t_i}^{i|V}\right]$$

Probabilistic tools: Gaussian copulas

- One factor Gaussian copula *(Basel 2)*
  - $V, \bar{V}_i, \; i = 1, \dots, n$ *independent Gaussian*

$$V_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
  - *Conditional default probabilities* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$
- Joint survival function

$$S(t_1, \dots, t_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\rho_i v - \Phi^{-1}(F_i(t_i))}{\sqrt{1-\rho_i^2}}\right)\right) \varphi(v)dv$$

  - *Copula*

$$C(u_1, \dots, u_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(u_i) - \rho_i v}{\sqrt{1-\rho_i^2}}\right)\right) \varphi(v)dv$$

Probabilistic tools : Clayton copula

- *Davis & Lo ; Jarrow & Yu ; Schönbucher & Schubert*
- Conditional default probabilities

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

- *V: Gamma distribution with parameter*
- Joint survival function

$$S(t_1, \ldots, t_n) = \int \prod_{i=1}^{n} \left(1 - p_{t_i}^{i|V}\right) \frac{1}{\Gamma(1/\theta)} e^{-V} V^{(1-\theta)/\theta} dV$$

- *Copula*

$$C(u_1, \ldots, u_n) = \left(u_1^{-\theta} + \ldots + u_n^{-\theta} - n + 1\right)^{-1/\theta}$$

Probabilistic tools: simultaneous defaults

- *Duffie & Singleton, Wong*
- Modelling of defaut dates $\tau_i = \min(\bar{\tau}_i, \tau)$
  - $Q(\tau_i = \tau_j) \geq Q\left(\tau \leq \min(\bar{\tau}_i, \bar{\tau}_j)\right) > 0$ *simultaneous defaults*
  - *Conditionally on* τ, τ_i *are independent*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid \tau) = \prod_{1 \leq i \leq n} Q(\tau_i \leq t_i \mid \tau)$$

- Conditional default probabilities

$$Q(\tau_i \leq t_i \mid \tau) = 1_{\tau > t_i} Q(\bar{\tau}_i \leq t_i) + 1_{\tau \leq t_i}$$

  - *Copula of default times*

$$C(u_1, \ldots, u_n) = E\left[\prod_{1 \leq i \leq n} Q\left(\tau_i \leq F_i^{-1}(u_i) \mid \tau\right)\right]$$

Probabilistic tools: Affine Jump Diffusion

- *Duffie, Pan & Singleton ;Duffie & Garleanu.*
- $n+1$ *independent affine jump diffusion processes:*

$$X_1, \ldots, X_n, X_c$$

- Conditional default probabilities:

$$Q(\tau_i > t \mid V) = q_t^{i|V} = V\alpha_i(t)$$

$$V = \exp\left(-\int_0^t X_c(s)ds\right), \quad \alpha_i(t) = E\left[\exp\left(-\int_0^t X_i(s)ds\right)\right].$$

- Survival function:

$$Q(\tau_1 > t, \ldots, \tau_n > t) = E\left[V^n\right] \times \prod_{i=1}^{n} \alpha_i(t).$$

  - *Explicitely known*

Probabilistic tools: conditional survivals

- Conditional survival functions and factors
  - *Example: survival functions up to first to default time*
  - *Conditional joint survival function easy to compute since:*

$$Q(\tau_1 > t_1, \tau_2 > t_2 \mid \tau_1 > t, \tau_2 > t) = \frac{Q(\tau_1 > t_1, \tau_2 > t_2)}{Q(\tau_1 > t, \tau_2 > t)}$$

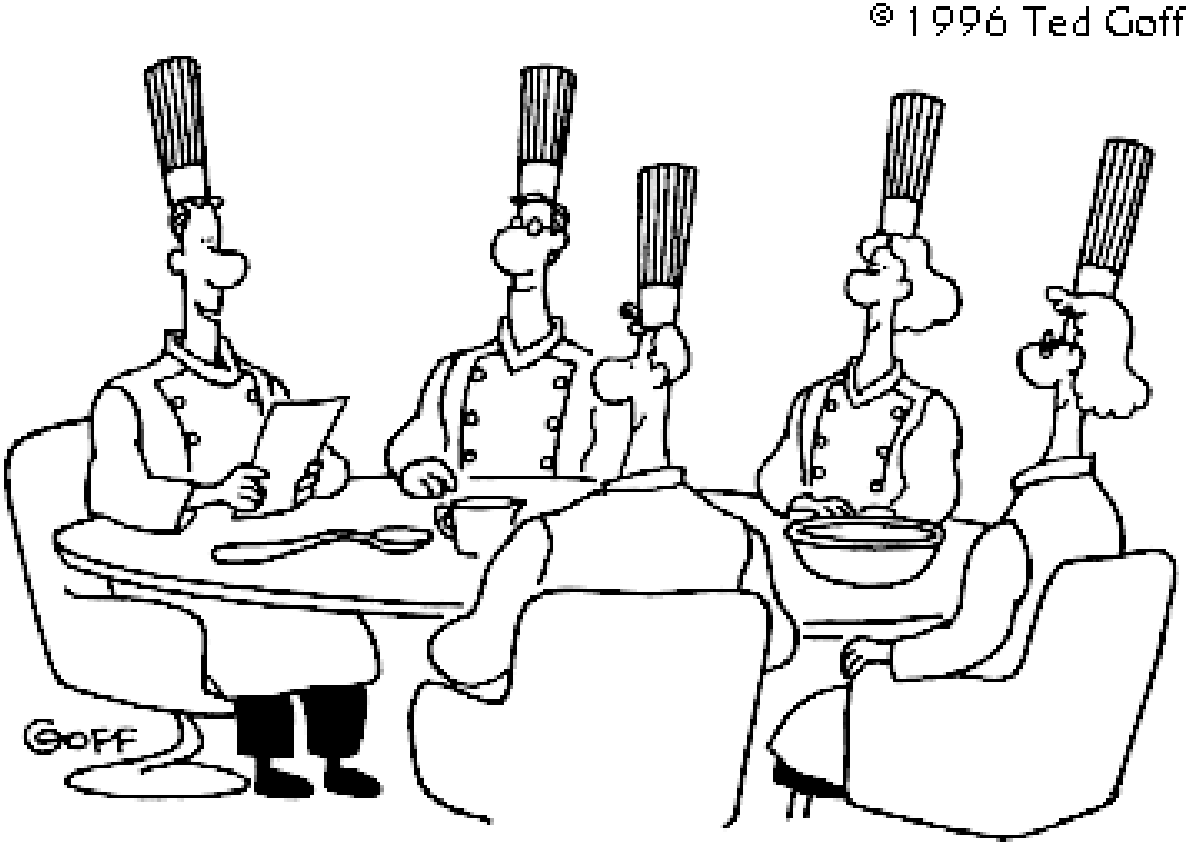
$$Q(\tau_1 > t_1, \tau_2 > t_2) = E\left[q_{t_1}^{1|V} \times q_{t_2}^{2|V}\right]$$

  - *However be cautious, usually:*

$$Q(\tau_1 > t_1, \tau_2 > t_2 \mid \tau_1 \wedge \tau_2 > t) \neq E\left[Q(\tau_1 > t_1, \tau_2 > t_2 \mid \tau_1 \wedge \tau_2 > t, V)\right]$$

«the whole is simpler than the sum of its parts »

Basket Valuation



"Our eggs are all in one basket, no milk has been spilt, and we have plenty of dough."

«Counting time is not so important as making time count»

Valuation of basket credit derivatives

- $N(t) = \sum_{1 \leq i \leq n} 1_{\{\tau_i \leq t\}} = \sum_{1 \leq i \leq n} N_i(t)$ *Number of defaults at t*
- τ^k *k^{th} to default time*
- $S^k(t) = Q(\tau^k > t)$ *Survival function of k^{th} to default*
- *Remark that* $\tau^k > t \Longleftrightarrow N(t) < k$
- *Survival function of* τ^k : $S^k(t) = \sum_{l \leq k-1} Q(N(t) = l)$
- *Computation of* $Q(N(t) = l)$
- *Use of pgf of N(t):* $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = \sum_{l=0}^{n} Q(N(t) = l)u^l$

Valuation of basket credit derivatives

- *Probability generating function of* $N(t)$: $\psi_{N(t)} = E\left[u^{N(t)}\right]$
  - $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = E\left[E\left[u^{N(t)} \mid V\right]\right]$ *iterated expectations*
  - $E\left[u^{N(t)} \mid V\right] = \prod_{1 \le i \le n} E\left[u^{N_i(t)} \mid V\right]$ *conditional independence*
  - $E\left[u^{N_i(t)} \mid V\right] = 1 - p_t^{i|V} + p_t^{i|V} \times u$ *binary random variable*
- $\psi_{N(t)}(u) = E\left[\prod_{i=1}^{n}\left(1 - p_t^{i|V} + p_t^{i|V} \times u\right)\right]$ *polynomial in u*
- *One can then compute* $Q(N(t) = k)$
- *Since* $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = \sum_{k=0}^{n} Q(N(t) = k)u^k$

Valuation of homogeneous baskets

- $i = 1, \ldots, n$ names
  - *Equal nominal (say 1) and recovery rate (say 0)*
- Payoff : 1 at *k*-th to default time if less than *T*
- Credit curves can be different
  - $S_i(t) = Q(\tau_i > t)$ *given from credit curves*
  - $S^k(t) = Q(\tau^k > t)$ *: survival function of* τ^k
  - $S^k(t)$ *computed from pgf of* $N(t)$

Valuation of homogeneous baskets

- Expected discounted payoff

$$E\left[B(\tau^k)1_{\tau^k \le T}\right] = -\int_0^T B(t)dS^k(t)$$

  - *From transfer theorem*
  - *B(t) discount factor*
- Integrating by parts

$$1 - B(T)S^k(T) + \int_0^T S^k(t)dB(t)$$

  - *Present value of default payment leg*
  - *Involves only known quantities*
  - *Numerical integration is easy*

Valuation of premium leg

- k^{th} to default swap, maturity T
  - $t_1, \ldots, t_{l-1}, t_l, \ldots, T$ *premium payment dates*
  - *Periodic premium p is paid until* τ^k
- l^{th} premium payment
  - $\tau^k > t_l$ *payment of p at date* t_l
  - *Present value:* $pB(t_l)S^k(t_l)$
  - $t_{l-1} \leq \tau^k \leq t_l$ *accrued premium of* $(\tau^k - t_{l-1})p$ *at* τ^k
  - *Present value:* $\displaystyle\int_{t_{l-1}}^{t_l} pB(t)(t - t_{l-1})dS^k(t)$
- PV of premium leg given by summation over l

Non homogeneous baskets

- $i = 1, \ldots, n$ *names*
- $M_i = (1 - \delta_i) N_i$ *loss given default for i*
- Payment at *k*th default of M_i if *i* is in default
  - *No simultaneous defaults*
  - *Otherwise, payoff is not defined*
- *i k*th default iff *k*-1 defaults before τ_i
  - $N^{(-i)}(\tau_i)$ *number of defaults (i excluded) at* τ_i
  - *k-1 defaults before* τ_i *iff* $N^{(-i)}(\tau_i) = k - 1$

Non homogeneous baskets

- *Guido Fubini*

Non homogeneous baskets

- (discounted) Payoff $\sum_{i=1}^{n} M_i B(\tau_i) 1_{\{N^{(-i)}(\tau_i)=k-1\}} 1_{\{\tau_i \leq T\}}$
- Upfront Premium
  - ... by iterated expectations theorem

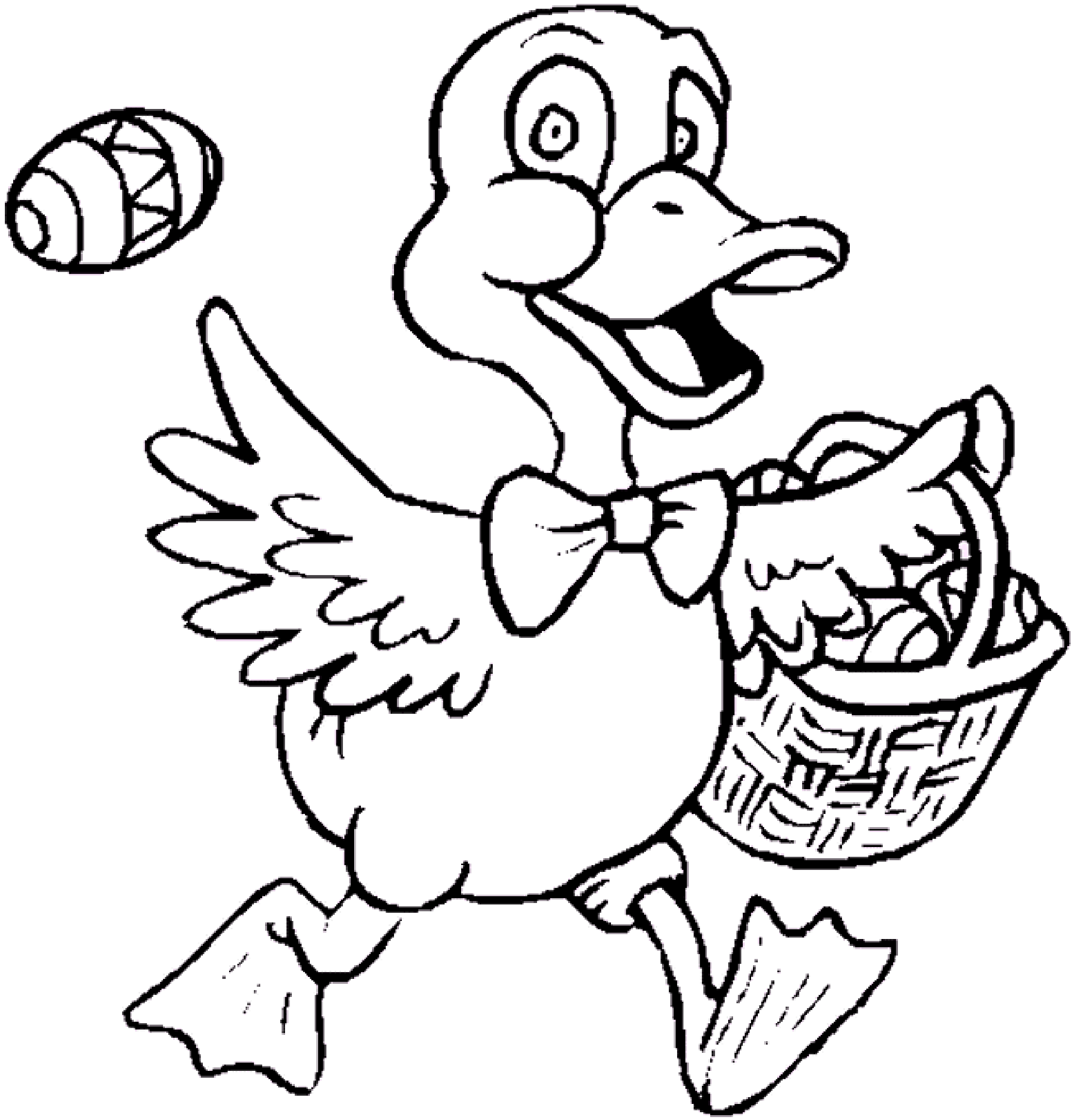
$$\sum_{i=1}^{n} M_i E\left[E\left[B(\tau_i) 1_{\{N^{(-i)}(\tau_i)=k-1\}} 1_{\{\tau_i \leq T\}} \mid V\right]\right]$$

  - ... by Fubini + conditional independence

$$\int_0^T B(t) Q(N^{(-i)}(t) = k-1 \mid V) dp_t^{i|V}$$

  - where $p_t^{i|V} = Q(\tau_i \leq t \mid V)$
  - $Q(N^{(-i)}(t) = k-1 \mid V)$: formal expansion of $\prod_{j \neq i} \left(1 - p_t^{j|V} + p_t^{j|V} u\right)$

First to default swap

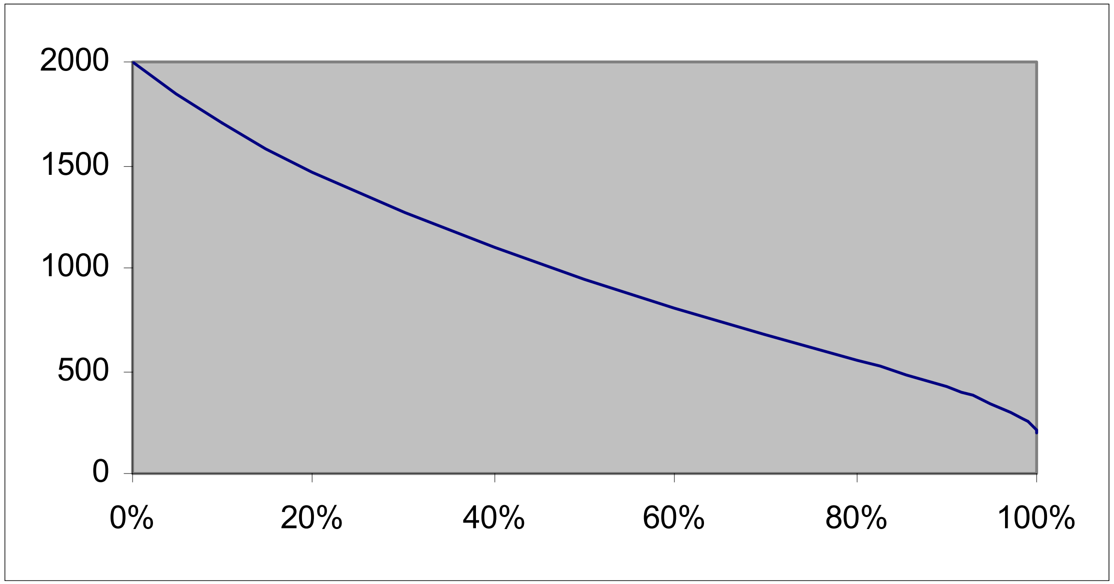


First to default swap

- *Case where* $k = 1$
- $Q\left(N^{(-i)}(t) = 0 \mid V\right) = \prod_{j \neq i} \left(1 - p_t^{j|V}\right)$ *no defaults for* $j \neq i$
- *premium* $= \sum_{i=1}^{n} M_i E\left[\int_0^T B(t) \prod_{j \neq i} \left(1 - p_t^{j|V}\right) dp_t^{i|V}\right]$
- $= \int_0^T \sum_{i=1}^{n} M_i B(t) E\left[\prod_{j \neq i} \left(1 - p_t^{j|V}\right) \frac{dp_t^{i|V}}{dt}\right] dt$ *(regular case)*
- *One factor Gaussian* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$
- *Archimedean* $p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$

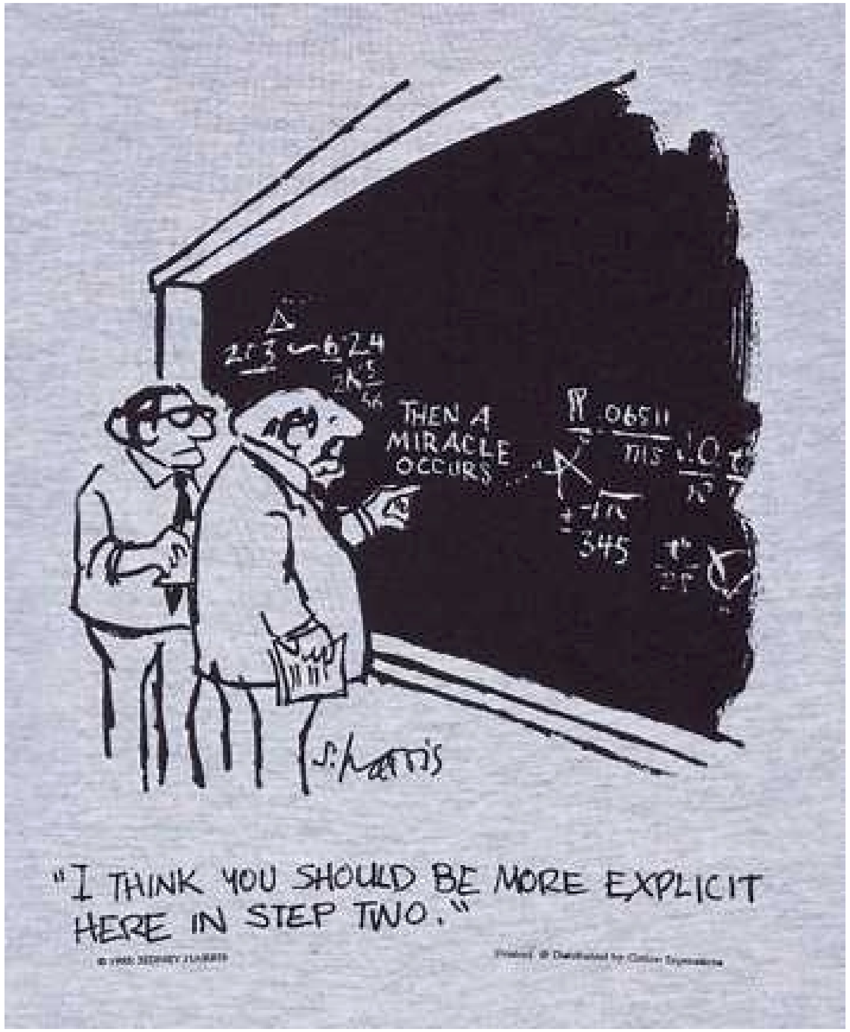
First to default swap

- One factor Gaussian copula
- n=10 names, recovery rate = 40%
- 5 spreads at 50 bps, 5 spreads at 350 bps
- maturity = 5 years
- x axis: correlation parameter, y axis: annual premium



Valuation of CDO's

«Everything should be made as simple as possible, not simpler»



- *Explicit premium computations for tranches*
- *Use of loss distributions over different time horizons*
- *Computation of loss distributions from FFT*
- *Involves integration par parts and Stieltjes integrals*

Credit loss distributions

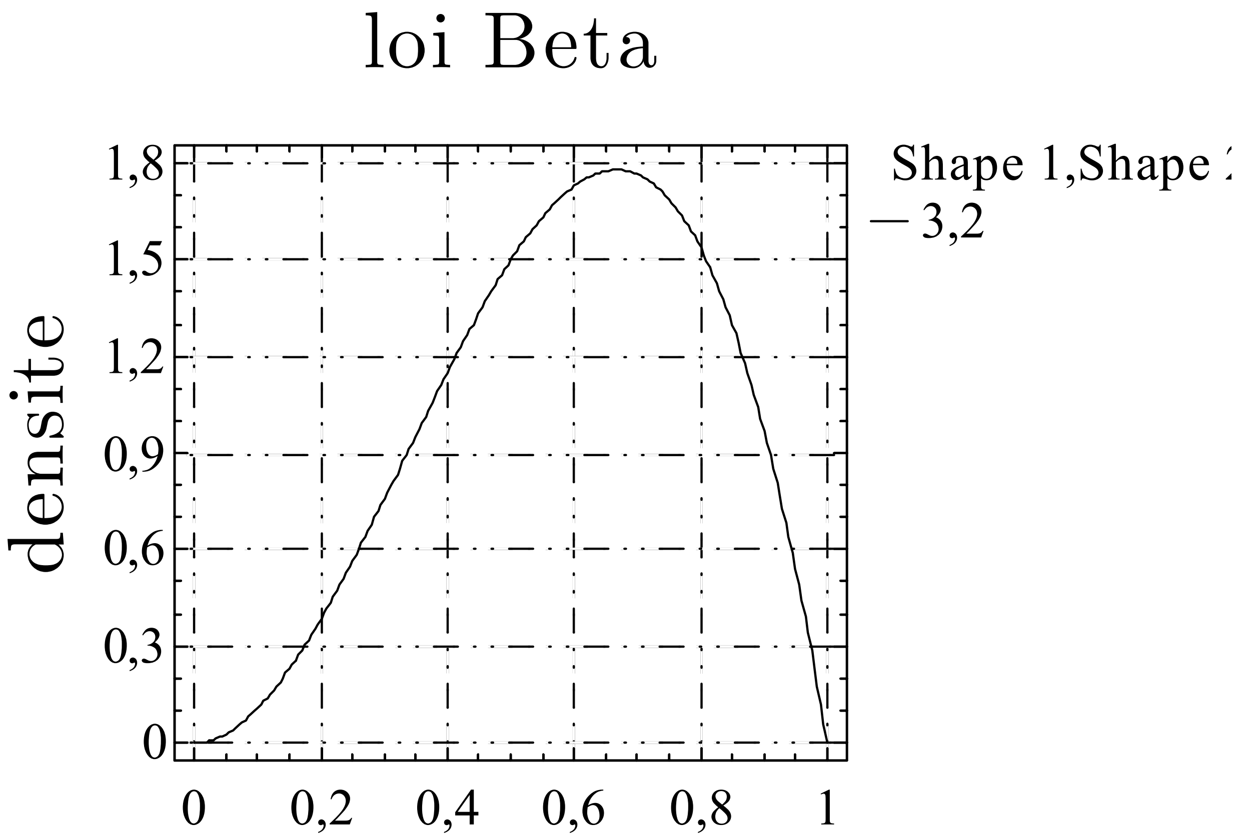
- *Accumulated loss at t:* $L(t) = \sum_{1 \le i \le n} N_i(1-\delta_i) N_i(t)$
  - *Where* $N_i(t) = 1_{\tau_i \le t},\ N_i(1-\delta_i)$ *loss given default*
- *Characteristic function* $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
- *By conditioning* $\varphi_{L(t)}(u) = E\left[\prod_{1 \le j \le n} \left(1 - p_t^{j|V} + p_t^{j|V} \varphi_{1-\delta_j}(N_j)\right)\right]$
- *If recovery rates follows a beta distribution:*

$$\varphi_{L(t)}(u) = E\left[\prod_{1 \le j \le n} \left(1 - p_t^{j|V} + p_t^{j|V} M(a_j, a_j + b_j, iN_j)\right)\right]$$

  - *where M is a Kummer function,* a_j, b_j *some parameters*
- *Distribution of L(t) is obtained by Fast Fourier Transform*

Credit loss distributions

- *Beta distribution for recovery rates*



Valuation of CDO's

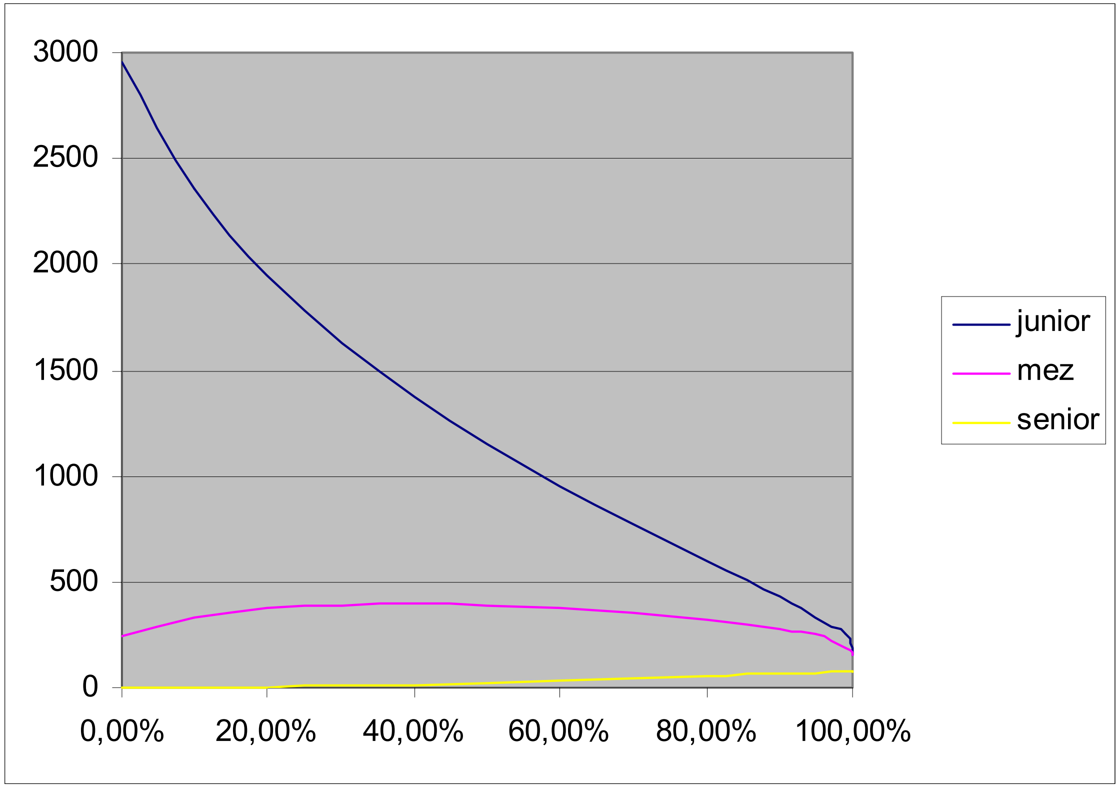
- Tranches with thresholds $0 \le A \le B \le \sum N_j$
- Mezzanine: pays whenever losses are between A and B
- Cumulated payments at time t: M(t)

$$M(t) = (L(t) - A))\,1_{[A,B]}(L(t)) + (B - A)1_{]B,\infty[}(L(t))$$

- Upfront premium: $E\left[\int_0^T B(t)dM(t)\right]$
  - B(t) discount factor, T maturity of CDO
- Stieltjes integration by parts $B(T)E[M(T)] + \int_0^T E[M(t)]dB(t)$
- where $E[M(t)] = (B - A)Q(L(t) > B) + \int_A^B (x - A)dF_{L(t)}(x)$

Valuation of CDO's

- *One factor Gaussian copula*
- *n=50 names, all at 100 bps, recovery = 40%*
- *maturity = 5 years, x axis: correlation parameter*
- *0-4%, junior, 4-15% mezzanine, 15-100% senior*



Conclusion

- Factor models of default times:
  - *Very simple computation of basket credit derivatives and CDO's*
  - *One can deal easily with a large range of names and dependence structures*