ESTIMATION OF A DYNAMIC HEDGE.

C. GOURIÉROUX[1]   J-P. LAURENT[2]

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Abstract

We focus on estimation of parameters used in dynamic hedging strategies and compare objective based inference and the maximum likelihood approach. When the financial model is misspecified the maximum likelihood estimation methodology may be misleading. Objective based estimators belong to the class of M-estimators, and are actually GMM estimators based on the tracking errors ; their asymptotic properties can be stated and compared with PML estimators ones through Monte Carlo simulations. If the objective based estimator does correct some bias, its variance is larger.

Résumé

Nous examinons l'estimation de paramètres utilisés dans des stratégies dynamiques de couverture et comparons les approches du maximum de vraisemblance et de l'inférence finalisée. Quand le modèle financier est mal spécifié, l'estimation par maximum de vraisemblance peut poser problème. Les estimateurs d'inférence finalisée présentés sont construits à partir des résidus de couverture. Leurs propriétés asymptotiques peuvent être établies et comparées avec celles des estimateurs du maximum de vraisemblance par des simulations. Si l'estimateur d'inférence finalisée corrige de certains biais, sa variance peut être plus grande.

Keywords : Objective based inference, implied hedging parameter, constrained dynamic hedge.

Mots clés : Inférence finalisée, paramètre de couverture implicite, couverture sous contrainte.

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[1]CREST and CEPREMAP.

[2]CREST.

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1 Introduction.

"To solve a financial problem it is preferable to use an estimation method based on a financial criterion instead of an estimation method based on a pure ad hoc statistical criterion such as a maximum likelihood technique".

Such an idea is common among financial practitioners, and seems a priori contradictory with classical statistical theory, which establishes that the maximum likelihood (ML) estimator has some asymptotic optimal properties. However, we intend to show that, when dealing with hedging or pricing problems, such an approach might be appropriate. The paper is organized as follows.

In section 2, we discuss the misspecification problems encountered when looking at hedging and pricing. In section 3, we present an objective based inference methodology (OBI) in order to deal with misspecifications of hedging strategies and pricing formulas. Section 4 introduces the concept of optimal hedging parameter in the context of dynamic hedging. Section 5 discusses the inference problems met in the previous framework. Finally, section 6 provides a numerical example, based on simulations and on a GARCH process for the underlying price.

2 Misspecification and statistical inference.

2.1 A gap between financial theory and econometrics of finance.

The pricing and hedging theories rely on some a priori knowledge of the stochastic evolution of asset prices. A theoretical specification of the conditional distribution of the asset prices, and the introduction of some additional assumptions concerning trading frequency, market completeness allow for the derivation of a risk neutral probability, and as a consequence of pricing formulas and hedging strategies. The a priori stochastic evolution is often parametrized, and the pricing formulas and hedging strategies also depend on this parameter.

This parameter has to be estimated from available data to allow for a practical implementation of the financial strategies. The estimate is usually derived by using a maximum likelihood method or a generalized method of moments, based on a modelling related to the one used in the theoretical step.

There is a mounting concern about the consistency and practical efficiency of the previous two steps procedure (see Melino (1994), Ghysels, Harvey and Renault (1995), Jacquier and Jarrow (1995), Renault (1996) for discussions). Indeed, the dynamic models introduced for theoretical purposes, such as the geometric Brownian motion (which underlies the Black-Scholes formula), the Ornstein-Uhlenbeck process, the square root process, usually do not provide a good fit to both underlying asset and derivative securities prices.

To improve this fit, the econometricians may modifiy the initial theoretical modelling in several directions.

On the one hand, they may enlarge the set of potential dynamics for the underlying asset prices, and consider some descriptive ARCH models to capture the main dynamic features of the conditional mean and volatility. But it is difficult to link the initial parameters and the auxiliary parameters used in this descriptive modelling.

On the other hand, to use the information contained in the derivative securities prices, they have to specify the joint distribution of the two kinds of prices. Moreover, to avoid degenerate maximum likekihood estimation, i.e. estimators with a variance equal to zero, they have to introduce in their models at least as many functionally independent error terms as assets of interest. This characteristic is inconsistent with the assumption of complete markets, which implies deterministic relationships between derivative securities prices and the underlying asset prices, the so-called pricing formulas[1].

This gap between theoretical and empirical specifications may presumably induce some bias in the estimated parameters, and more importantly, some bias in pricing formulas and hedging strategies.

2.2 How can we deal with misspecification ?

Faced to this misspecification problem, we might have three attitudes :

Firstly, we may simply ignore it. Implicitely, we consider that the various misspecifications are "small enough" and thus derivative securities prices and hedging strategies provided by the theory are "reliable enough". The deviations from the current pricing and hedging theory may then be seen as due to measurement errors or temporary departures from equilibrium. This may be considered as "largely" true, due precisely to the achievements of financial mathematics and econometrics. However, in those times where a lot of people wonder about the reliability of internal models for managing the financial risks of derivative securities, it seems reasonable to catch some distance between models and facts.

A second reaction which is certainly the most sensible in the long run, consists in improving the various specifications to deal with theoretical models which better fit the data.

[1] Even with factor models and stochastic volatility models, the dimension of randomness (i.e. the number of factors, or the dimension of the Brownian motion) is assumed to be constant, independent of the number of derivative securities based on the same underlying asset.

For instance, in a stochastic volatility model, it implies a deterministic relationship between the price of the basic asset and the prices of two european calls of different strikes and/or maturities.

A third reaction, that motivates the present paper, is to estimate the parameters in a way that "reduces" the consequences of misspecification for pricing or hedging, and to compare those estimates with standard ones (let us say maximum likelihood ones). We would also like to compare the pricing prediction error and the hedging efficiency corresponding to the two kinds of estimates.

3 Objective Based Inference.

3.1 An example.

To make the previous discussion more specific, let us consider an illustrative example. Let us assume that the underlying asset price S_t follows a stochastic volatility model. Moreover, we assume that hedging occurs at given discrete times t_i.

These assumptions differ from the standard assumptions of the Black-Scholes model, both regarding the dynamics of the underlying asset price and the trading specification. There does not exist a unique no arbitrage price for a call option on the underlying asset (incompleteness) and the numerical computation of an optimal hedging strategy cannot be considered as an easy task.

It is tempting to keep using Black-Scholes pricing formulas and hedging strategies even if the underlying model is misspecified. They depend on the unknown volatility parameter (let us say σ). We may try to estimate σ by a pseudo maximum likelihood method :

$$\hat{\sigma}_1^2 = \frac{1}{T}\sum_{t=1}^{T}\left(\Delta \log S_t - \frac{1}{T}\sum_{t=1}^{T}\Delta \log S_t\right)^2 .$$

However, because of the inconsistencies between the assumptions and the pricing formulas and hedging strategies, we might wonder whether $\hat{\sigma}_1^2$ is indeed the best input parameter σ.

Firstly, the PML method is a partial method, which does not take into account the observations of derivative asset prices. Moreover, $\hat{\sigma}_1^2$ converges to the marginal variance, while we are mainly interested in conditional effects. Thus, we might wonder whether we should not correct the PML estimate for stochastic volatility, discrete time trading and reduce hedging errors.

A similar idea applies when using implied volatilities as inputs of a Black-Scholes delta hedging strategy, while the assumptions of Black-Scholes model are violated. Should not we correct these implied volatilities in order to improve hedging efficiency ? At this stage, we have implicitely introduced an economic criterion which is the hedging efficiency of the different estimates.

3.2 Objective Based Estimators of the Volatility.

As an illustration, we will use the Black-Scholes modelling. We introduce some objective based estimators, using a given pricing formula (let's say Black-Scholes as an example) or hedging formula, and some economic criterion for measuring efficiency.

3.2.1 Pricing estimators.

A pricing estimator is defined by minimizing of the deviation between some observed option prices, $P(t_i, T_i, K_i)$ and their theoretical values proposed by the Black-Scholes model, $g^{BS}(S_{t_i}, T_i, K_i, r_{t_i}, \sigma)$ (say). The program to be solved is :

$$\hat{\sigma}_2^2 = \arg\min_{\sigma^2} \psi_2(\sigma), \tag{3.1}$$

$$\psi_2(\sigma) = \sum_{i=1}^{n} \left[P(t_i, T_i, K_i) - g^{BS}(S_{t_i}, T_i, K_i, r_{t_i}, \sigma) \right]^2 .$$

A slightly different pricing estimator should have been defined by minimizing the mean square error of the difference between the observed values of a traded options portfolio and the theoretical ones over some time interval. Let us denote by $V(t_i)$ and $V^{BS}(t_i, \sigma)$, the observed and theoretical (Black-Scholes) values of this portfolio :

$$\hat{\sigma}_2^2 = \arg\min_{\sigma^2} \sum_{i=1}^{n} \left[V(t_i) - V^{BS}(t_i, \sigma) \right]^2 . \tag{3.2}$$

It corresponds to the idea of the Bank for International Settlements' proposal for validation of internal models by historical simulation. The $\hat{\sigma}_2^2$ estimator ensures that observed and theoretical portfolio values are not too far away, and may be interesting for risk management.

3.2.2 Hedging estimators.

We may alternatively select a value for the volatility parameter in order to be close to a perfect hedge. Let us consider a given hedging horizon N, and the problem of hedging the cash-flow $[S_{t+N} - K]^+$ of an european call with strike K. We may use a hedging strategy with an initial investment V_0, some updating frequency between t and $t + N$, and hedging ratios corresponding to the Black-Scholes model for the updating date $t + n$. Let us denote by $V_{t+N}(V_0, \sigma)$ the value of this hedging portfolio at $t+N$. A hedging estimator $\hat{\sigma}_3^2$ minimizes the hedging errors [also called the *tracking errors*] :

$$\hat{\sigma}_3^2 = \arg\min_{\sigma^2} \psi_3(\sigma), \tag{3.3}$$

$$\psi_3(\sigma) = \sum_{t=1}^{T} \left[(S_{t+N} - K)^+ - V_{t+N}(V_0, \sigma) \right]^2 .$$

Therefore we have introduced three different estimation methods, the first one based on a pure statistical criterion (pseudo-likelihood), and the two other ones on some financial (pricing or hedging) criteria.

3.3 Estimators of parameters in well and misspecified models.

To state some statistical properties of these estimators, we need some knowledge about the joint pdf of asset and derivative prices.

If the Black-Scholes model is well-specified, the three previous methods provide consistent estimators of the volatility parameter, with some asymptotic efficiency property for the maximum likelihood approach.

If the Black-Scholes model is misspecified, it is no more possible to give a meaning to the true volatility parameter σ_0^2, since the volatility is usually time and path dependent, but it remains possible to implement the previous estimation methods. They will provide estimators converging to different limit values. We now discuss the interpretations of these limit values in our Black-Scholes example.

The estimator $\hat{\sigma}_1^2$ is computed as if the log-normal model was correct, and this estimator will tend to the marginal variance of the return [the historical volatility in the financial terminology] and not to the conditional variance [the volatility], which is generally random.

The pricing estimator will tend to a limit value $\sigma_{2\infty}^2$ for which $g^{BS}(S_t, T, K, r_t, \sigma_{2\infty})$ is the best pricing formula among the constrained set of pricing formulas $[g^{BS}(S_t, T, K, r_t, \sigma), \sigma$ varying].

Similarly the hedging estimator $\hat{\sigma}_3^2$ will tend to a limit $\sigma_{3\infty}^2$ such that the corresponding hedging strategies is optimal in the class of Black-Scholes hedging strategies, for the criterion "expected squared tracking error".

Under misspecification, the estimators $\hat{\sigma}_j^2(\psi_j)$ associated with the different criteria ψ_j tend to values $\sigma_\infty^2(\psi_j)$ depending on the criterion and generally with different statistical or financial interpretations. These limit values are called **pseudo-true values** (in the statistical terminology) or **implied parameters** (in the financial terminology).

The implied values depend on the problem of interest through the choice of the criterion. Therefore we have implied pricing volatilities [which may depend on the kind of asset to price] with criterion ψ_2, implied hedging volatilities (which may depend on the asset to hedge, on the hedging horizon, on the initial investment) with a criteria of ψ_3 type.

The discussion is summarized in diagrams 1 and 2.

diagram 1 : Well-specified model

diagram 2 : Misspecified model

Moreover when looking for the best hedging strategy or pricing formula in a given constrained class, it may be interesting to weaken the constraints by enlarging this class. For instance for a pricing problem we may replace the optimisation problem (3.1) by :

$$\left(\hat{\hat{\sigma}}_2^2,\ \hat{\alpha}_2\right) = \arg\min_{\alpha,\sigma^2} \sum_{i=1}^{n} \left[P[t_i, T_i, K_i] - g^{BS}[S_{t_i}, T_i, K_i, r_{t_i} + \alpha, \sigma)\right]^2,$$

allowing for a joint estimation of an implied pricing volatility and an implied pricing spread in the short term interest rate. By construction the pricing formula associated with $\left(\hat{\hat{\sigma}}_2^2,\ \hat{\alpha}_2\right)$ is preferable to the more constrained pricing formula associated with equation 3.1 and $(\hat{\sigma}_2^2,\ 0)$.

3.4 A modelling principle.

We may now describe a modelling principle (Objective Based Inference) which may be followed in the case of misspecified models. In this modelling approach the main roles will be for the criterion and the class of constrained strategies. The main steps of this approach are given below.

i) We first precise what is the problem of interest (for instance the pricing of some european calls), and consequently the criterion.

ii) Then we introduce a class of hedging strategies or pricing formulas (for instance the Black-Scholes pricing formulas $g^{BS}(t, T, K, r, \sigma)$) and precise a parametrisation (for instance $r = r_t$ observed short term interest rate, σ free parameter).

iii) Then we may compute the objective based estimate, in the example :

$$\hat{\sigma}^2 = \arg\min_{\sigma^2} \sum_{i=1}^{n} \left[P(t_i, T_i, K_i) - g^{BS}(t_i, T_i, K_i, r_{t_i}, \sigma)\right]^2,$$

and derive the best hedging strategy or pricing formula in the previous class.

At this level, the previous class of pricing formulas or hedging strategies may be too constrained, and the best element of this class far to give an accurate result.

iv) In a second step, we will enlarge the class of hedging strategies or pricing formulas by directly modifying the form of the strategies, either introducing some additional parameters (for example the implied pricing spread for short term interest rate), or allowing some previously introduced parameter to depend on some lagged variables (for instance we may introduce a stochastic implied volatility $\sigma = \sigma(\Delta \log S_{t-1})$ depending on the lagged return). Several problems occur at this level :

- Is it useful to enlarge the class of hedging strategies (or pricing formulas) ?
- If, the answer is positive, what is the best direction for enlarging it ?

4 Dynamic hedging.

Objective Based Inference (OBI) is applied in the following sections to the dynamic mean-variance hedge of a given cash-flow. We first recall the main features of this problem, then discuss objective based inference applied to it.

4.1 Optimal dynamic mean-variance hedging.

We consider the hedging at a given date t of a stochastic cash-flow H_{t+N} delivered at date $t+N$. N is the hedging horizon. This stochastic cash-flow H_{t+N} is approximated by the value of a portfolio containing some pieces of a risk-free asset and of p risky assets, and regularly updated at discrete dates $t, t+1, \ldots, t+N-1$. We denote by S_{t+n} the vector of prices of the p risky assets at date $t+n$, $S_{0,t+n} = \prod_{u=0}^{n-1}(1+r_{t+u})$ the price of the risk-free asset, where r_{t+u} is the short term interest rate at $t+u$, ΔS_{t+n}, the quantity $S_{t+n} - (1+r_{t+n-1})S_{t+n-1}$.

The number of pieces in the hedging portfolio at date $t+n$ are δ_{t+n} for the risky assets and $\delta_{0,t+n}$ for the risk-free asset. We assume that this hedging portfolio is self-financed with an initial investment denoted by $V_{0,t}$. Then the value of this porfolio at $t+n$ satisfies the recursive equation :

$$\left\{ \begin{array}{lcl} V_{t+n}(\delta) & = & (1+r_{t+n-1})V_{t+n-1}(\delta) + \Delta S'_{t+n}\delta_{t+n-1}, \quad n = 1, \ldots, N, \\ V_t(\delta) & = & V_{0,t}. \end{array} \right. \tag{4.1}$$

Definition 4.1 *: An optimal hedging strategy for the stochastic cash-flow H_{t+N} and the hedging horizon N, is a sequence of allocations $\left(\delta^*_{0,t+n,}, \delta^*_{t+n}\right)$, $n = 0, \ldots, N-1$, such that :*

i) $V_{0,t} = \delta_{0,t}S_{0,t} + S'_t\delta_t$ (Initial budget constraint) ;

ii) the self-financing condition 4.1 is satisfied ;

iii) $(\delta_{0,t+n}, \delta_{t+n})$ are measurable with respect to the information available at time $t+n$;

iv) the expected squared hedging error is minimized :

$$\left(\delta^*_{0,t+n}, \delta^*_{t+n}\right) = \arg \min_{\delta_{0,t+n};\delta_{t+n}} E_t \left(H_{t+N} - V_{t+N}(\delta)\right)^2,$$

where the admissible allocations, on which the optimization is performed, may be submitted to some a priori constraints.

4.2 Constrained and unconstrained optimal dynamic hedging.

An unconstrained hedging problem arises when δ_{t+n} is not a priori specified. Unconstrained dynamic mean-variance hedging has been extensively adressed by the litterature [See Follmer-Schweizer (1991), Duffie-Richardson (1991), Schweizer (1994, 1995a, 1995b), Gouriéroux-Laurent (1995), Gouriéroux-Laurent-Pham (1996)]. Except for very simple cases, it is not possible to derive analytic expressions for the optimal hedging strategies in an incomplete market framework. Moreover, such an optimal unconstrained hedging strategy is not robust to misspecifications in the pdf of asset prices.

On the contrary, in a constrained problem, $\delta_{t+n}(\theta)$ is specified to belong to a class of functions depending on a finite dimension parameter, $\theta \in \Theta$. For example, it may be the Black-Scholes delta depending on the parameter σ. The constrained dynamic hedging problem becomes a parametric problem :

$$\min_{\theta} E_t \left[H_{t+N} - V_{t+N}(\theta)\right]^2 \tag{4.2}$$

submitted to the self financing constraint :

$$V_{t+n}(\theta) = (1 + r_{t+n-1})V_{t+n-1}(\theta) + \Delta S'_{t+n}\delta_{t+n-1}(\theta), \tag{4.3}$$

and a given initial investment, $V_t = V_{0,t}$.

4.3 Optimal value of the unknown parameter.

The optimal value of the parameter, solution of the problem (4.2),(4.3), depends on the initial investment $V_{0,t}$ and on the information available at time t. Generally it is a random variable $\theta^*(t, V_{0,t})$ (because of the dependence in this information). How to circumvent this difficulty and to establish a link with the constant implied parameters introduced in section 3 ? It is useful to introduce some dynamics in the hedging problem itself and to define sequences of hedging problems.

Definition 4.2 : *A sequence of hedging problems, indexed by t, with fixed horizon N is defined by :*

- an increasing sequence of informations I_t,

- an adapted square integrable price process for the assets, $S_{0,t}, S_t$,

- a payoff process H_{t+N}, assumed to be square integrable and I_{t+N}-measurable,

- an initial investment process $V_{0,t}$, square integrable and I_t-measurable,

- a hedging strategy process $\delta^t(\theta) = (\delta^t_0(\theta), \delta^t_1(\theta), ..., \delta^t_n(\theta), ..., \delta^t_{N-1}(\theta))$, the $\delta^t_n(\theta)$ being I_{t+n}-measurable, and

- a sequence of minimisation problems : $\min_{\theta} E_t \left[H_{t+N} - V^t_N(\theta)\right]^2$, where $V^t_N(\theta)$ is defined by :

$$V^t_N(\theta) = V_{0,t}S_{0,t+N}/S_{0,t} + \sum_{n=1}^{N} S_{0,t+N}/S_{0,t+n}\Delta S'_{t+n}\delta^t_{n-1}(\theta).$$

A sequence of hedging problems indexed by t, leads to a net terminal wealth process for the investor $H_{t+N} - V_N^t(\theta)$, to a (squared) tracking error process $\psi_t(\theta, V_{0,t})$,

$$\psi_t(\theta, V_{0,t}) = \left(H_{t+N} - V_N^t(\theta)\right)^2, \tag{4.4}$$

and to a hedging parameter process,

$$\theta^*(t, V_{0,t}) = \arg\min_\theta E_t\psi_t(\theta, V_{0,t}).$$

Clearly, the hedging parameter may be path dependent and is an adapted process. It will be important in practice to consider a sequence of hedging problems with some "stationarity" properties which may concern either the tracking error, or the hedging parameter process.

Definition 4.3 *: A sequence of hedging problems is stationary, if the tracking error process $\psi_t(\theta, V_{0,t})$ is stationary.*

Definition 4.4 *: A sequence of hedging problems is with constant hedging parameter if*

$$\theta^*(t, V_{0,t}) = \theta^*(V_{0,t}),$$

depends only on t, I_t, through the initial investment.[2].

If the sequence of hedging problems is with constant hedging parameter, the optimal parameter θ^* is simultaneously solution of all the conditional minimisation problems (4.3), and also of the marginal problems :

$$\min_\theta E\left[w_t\psi_t(\theta, V_{0,t})\right],$$

where w_t are any positive adapted weights.

Let us now consider a specific hedging problem at a given date, $\left(\bar{H}_{u+N}, \bar{V}_{0,u}, \bar{\delta}^u(\theta)\right)$(say). Such a problem may be nested in several sequences of hedging problems $(H_{t+N}, V_{0,t}, \delta^t(\theta))$ as soon as we have : $H_{u+N} = \bar{H}_{u+N}, V_{0,u} = \bar{V}_{0,u}, \delta^u(\theta) = \bar{\delta}^u(\theta)$, a.s. For estimation purpose, it will be preferable to retain such a sequence with stationary tracking errors and constant hedging parameter (if such a sequence exists).

4.3.1 Best initial investment.

In Duffie-Richardson (1991), Gouriéroux-Laurent (1995), it is shown that there exists a best initial investment $V_0^*(t)$ corresponding to the following problem :

$$V_0^*(t) = \arg\min_{V_0} E_t\psi_t(\theta^*(t, V_0), V_0).$$

As in definition 4.4, it is possible to define a sequence of hedging problems with constant best initial investment.

[2]A sufficient condition is that the process $E_t\psi_t(\theta, V_0)$ is constant, for any V_0.

4.4 Linear sets of hedging strategies.

Finally we have to discuss the choice of the constrained forms of the hedging coefficients in practice. Two different lines may be followed.

The first one consists in retaining a classical parametric form, even if misspecified. It is the line described in section 3, using the Black-Scholes deltas, and which will be further followed in the Monte-Carlo studies.

A second one consists in introducing a more descriptive class of strategies. Let us consider the example given in GOURIÉROUX-LAURENT (1995) of an hedging based on a single risky asset. We might define at date $t+n$ two regimes depending on the lagged return :

$$\begin{cases} \text{regime 1 :} & \Delta \log S_{t+n-1} > 0; \\ \text{regime 2 :} & \Delta \log S_{t+n-1} < 0; \end{cases}$$

then look for a strategy $\theta_{j,t+n}$ at $t+n$, depending on the regime. We have :

$$\delta_{t+n-1}(\theta) = \sum_{j=1}^{2} \theta_{j,t+n} \varepsilon_{j,t+n}, \tag{4.5}$$

where $\varepsilon_{j,t+n}$ is the regime j indicator, equal to one of this regime has been realized, to zero otherwise. The dependence of δ_{t+n-1} on θ is linear.

Definition 4.5 *: A linear set of hedging strategies is $\delta_t(\theta) = Z_t\theta$, where Z_t is an adapted process.*

5 Inference.

5.1 Estimation of a constant hedging parameter.

We are now defining more precisely the hedging estimators.

Definition 5.1 *: A hedging estimator of the constant hedging parameter θ, associated with a given stationary sequence of hedging problems and some stationary weight process w_t is given by :*

$$\hat{\theta}_T(V_0) = \arg\min_{\theta} \frac{1}{T} \sum_{t=1}^{T} w_t \psi_t(V_0;\theta), \tag{5.1}$$

where $\psi_t(V_0;\theta)$ is the tracking error defined in equation 4.4.

$\hat{\theta}_T(V_0)$ may be seen as a M-estimator [See HUBER (1981), GALLANT (1987), GOURIÉROUX-MONFORT (1995), chapter 8], and the usual properties of such estimators apply under some standard regularity conditions (including the stationarity of the weighted tracking errors). Let us denote by $\Psi_t(V_0;\theta) = w_t\psi_t(V_0;\theta)$, the weighted tracking error.

Proposition 5.1 *:*

i) $\hat{\theta}_T(V_0)$ *is a consistent estimator of the constant implied parameter* $\theta^*(V_0)$*.*

ii) It is asymptotically normal :

$$\sqrt{T}\left[\hat{\theta}_T(V_0) - \theta^*(V_0)\right] \xrightarrow{d} N[0, \Omega(V_0)],$$

where :

$$\begin{aligned}
\Omega(V_0) &= J(V_0)^{-1} I(V_0) J(V_0)^{-1}, \\
J(V_0) &= E\left\{ -\frac{\partial^2 \Psi_t}{\partial\theta\partial\theta'}[V_0; \theta^*(V_0)] \right\}, \\
I(V_0) &= V\left\{ \frac{\partial \Psi_t}{\partial\theta}[V_0; \theta^*(V_0)] \right\} + 2\sum_{h=1}^{\infty} \; Cov \left\{ \frac{\partial \Psi_t}{\partial\theta}[V_0; \theta^*(V_0)], \frac{\partial \Psi_{t+h}}{\partial\theta}[V_0; \theta^*(V_0)] \right\}.
\end{aligned}$$

Similarly, we may introduce hedging estimators of the best initial investment.

Definition 5.2 *: A hedging estimator of the constant best initial investment* V_0^**, associated with a given stationary sequence of hedging problems and some stationary weight process* w_t *is given by :*

$$\hat{V}_0^* = \arg\min_{V_0} \frac{1}{T}\sum_{t=1}^{T} w_t \psi_t(\hat{\theta}_T(V_0), V_0).$$

It is also straightforward to derive the asymptotic properties of the best initial investment. When the hedging parameter, $(\theta^*(t, V_0^*), V_0^*(t))$, is time and path independent, the joint estimator $\left(\hat{\theta}_T(\hat{V}_0^*), \hat{V}_0^*\right)$, is consistent, asymptotically normal, with an asymptotic covariance matrix given by $\Omega^* = J^{*-1} I^* J^{*-1}$, where I^*, J^* are the analogues of $I(V_0)$,$J(V_0)$ in property (5.1), but deduced from the first and second order derivatives with respect to the couple (θ, V_0). In particular $\hat{\theta}_T(\hat{V}_0^*)$ and $\hat{V}_0^*$ are in general asymptotically correlated.

Eventually, we can state the following convergence property, under standard regularity conditions for the convergence of M-estimators, when the sequence of hedging problems is stationary but not with constant hedging parameter :

Proposition 5.2 *: The hedging estimator* $\hat{\theta}_T(V_0)$*, associated with a given stationary sequence of hedging problems and some stationary weight process* w_t *(but not necessarily with constant hedging parameter), will converge to* $\theta_0^*(V_0)$ *the solution of the marginal problem :*

$$\theta_0^*(V_0) = \arg\min_{\theta} E[\Psi_t(V_0, \theta)].$$

5.2 Linear case.

Some important simplifications arise for *linear sets of hedging strategies* introduced in Definition 4.5. In such a case the optimisation problem becomes :

$$\min_{\theta} \frac{1}{T} \sum_{t=1}^{T} w_t \left[H_{t+N} - V_0 S_{0,t+N} - \sum_{n=1}^{N} \frac{S_{0,t+N}}{S_{0,t+n}} \Delta S'_{t+n} Z_{t+n-1} \theta \right]^2$$

This is a regression problem and the least squares estimator of θ is given by :

$$\hat{\theta}_T(V_0) = \left[\sum_{t=1}^{T} w_t X'_{t+N} X_{t+N} \right]^{-1} \sum_{t=1}^{T} w_t X'_{t+N} \left(H_{t+N} - V_0 S_{0,t+N} \right), \tag{5.2}$$

where $X_{t+N} = \sum_{n=1}^{N} \frac{S_{0,t+N}}{S_{0,t+n}} \Delta S'_{t+n} Z_{t+n-1}$.

We can notice that : $\hat{\theta}_T(V_0) = \hat{\theta}_{1T} - V_0 \hat{\theta}_{2T}$, where :

$$\left\{ \begin{array}{lcl} \hat{\theta}_{1T} & = & \left[\sum_{t=1}^{T} w_t X'_{t+N} X_{t+N} \right]^{-1} \sum_{t=1}^{T} w_t X'_{t+N} H_{t+N}, \\ \hat{\theta}_{2T} & = & - \left[\sum_{t=1}^{T} w_t X'_{t+N} X_{t+N} \right]^{-1} \sum_{t=1}^{T} w_t X'_{t+N} S_{0,t+N}. \end{array} \right. \tag{5.3}$$

The estimators $\hat{\theta}_T(V_0)$, V_0 varying, are known as soon as we know the two estimators $\hat{\theta}_{1T}, \hat{\theta}_{2T}$. Similarly the asymptotic variance-covariance matrices $\Omega(V_0)$, V_0 varying, are easily estimated as soon as we know consistent estimates of the variance-covariance matrix of $\begin{bmatrix} \hat{\theta}_{1T} \\ \hat{\theta}_{2T} \end{bmatrix}$.

5.3 Hedging and PML estimators.

Let us consider a comprehensive financial model, including a parameterized description of the underlying asset prices. We denote by $\ell(S_t/\underline{S_{t-1}};\theta)$ the conditional pdf at time t. From this financial modelling we may deduce the corresponding hedging coefficient $\delta_{t+n}(\theta)$ for hedging a cash-flow based on S_{t+N}. Then we compute the maximum likelihood estimator of θ :

$$\hat{\hat{\theta}}_T = \arg\max_{\theta} \frac{1}{T} \sum_{t=1}^{T} \log \ell(S_t/\underline{S_{t-1}};\theta), \tag{5.4}$$

and in parallel the hedging estimators introduced in Definition 5.1.

5.3.1 Well specified model.

If the comprehensive financial model is well specified, all these estimators converge to the true value θ_0 of the parameter, and they are jointly asymptotically normal :

$$\sqrt{T}\begin{bmatrix} \hat{\hat{\theta}}_T - \theta_0 \\ \hat{\theta}_T(V_0) - \theta_0 \end{bmatrix} \xrightarrow{d} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} I_{11}(\theta_0) & I_{12}(\theta_0) \\ I_{21}(\theta_0) & \Omega(V_0;\theta_0) \end{bmatrix}\right]. \ (say).$$

Moreover since the maximum likelihood estimator is asymptotically efficient, we get :

$$\mathrm{Cov}_{as}\left[\hat{\hat{\theta}}_T, \hat{\theta}_T(V_0) - \hat{\hat{\theta}}_T\right] = 0. \tag{5.5}$$

Equivalently, let us consider the theoretical linear regression of the hedging estimator on the maximum likelihood estimator,

$$\hat{\theta}_T(V_0) = A\hat{\hat{\theta}}_T + b + v_T, \tag{5.6}$$

where $E_{as}v_T = 0$, $\mathrm{Cov}_{as}\ (v_T, \hat{\hat{\theta}}_T) = 0$; we get $A = Id$, because of equation (5.5), and $b = 0$ because of the convergence to the true value.

The affine transformation $\hat{\hat{\theta}}_T \longrightarrow A\hat{\hat{\theta}}_T + b$ may be considered as a crude correction of the maximum likelihood estimator to take into account for the difference between an hedging problem and a maximum likelihood problem. If the model is well specified such a correction is not necessary since $A = Id, b = 0$.

5.3.2 Misspecified model.

If the comprehensive financial model is misspecified, the parameter of interest is the implied hedging parameter associated with a given initial investment V_0. The bivariate vector $\begin{bmatrix} \hat{\hat{\theta}}_T \\ \hat{\theta}_T(V_0) \end{bmatrix}$ is still asymptotically normal :

$$\sqrt{T}\begin{bmatrix} \hat{\hat{\theta}}_T - \theta_{0\infty} \\ \hat{\theta}_T(V_0) - \theta^*(V_0) \end{bmatrix} \xrightarrow{d} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} I_{11}(\ell_0) & I_{12}(\ell_0) \\ I_{21}(\ell_0) & \Omega(V_0, \ell_0) \end{bmatrix}\right]. \tag{5.7}$$

But the PML estimator is not in general a consistent estimator of the implied hedging parameter $\theta^*(V_0)$, and no more satisfies the orthogonality condition (5.5). When we regress $\hat{\theta}_T(V_0)$ on $\hat{\hat{\theta}}_T$, we get :

$$\begin{cases} A &= \mathrm{Cov}_{as}\left[\hat{\theta}_T(V_0), \hat{\hat{\theta}}_T\right]\left[V_{as}(\hat{\hat{\theta}}_T)\right]^{-1} \neq Id \text{ (in general)}, \\ B &= \theta^*(V_0) - A\theta_{0\infty} \neq 0 \text{ (in general)}. \end{cases}$$

It may be interesting to study the bias of the PML estimator, i.e., of the difference between $\theta_{0\infty}$ and $\theta^*(V_0)$, for instance when the misspecification is not too large. Let us assume that the true conditional distribution is of the form $\ell(S_t/\underline{S_{t-1}};\theta_0;\alpha_0)$, with α_0 small, whereas the misspecified model used for PML estimation purpose is $\ell(S_t/\underline{S_{t-1}};\theta,0)$.

Proposition 5.3 *:* $\lim\limits_{\alpha_0\to 0}\dfrac{\theta^*\left(V_0\right)-\theta_{0\infty}}{\alpha_0}=$

$$\left[E\frac{\partial^2}{\partial\theta\partial\theta'}\log l\right]^{-1} Cov\left(\frac{\partial}{\partial\theta}\log l,\frac{\partial}{\partial\alpha}\log l\right)-\left[E\left(\frac{\partial^2}{\partial\theta\partial\theta'}\Psi_t\left(\theta_0\right)\right)\right]^{-1} Cov\left(\frac{\partial}{\partial\theta}\Psi_t\left(\theta_0\right),\frac{\partial\log}{\partial\alpha}l\right)$$

where l stands for $l(S_t\mid\underline{S_{t-1}};\theta_0,0)$ and the expectations and covariances are taken for $\theta=\theta_0$ and $\alpha_0=0$.

The proof is provided in appendix 1. The size of this local bias depends on the misspecification through α_0, but also of the curvature of the financial criterion used to define the parameter of interest through $\mathrm{E}\left[\dfrac{\partial^2\Psi_t}{\partial\theta\partial\theta'}(\theta_0)\right]^{-1}$.

5.4 Check for the constancy of the implied parameter.

The previous estimation procedure is only meaningful for a constant implied parameter. Therefore we have to develop statistical procedures to check this condition, and, if it is rejected, to detect the omitted effect. This problem is a test for an omitted variable Z_t (say) in the expression of $\theta^*(t,V_0)$. We may propose two kinds of procedures to test the null hypothesis : $H_0=\{\theta^*(t,V_0)=\theta^*(V_0)\}$ against the hypothesis $H=\{\theta^*(t,V_0)=\theta^*(Z_t;V_0)\}$, depending if we develop parametric or semi-parametric approaches. They are based on the same idea of modifying the weights in the criterion function.

5.4.1 A parametric approach

Let us consider an adapted positive stationary process $\lambda_t(Z_t)$ (say). Another hedging estimator corresponds to the weights $w_t=\lambda_t(Z_t)$:

$$\hat\theta_T(V_0;\lambda)=\arg\min_\theta\frac{1}{T}\sum_{t=1}^{T}\lambda_t(Z_t)\psi_t(\theta). \qquad (5.8)$$

Under the null hypothesis H_0 the two estimators $\hat\theta_T(V_0)$ and $\hat\theta_T(V_0;\lambda)$ converge to the same value $\theta^*(V_0)$, whereas they generally tend to different values when $\theta^*(t,V_0)$ actually depends on Z_t. Hence, we can introduce a misspecification test based on the test statistic :

$$\xi_T(\lambda)=\left[\hat\theta_T(V_0)-\hat\theta_T(V_0;\lambda)\right]'\left(\hat V\left[\hat\theta_T(V_0)-\hat\theta_T(V_0,\lambda)\right]\right)^{-}\left[\hat\theta_T(V_0)-\hat\theta_T(V_0;\lambda)\right], \qquad (5.9)$$

where $\hat{V}\left[\hat{\theta}_T(V_0) - \hat{\theta}_T(V_0;\lambda)\right]$ is an estimator of the variance-covariance matrix of the difference between the estimators, and "-" denotes a generalized inverse. Under the null hypothesis this statistics is asymptotically chi-square distributed with d degrees of freedom, where d is the rank of $V\left\{\hat{\theta}_T(V_0) - \hat{\theta}_T(V_0;\lambda)\right\}$ under the null. Therefore this test consists :

$$\text{in accepting } H_0, \text{ if } \xi_T(\lambda) < \mathcal{X}^2_{95\%}(d),$$

$$\text{in rejecting it, otherwise.}$$

5.4.2 A non parametric approach.

Another idea is to directly estimate the functional form $z \rightarrow \theta^*(z, V_0)$ as if H_0 were not satisfied and to compare this estimated function to the constant parameter estimator $\hat{\theta}(V_0)$.

A consistent functional estimator may be derived using a kernel M-estimator [Gozalo-Linton (1994), Gouriéroux-Monfort-Tenreiro (1994)]. We introduce a kernel K, compute for each value z :

$$\hat{\theta}_T(z;V_0) = \arg\min_\theta \sum_{t=1}^T \frac{1}{h_T} K\left(\frac{Z_t - z}{h_T}\right) \psi_t(\theta), \tag{5.10}$$

where the bandwidth h_T tends to zero at a suitable rate, and derive a functional residual plot giving the discrepancy $\hat{\theta}_T(z;V_0) - \hat{\theta}_T(V_0)$ as a function of z.

Except for a linear hedging problem, the criterion function ψ_t has a complicated expression and the solution $\hat{\theta}_T(z;V_0)$ has to be derived by a numerical algorithm. As it must be computed for a large number of z values (and several potentially omitted variables), it may be time consuming to get the previous kernel residual plot. It has been proposed in Gouriéroux-Monfort-Tenreiro (1994) to replace $\hat{\theta}_T(z;V_0)$ by an approximation computed in a neighbourhood of the null hypothesis. For such a purpose we replace the criterion function $\psi_t(\theta_0)$ by its second order expansion around $\hat{\theta}_T(V_0)$:

$$\tilde{\psi}_t(\theta) = \psi_t\left[\hat{\theta}_T(V_0)\right] + \frac{\partial \psi_t}{\partial \theta'}\left[\hat{\theta}_T(V_0)\right]\left[\theta - \hat{\theta}_T(V_0)\right] + \frac{1}{2}\left[\theta - \hat{\theta}_T(V_0)\right]' \frac{\partial^2 \psi_t}{\partial\theta\partial\theta'}\left[\hat{\theta}_t(V_0)\right]\left[\theta - \hat{\theta}_T(V_0)\right],$$

and introduce the functional estimator :

$$\tilde{\theta}_T(z;V_0) = \arg\min_\theta \sum_{t=1}^T \frac{1}{h_T} K\left(\frac{Z_t - z}{h_T}\right) \tilde{\psi}_t(\theta), \tag{5.11}$$

which has the explicit expression :

$$\tilde{\theta}_T(z;V_0) - \hat{\theta}_T(V_0) = -\left\{\sum_{t=1}^T \frac{1}{h_T} K\left(\frac{Z_t - z}{h_T}\right) \frac{\partial^2 \psi_t}{\partial\theta\partial\theta'}\left[\hat{\theta}_T(V_0)\right]\right\}^{-1} \sum_{t=1}^T \frac{1}{h_T} K\left(\frac{Z_t - z}{h_T}\right) \frac{\partial \psi_t}{\partial \theta}[\hat{\theta}_T(V_0)]. \tag{5.12}$$

Then the previous kernel residual plot based on $\hat{\theta}_T(z;V_0) - \hat{\theta}_T(V_0)$ may be replaced by the one based on $\tilde{\theta}_T(z;V_0) - \hat{\theta}_T(V_0)$.

$$\tilde{\theta}_T(z;V_0) - \hat{\theta}_T(V_0) = -\left\{\sum_{t=1}^{T} \frac{1}{h_T} K\left(\frac{Z_t - z}{h_T}\right) \frac{\partial^2 \psi_t}{\partial\theta\partial\theta'}, [\hat{\theta}_T(V_0)]\right\}$$

$$\sum_{t=1}^{T} \frac{1}{h_T} K\left(\frac{Z_t - z}{h_T}\right) \frac{\partial^2 \psi_t}{\partial\theta\partial\theta'} (\hat{\theta}_T(V_0)) \left(\frac{\partial^2 \psi_t}{\partial\theta\partial\theta'} (\hat{\theta}_T(V_0))\right)^{-1} \frac{\partial \psi_t}{\partial\theta} [\hat{\theta}_T(V_0)].$$

may be seen as a regressogram of some normalized residuals [Chesher-Irish (1987)] on the Z variable. We will not discuss more deeply this nonparametric approach. The confidence bounds associated with these functional estimators may be found in Gouriéroux-Monfort-Tenreiro (1994)].

6 A Monte Carlo study.

In this last section, we apply objective based inference to the hedging of an option on the basis of simulated data. We assume that the risky asset prices presents some heteroscedasticity, while the hedging strategy remains a standard Black-Scholes strategy applied in discrete time. Our purpose is to compare the two statistical PML and OBI procedures in a reasonable framework. In order to get meaningful results we focus on a financial model where the misspecification is not severe. We first present the sequence of hedging problems, compare the hedging and PML estimators of the volatility parameter, analyse the hedging efficiency of the two estimators and provide some insight for improvement of the Black-Scholes delta hedging strategy.

6.1 The sequence of trading problems.

Regarding the true dynamics of prices and interest rates, we consider some usual specifications including the Black-Scholes evolution model as a particular case. The interest rate, r_t, is assumed to follow an AR(1) model :

$$r_t - r_{t-1} = \lambda_r(\gamma_r - r_{t-1}) + \sigma_r \varepsilon_{1t},\ t = 1, \ldots, T,\ \text{where } \varepsilon_{1t} \sim IIN(0,1). \tag{6.1}$$

In the simulations the initial value, r_0, of r_t, and the "true" parameters have been set to the following values :

r_0	λ_r	γ_r	σ_r
10 %	0.15/365	10%	$2\%/\sqrt{365}$

The price $S_{0,t}$ of the riskless asset is then recursively computed as $S_{0,t} = S_{0,t-1}(1+r_{t-1}/365)$, $S_{0,0} = 1$. The underlying asset return is assumed to follow a GARCH(1,1) model :

$$\log S_t - \log S_{t-1} = \mu_S + \sigma_{t-1}\varepsilon_{2t}, \qquad \varepsilon_{2t} \sim IIN(0,1), \tag{6.2}$$

$$\sigma_t^2 = \omega_S + (\alpha_S \varepsilon_{2,t}^2 + \beta_S)\sigma_{t-1}^2,\ t = 1, \ldots, T, \tag{6.3}$$

whose initial value and true parameters are given in the following table :

S_0	σ_0	μ_S	α_S	β_S	ω_S
100	$15\%/\sqrt{365}$	15%/ 365	0.3	0.3	$\sigma_0^2 \times (1 - \alpha_S - \beta_S)$

Since $\alpha_S + \beta_S = 0.6$, the volatility persistence is not very important and the Black-Scholes log-normality assumption is not a severe misspecification in our simulation framework. The cash-flow (payoff process) to be hedged corresponds to a call option :

$$H_{t+N} = \left[S_{t+N}\frac{S_0}{S_t} - K\right]^+. \tag{6.4}$$

The factor $\frac{S_0}{S_t}$ is introduced to stationarize the payoff process. The hedging horizon is taken equal to $N = 25$, which corresponds approximately to one month for daily returns taking into account non trading days.

The initial investment process $V_{0,t}$ is assumed to be constant and equal to V_0.

Finally we first retain a class of parametric hedging strategies deduced from the misspecified Black-Scholes model, and normalized in accordance with equation 6.4 :

$$\delta_n^t(\sigma) = \frac{S_0}{S_t}\Phi\left\{\frac{1}{\sigma\sqrt{\frac{25-n}{365}}}\left[\log\left(\frac{S_{t+n}}{S_t}\frac{S_0}{K}\right) + \left(\frac{\sigma^2}{2} + r_{t+n}\right)\frac{25-n}{365}\right]\right\}. \tag{6.5}$$

We assume a daily rebalancing of the hedging portfolio. As for the risky asset price dynamics, we do not depart too much from the Black-Scholes assumption of continuous time trading. The number of observations (S_t, r_t) in a given sample is taken equal to $T = 175$.

As before, the hedging parameter is defined by :

$$\sigma^*(t, V_0, K) = \arg\min_{\sigma} E_t\psi_t(\sigma, V_0, K). \tag{6.6}$$

The previous sequence of hedging problems is not with constant parameter and is not stationary. In particular, $\sigma^*(t, V_0, K)$ will depend on the current asset return volatility σ_t. As the usual implied volatility (based on a pricing estimator, following our terminology), the *implied hedging volatility* $\sigma^*(t, V_0, K)$ may depend on the exercise price K.

6.2 Comparison of hedging estimators and of PML estimators.

For any simulated path $(r_t^s, S_t^s), t = 1, \ldots, T = 175,\ s = 1, \ldots, S$, we compute the PML estimator of σ, i.e., the historical volatility $\hat{\sigma}_{PML}^s$, and some hedging estimators $\hat{\sigma}_{OBI}^s(V_0, K)$ with weight $w_t = 1$:

$$\hat{\sigma}_{PML}^s = \frac{1}{T}\sum_{t=1}^{T}\left(\Delta \log S_t^s - \frac{1}{T}\sum_{t=1}^{T}\Delta \log S_t^s\right)^2, \tag{6.7}$$

$$\hat{\sigma}_{OBI}^s(V_0, K) = \arg\min_{\sigma} \sum_{t=1}^{T-N} \psi_t^s(\sigma, V_0, K), \tag{6.8}$$

where $\psi_t^s(\sigma, V_0, K)$ stands for the simulated tracking error.

Though ψ_t is not stationary, it has a stationary limit. Thus, $\hat{\sigma}_{OBI}(V_0, K)$ will converge to $\arg\min E_0[\psi_\infty(\sigma, V_0, K)]$, where $\psi_\infty(\sigma, V_0, K)$ stands for the stationary limit of the tracking error.

The objective based estimator has been derived through a grid search. We have computed the empirical mean, standard deviation, correlation of the hedging estimator and of the PML estimator, for several exercise prices K :

Table 1 : PML and OBI[3].

		m_{PML}	σ_{PML}	m_{OBI}	σ_{OBI}	ρ
$K = 96$	$V_0 = 1.43$	14.8%	1.6%	17.8%	14.8%	0.05
$K = 100$	$V_0 = 1.43$	14.8%	1.6%	16.2%	7.3%	0.35
$K = 104$	$V_0 = 1.43$	14.8%	1.6%	15.2%	6.6%	0.22

As expected, the mean of the PML estimate is close to the pseudo true value 15%, whereas the correction for misspecification leads to overestimate σ. The variance of the hedging estimator is also much larger. For at and in the money options, the two estimators exhibit positive correlation.

We report in table 2 the mean of the hedging estimator for different exercise prices (based on 300 samples for each strike) and for an initial investment $V_0 = 1.43$[4], to exhibit a smile effect. The hedging estimator is smaller in average for at the money options than for in or out of the money options.

[3] Monte Carlo study with $S = 300$ simulated paths.

[4] We have computed the estimators of the implied hedging volatility parameter, with a strike independent initial investment V_0. Since the objective based estimator depends on this initial investment, we might also have estimated the couple $(V_0^*(K),\ \sigma(K, V_0^*(K))$ by minimizing the tracking error. We would have then derived a different smile curve $\sigma(K, V_0^*(K))$.

Table 2 : The smile effect

K	94	96	98	99	100	102	104	106
m_{OBE}	16.6%	17.8%	16.2%	13.7%	16.2%	14.9%	15.2%	18.1%

This smile effect is due to the departures from Black-Scholes modelling, i.e. stochastic volatility and hedging in discrete time [see Bossaerts and Hillion (1995)].

It is also possible to give the crude corrections for the PML estimation, i.e the coefficients $a(K, V_0), b(K, V_0)$ of the regression of the hedging estimator on the PML estimator. The following table provides some results for $V_0 = 1.43$.

Table 3 : Correction coefficients for PML estimation.

K	96	100	104
$a(K, V_0)$	0.45	1.8	0.88
$b(K, V_0)$	11.0	-10.0	2.1

6.3 Comparison of the hedging accuracies.

The comparisons of the previous subsection are based on the two kinds of estimators. We now examine how the hedging accuracy $E_0[\psi_0(\sigma, V_0, K)]$ depends on σ, for $V_0 = 1.43$, $K = 100$.

Figure 1 : The criterion function

It may be seen on figure 1 that the hedging criterion is nearly flat around its minimum, i.e. the best hedging parameter $\sigma(0, V_0, K)$. We can also see that the values of the hedging criterion are almost the same for $\sigma = 14.8\%$ (mean of the PML estimator) and $\sigma = 16.2\%$ (mean of the hedging estimator). Moreover, due to the flatness of the hedging criterion, some increase in the variance of the estimators will have little effect on the hedging efficiency.

6.4 The extension of the class of strategies.

To improve the hedging estimator and the corresponding hedging accuracy, we may enlarge the class of strategies. We introduce an implied hedging volatility which may depend on the current squared excess return :

$$\sigma_{t+n}^2(\theta) = \overline{\sigma}_0^2 + \theta \left[\left(\log \frac{S_{t+n}}{S_{t+n-1}} - \mu_S \right)^2 - \overline{\sigma}_0^2 \right],$$

where θ is between 0 and 1, μ_S is defined as previously and $\overline{\sigma}_0$ is taken as the mean of the hedging estimator, 16.2%, for $K = 100$. We keep $K = 100$, $V_0 = 1.43$. The expression of the hedging strategy remains otherwise unchanged.
When $\theta = 0$, $\sigma_{t+n}^2(\theta) = \overline{\sigma}_0^2$ (which corresponds approximately to the minimization of the criterion function in figure 1) the implied hedging volatility does not depend on current squared excess returns. On the contrary, when $\theta = 1$ the hedging volatility only takes into account current squared excess returns and not the historical volatility $\overline{\sigma}_0^2$. The optimal value of θ is 0.1(see figure 2) and the hedging is slightly improved when current squared returns are introduced in the implied volatility[5].

[5]We have only proceeded to a marginal optimization on parameter θ leaving parameter $\overline{\sigma}_0^2$ unchanged. We might have also estimated jointly the two parameters.

Figure 2 : The criterion function and the implied volatility parameter θ

7 Concluding remarks.

It is not surprising that the application of the maximum likelihood method under misspecification may be misleading. The ML estimators are unconsistent and are not the best inputs of a hedging strategy derived from the misspecified financial model. It may be useful to introduce hedging estimators to correct for the bias.

However, some simple Monte Carlo experiments show that the previous remark must be mitigated. When the financial model providing the hedging strategy is not too misspecified, the ML estimator can be a serious competitor to objective based inference estimators. The bias (difference between the mean of the ML estimator and the best hedging parameter) appears to be small. In our example of hedging a call option in a GARCH framework, the hedging criterion is almost flat around its minimum and the bias induces only a small decrease of hedging efficiency. Moreover the variance of the ML estimator proves to be much smaller than that of the objective based estimator. Eventually, the hedging efficiency of the two estimators are nearly the same. Of course, the larger the sample of observed data, the better would be the relative performance of the objective based estimator.

These Monte Carlo experiments are only a first step towards a better understanding of statistical inference when applied to hedging strategies based on misspecified models. Such a better understanding is required in order to improve the efficiency and the reliability of risk management models and their ability to correctly handle non linear payoffs.

A Appendix 1 : determination of the local bias.

The limit points $\theta_{0\infty}$ and $\theta^*(V_0)$ are solutions of the two limit optimization problems :

$$\begin{aligned} \theta_{0\infty} &= \arg\max_{\theta} E\left[\log l\left(S_t \mid \underline{S_{t-1}}; \theta, 0\right)\right], \\ \theta^*(V_0) &= \arg\max E\left[\psi_t(\theta)\right], \end{aligned}$$

where the expectations are taken conditional on θ_0, α_0,. They satisfy the first order conditions :

$$\begin{aligned} E\left[\frac{\partial \log l}{\partial \theta}\left(S_t \mid \underline{S_{t-1}}; \theta_{0\infty}, 0\right)\right] &= 0, \\ E\frac{\partial \psi_t}{\partial \theta}\left[\theta^*(V_0)\right] &= 0. \end{aligned}$$

When $\alpha_0 = 0$, the two solutions $\theta_{0\infty}$ and $\theta^*(V_0)$ coincide with θ_0. We can derive their first order expansion for small values of α_0. This provides the following equations :

$$E\left[\frac{\partial^2 \log l}{\partial \theta \partial \theta'}\right](\theta_{0\infty} - \theta_0) + E\left[\frac{\partial \log l}{\partial \theta}\frac{\partial \log l}{\partial \alpha'}\right]\alpha_0 = \mathrm{o}(\alpha_0),$$

$$E\left[\frac{\partial^2 \Psi_t}{\partial \theta \partial \theta'}\right](\theta^*(V_0) - \theta_0) + E\left[\frac{\partial \Psi_t}{\partial \theta}\frac{\partial \log l}{\partial \alpha'}\right]\alpha_0 = \mathrm{o}(\alpha_0),$$

where the expectations are taken conditional on $\theta_0, \alpha_0 = 0$. Since

$$E\left[\frac{\partial \log l}{\partial \alpha'}\left(S_t \mid \underline{S_{t-1}}; \theta_0, 0\right)\right] = 0,$$

we deduce the property.

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