Credit risk and credit derivatives: a risk assessment and pricing framework

Entretiens de Banque & Marchés
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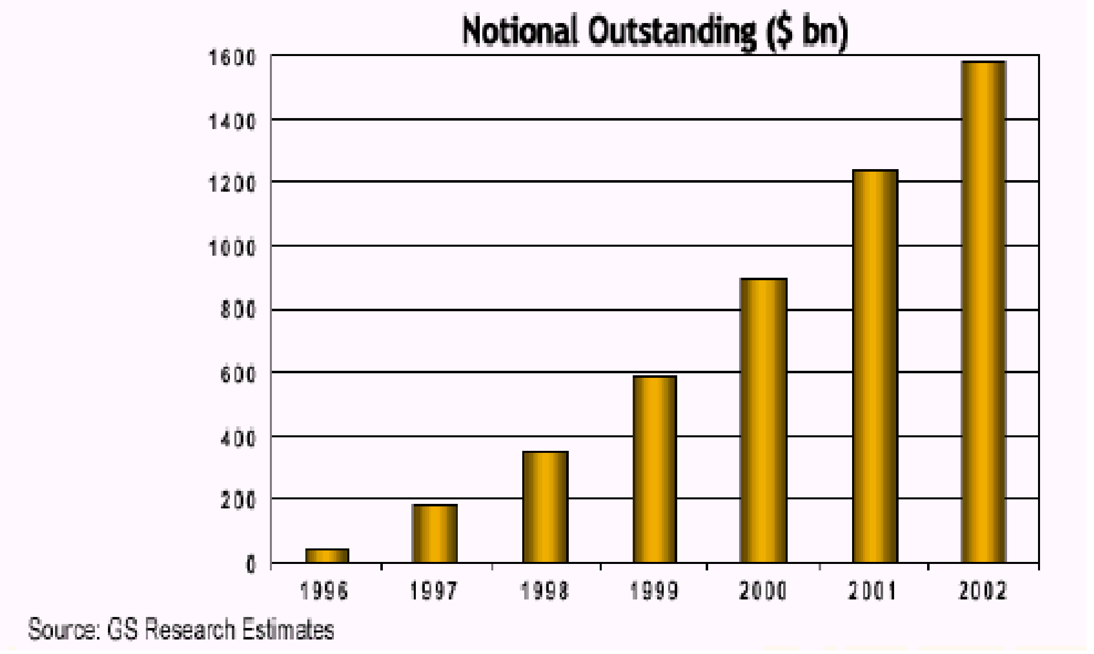
Le papier « basket defaults swaps, CDO's and Factor Copulas » est disponible sur www.defaultrisk.com

Les articles « aggregating Basel II » et « I will survive: analytical pricing of basket credit derivatives » seront bientôt disponibles sur mon site.

Changes in our environment



CDO in Europe

Year	1996	1997	1998	1999	2000	2001	2002
Number	1	3	3	24	50	133	144
Volume ($ bn)	5	5.7	4.5	29.2	63.2	106	143.4

Source: Moody's Investor Service

- *from Thierry Roncalli, GRO Crédit Lyonnais*

Changes in our environment

- *« We are witnessing an impressive escalation in analytical resources devoted to more effective management of credit risk*
- *…the new millenium has a fast paced start to further new developments and techniques for the analytical treatment of credit-risk management. »*
- *From Edward L. Altman*

	Retail	Corporate	OTC derivatives	Credit derivatives
Risk assessement	Factor models Continuous distributions	Factor models of default times	Stochastic exposure	Similar to corporate
Risk transfer	Securitization	CDS, CDO's, basket default swaps	Contingent Default swaps	Basket default swaps

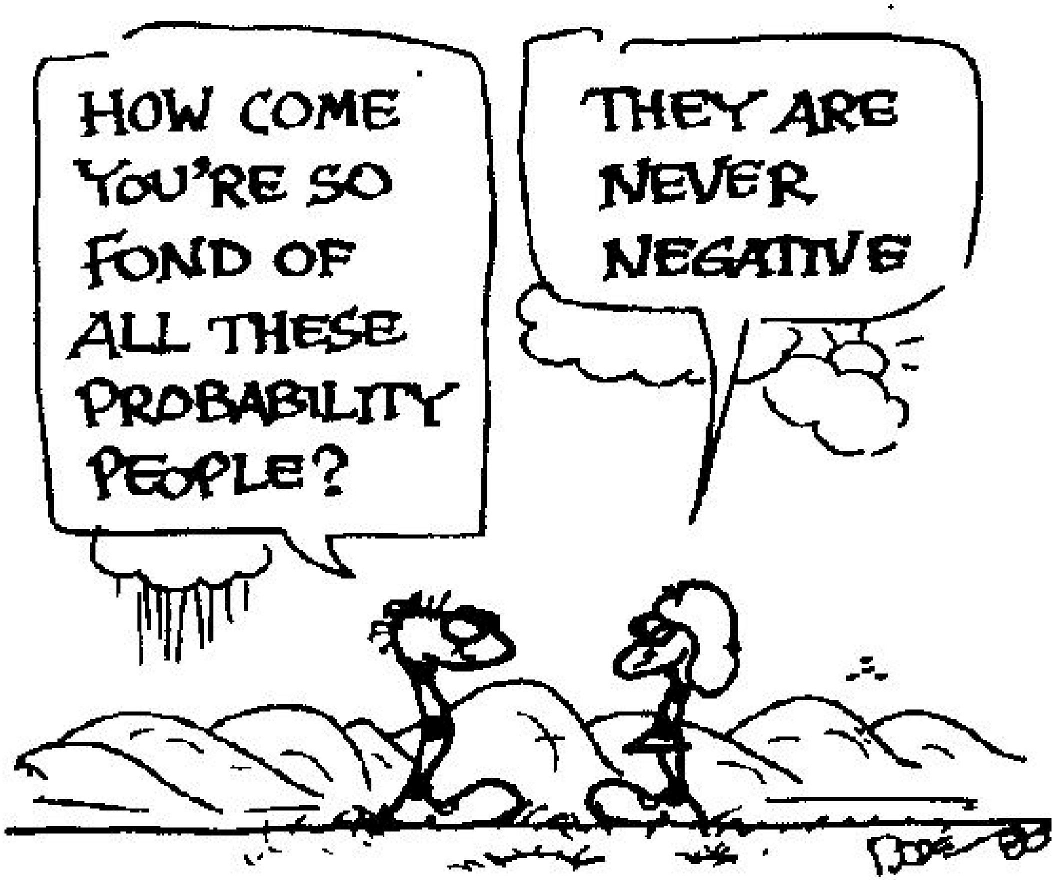
Changes in our environment

- What are we looking for ?
  - *Not really a new model for defaults*
    - Already a variety of modelling approaches to default
  - *Rather a framework where:*
    - one can easily deal with a large number of names,
    - Tackle with different time horizons,
    - Compute loss distributions, measures of risk (VaR, ES) …
    - And credit risk insurance premiums (pricing of credit derivatives).
- Straightforward approach:
  - *Direct modelling of default times*
  - *Modelling of dependence through copulas*
  - *Default times are independent conditionnally on factors*

Overview

- *Probabilistic tools*
  - Survival functions of default times
  - Factor copulas
- *Credit loss distributions and risk measures*
    - *Loss distributions over different time horizons*
    - *Information flow and credit migration*
    - *Risk measures : retail and corporate portfolios*
- *Valuation of basket credit derivatives*
  - Moment generating functions
    - *Distribution of k-th to default time*
  - k^{th} to default swaps
  - Valuation of CDO tranches
- *Valuation of default swaps on OTC derivatives*
- *Counterparty risk on credit default swaps*

Probabilistic tools



by Andrejs Dunkels

Probabilistic tools: survival functions

- $i = 1, \dots, n$ names
- $\tau_1, \dots, \tau_n$ default times
- Marginal distribution function $F_i(t) = Q(\tau_i \leq t)$
- Marginal survival function $S_i(t) = Q(\tau_i > t)$
  - *Risk-neutral probabilities of default*
    - Obtained from defaultable bond prices or CDS quotes
  - *« Historical » probabilities of default*
    - Obtained from time series of default times

Probabilistic tools: survival functions

- Joint survival function

$$S(t_1, \ldots, t_n) = Q(\tau_1 > t_1, \ldots, \tau_n > t_n)$$

  - *Needs to be specified given marginals*
- (Survival) Copula of default times

$$C(S_1(t_1), \ldots, S_n(t_n)) = S(t_1, \ldots, t_n)$$

  - *C characterizes the dependence between default times*
- We need tractable dependence between defaults
  - *Parsimonious modelling*
  - *Semi-explicit computations for portfolio credit derivatives*

Probabilistic tools: factor copulas

- Factor approaches to joint distributions
  - *V low dimensional factor, not observed « latent factor »*
  - *Conditionally on V default times are independent*
  - *Conditional default probabilities*

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right), \quad q_t^{i|V} = Q\left(\tau_i > t \mid V\right).$$

  - *Conditional joint distribution*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid V) = \prod_{1 \leq i \leq n} p_{t_i}^{i|V}$$

  - *Joint survival function (implies integration wrt V)*

$$Q(\tau_1 > t_1, \ldots, \tau_n > t_n) = E\left[\prod_{i=1}^{n} q_{t_i}^{i|V}\right]$$

Probabilistic tools: Gaussian copulas

- One factor Gaussian copula *(Basel 2)*
  - $V, \bar{V}_i,\ i = 1, \ldots, n$ *independent Gaussian*

$$V_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
  - *Conditional default probabilities* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$
- Joint survival function

$$S(t_1, \ldots, t_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\rho_i v - \Phi^{-1}(F_i(t_i))}{\sqrt{1-\rho_i^2}}\right)\right)\varphi(v)dv$$

  - *Copula*

$$C(u_1, \ldots, u_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(u_i) - \rho_i v}{\sqrt{1-\rho_i^2}}\right)\right)\varphi(v)dv$$

Probabilistic tools : Clayton copula

- *Davis & Lo ; Jarrow & Yu ; Schönbucher & Schubert*

- Conditional default probabilities

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

- *V: Gamma distribution with parameter*

- Joint survival function

$$S(t_1, \ldots, t_n) = \int \prod_{i=1}^{n} \left(1 - p_{t_i}^{i|V}\right) \frac{1}{\Gamma(1/\theta)} e^{-V} V^{(1-\theta)/\theta} dV$$

- *Copula*

$$C(u_1, \ldots, u_n) = \left(u_1^{-\theta} + \ldots + u_n^{-\theta} - n + 1\right)^{-1/\theta}$$

Probabilistic tools: simultaneous defaults

- *Duffie & Singleton, Wong*

- Modelling of defaut dates $\tau_i = \min(\bar{\tau}_i, \tau)$
  - $Q(\tau_i = \tau_j) \geq Q\left(\tau \leq \min(\bar{\tau}_i, \bar{\tau}_j)\right) > 0$ *simultaneous defaults*
  - *Conditionally on* τ, τ_i *are independent*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid \tau) = \prod_{1 \leq i \leq n} Q(\tau_i \leq t_i \mid \tau)$$

- Conditional default probabilities

$$Q(\tau_i \leq t_i \mid \tau) = 1_{\tau > t_i} Q(\bar{\tau}_i \leq t_i) + 1_{\tau \leq t_i}$$

  - *Copula of default times*

$$C(u_1, \ldots, u_n) = E\left[\prod_{1 \leq i \leq n} Q\left(\tau_i \leq F_i^{-1}(u_i) \mid \tau\right)\right]$$

Probabilistic tools: Affine Jump Diffusion

- *Duffie, Pan & Singleton ;Duffie & Garleanu.*
- $n+1$ *independent affine jump diffusion processes:*

$$X_1, \ldots, X_n, X_c$$

- Conditional default probabilities:

$$Q(\tau_i > t \mid V) = q_t^{i|V} = V\alpha_i(t)$$

$$V = \exp\left(-\int_0^t X_c(s)ds\right), \quad \alpha_i(t) = E\left[\exp\left(-\int_0^t X_i(s)ds\right)\right].$$

- Survival function:

$$Q(\tau_1 > t, \ldots, \tau_n > t) = E\left[V^n\right] \times \prod_{i=1}^n \alpha_i(t).$$

- *Explicitely known*

Credit loss distributions and risk measures

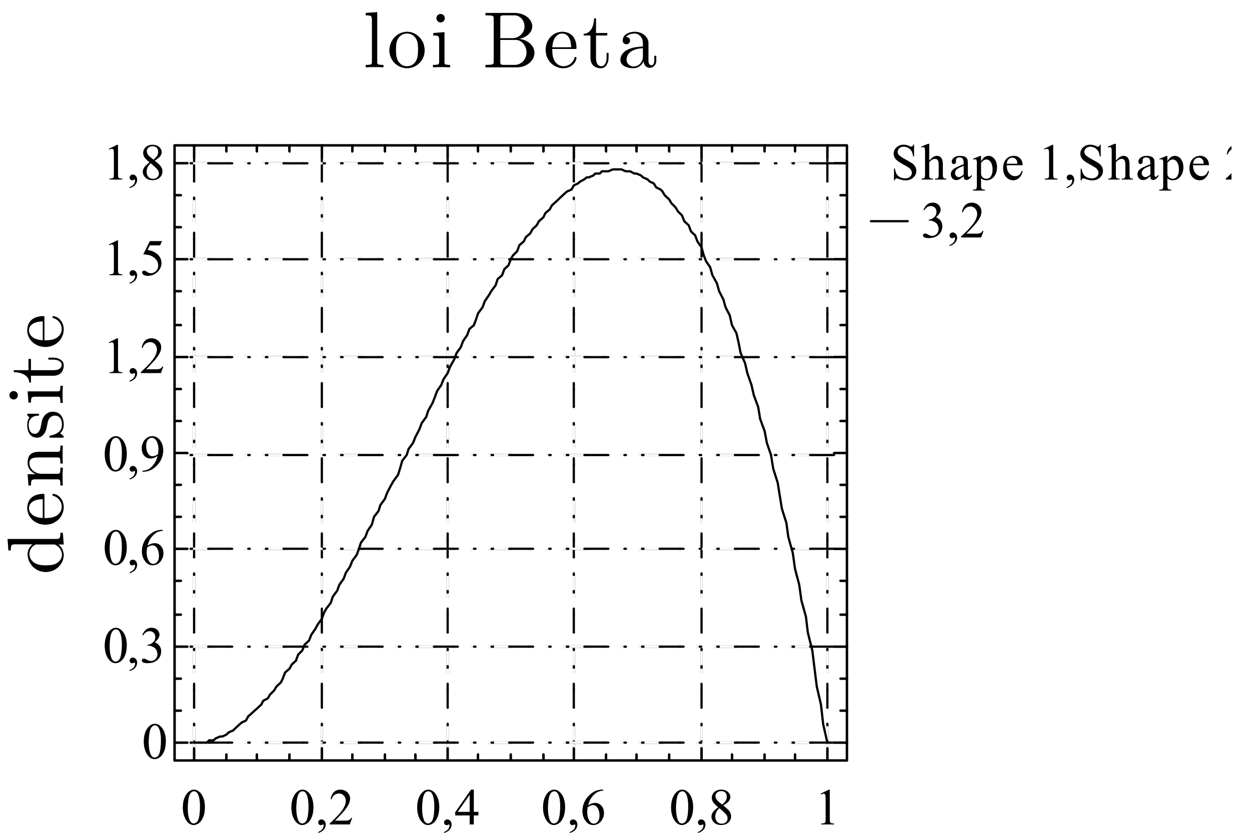
- Accumulated loss at t: $L(t) = \sum_{1 \le i \le n} N_i(1-\delta_i)N_i(t)$
  - Where $N_i(t) = 1_{\tau_i \le t}, \; N_i(1-\delta_i)$ loss given default
- Characteristic function $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
- By conditioning $\varphi_{L(t)}(u) = E\left[\prod_{1 \le j \le n}\left(1 - p_t^{j|V} + p_t^{j|V}\varphi_{1-\delta_j}(N_j)\right)\right]$
- If recovery rates follows a beta distribution:

$$\varphi_{L(t)}(u) = E\left[\prod_{1 \le j \le n}\left(1 - p_t^{j|V} + p_t^{j|V}M(a_j, a_j + b_j, iN_j)\right)\right]$$

  - where M is a Kummer function, a_j, b_j some parameters
- Distribution of L(t) is obtained by Fast Fourier Transform

Credit loss distributions and risk measures

- *Beta distribution for recovery rates*



Credit loss distributions and risk measures

- Modelling of information

$$\mathcal{H}_{i,t} = \sigma\left(1_{\tau_i \leqslant s}, s \leqslant t\right), \ \mathcal{H}_t = \bigvee_{1 \leqslant i \leqslant n} \mathcal{H}_{i,t}$$

  - *Information provided by observation of default times*

- Credit migration:
  - *modelling of conditional survival functions*

$$Q\left(\tau_i \geq T \mid \mathcal{H}_t\right)$$

  - *involves partial derivatives of the survival function:*

$$Q\left(\tau_1 > T \mid \tau_1 > t, \tau_2 = t_2\right) = \frac{\partial S}{\partial t_2}(T, t_2) \Big/ \frac{\partial S}{\partial t_2}(t, t_2)$$

  - *can be computed analytically under factor assumption*

Credit loss distributions and risk measures

- market value of survived loans (credit migration)

$$\sum_{1\le i\le n} N_i B_i(t)(1-N_i(t))$$

  - N_i *nominal exposure on name i*
  - $B_i(t)$ *market value of survived loan on name i*
  - *involves conditional survival distributions* $Q\left(\tau_i \ge T \mid \mathcal{H}_t\right)$
- Total loss

$$\hat{L}(t) = \sum_{1\le i\le n} N_i\left[(1-\delta_i)N_i(t) + B_i(t)(1-N_i(t))\right]$$

  - *Need of some Monte Carlo simulation for* $\hat{L}(t)$

Credit loss distributions and risk measures

- Coherent risk measures :

$$
\begin{aligned}
& X \leqslant Y \Rightarrow \rho(X) \leqslant \rho(Y), \\
& \forall \lambda > 0, \quad \rho(\lambda X) = \lambda \rho(X), \\
& \rho(X+Y) \leqslant \rho(X) + \rho(Y) \\
& \forall a \in \mathbb{R}, \quad \rho(X+a) = \rho(X) - a.
\end{aligned}
$$

- VaR (not sub-additive)

$$VaR_\alpha\left(\hat{L}(t)\right) = \inf\left(x, Q\left(\hat{L}(t) \leqslant x\right) \geqslant 1-\alpha\right)$$

- Expected Shortfall (coherent risk measure)

$$ES_\alpha\left(\hat{L}(t)\right) = \frac{1}{\alpha}\int_0^\alpha VaR_u\left(\hat{L}(t)\right) du$$

Credit loss distributions and risk measures

- Credit loss distributions and risk measures
  - *Retail portfolios: continuous smooth distributions*

$$ES_\alpha\left(\hat{L}(t)\right) = E\left[\hat{L}(t) \mid \hat{L}(t) \geqslant VaR_\alpha\left(\hat{L}(t)\right)\right]$$

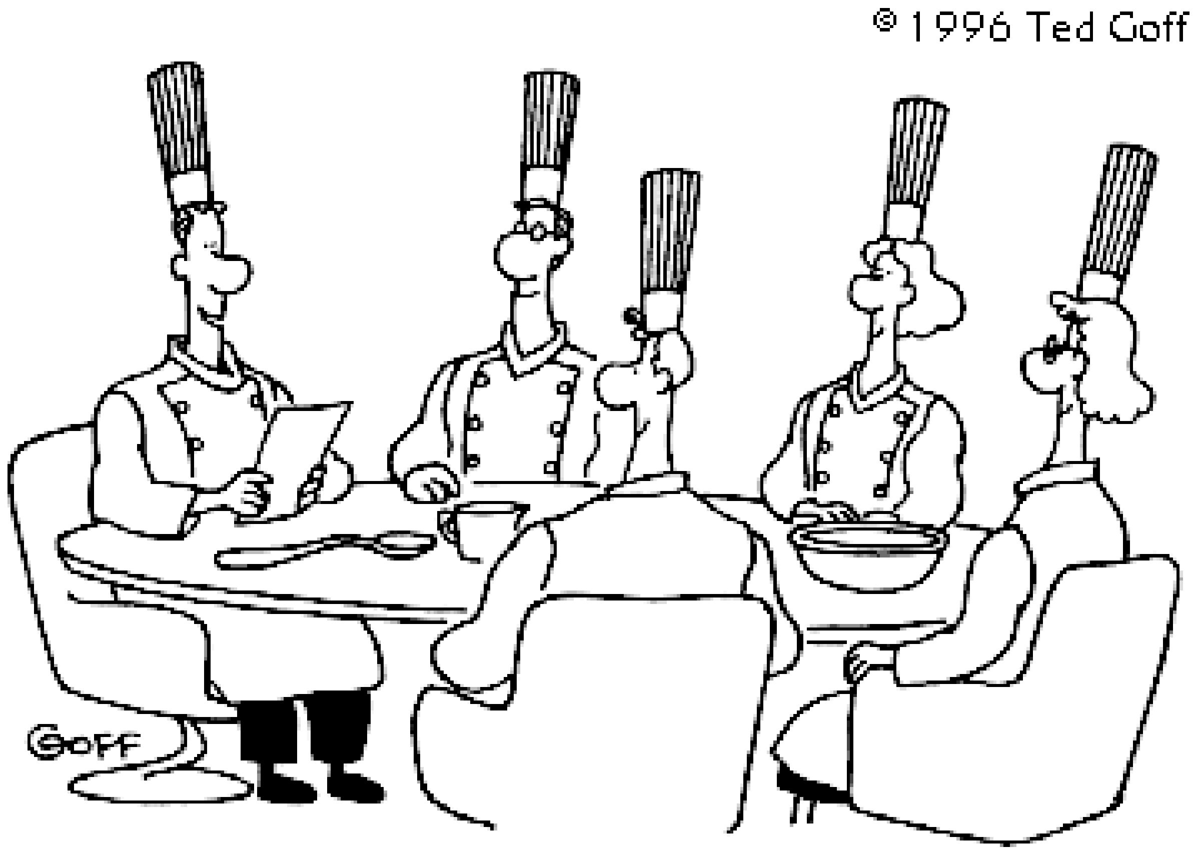
  - *more details in « aggregating Basel II »*
  - *corporate portfolios: discrete distributions*

- Risk contributions for continuous distributions

$$\frac{\partial ES_\alpha\left(\hat{L}(t)\right)}{\partial N_i} = E\left[(1-\delta_i)N_i(t) + B_i(t)(1-N_i(t)) \middle| \hat{L}(t) \geq VaR_\alpha\left(\hat{L}(t)\right)\right]$$

  - *Differentiability issues for discrete distributions* *«the whole is simpler than the sum of its parts »*

Basket Valuation



"Our eggs are all in one basket, no milk has been spilt, and we have plenty of dough."

«Counting time is not so important as making time count»

Valuation of basket credit derivatives

- $N(t) = \sum_{1 \le i \le n} 1_{\{\tau_i \le t\}} = \sum_{1 \le i \le n} N_i(t)$ *Number of defaults at t*
- τ^k *k^{th} to default time*
- $S^k(t) = Q(\tau^k > t)$ *Survival function of k^{th} to default*
- *Remark that* $\tau^k > t \Longleftrightarrow N(t) < k$
- *Survival function of* τ^k : $S^k(t) = \sum_{l \le k-1} Q(N(t) = l)$
- *Computation of* $Q(N(t) = l)$
- *Use of pgf of N(t):* $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = \sum_{l=0}^{n} Q(N(t) = l) u^l$

Valuation of basket credit derivatives

- *Probability generating function of* $N(t)$: $\psi_{N(t)} = E\left[u^{N(t)}\right]$
    - $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = E\left[E\left[u^{N(t)} \mid V\right]\right]$ *iterated expectations*
    - $E\left[u^{N(t)} \mid V\right] = \prod_{1 \le i \le n} E\left[u^{N_i(t)} \mid V\right]$ *conditional independence*
    - $E\left[u^{N_i(t)} \mid V\right] = 1 - p_t^{i|V} + p_t^{i|V} \times u$ *binary random variable*
- $\psi_{N(t)}(u) = E\left[\prod_{i=1}^{n}\left(1 - p_t^{i|V} + p_t^{i|V} \times u\right)\right]$ *polynomial in u*
- *One can then compute* $Q(N(t) = k)$
- *Since* $\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = \sum_{k=0}^{n} Q(N(t) = k) u^k$

Valuation of homogeneous baskets

- $i = 1, \dots, n$ names
  - *Equal nominal (say 1) and recovery rate (say 0)*
- Payoff : 1 at *k*-th to default time if less than *T*
- Credit curves can be different
  - $S_i(t) = Q(\tau_i > t)$ *given from credit curves*
  - $S^k(t) = Q(\tau^k > t)$ *: survival function of* τ^k
  - $S^k(t)$ *computed from pgf of* $N(t)$

Valuation of homogeneous baskets

- Expected discounted payoff

$$E\left[B(\tau^k)1_{\tau^k \leq T}\right] = -\int_0^T B(t)dS^k(t)$$

  - *From transfer theorem*
  - *B(t) discount factor*
- Integrating by parts

$$1 - B(T)S^k(T) + \int_0^T S^k(t)dB(t)$$

  - *Present value of default payment leg*
  - *Involves only known quantities*
  - *Numerical integration is easy*

Valuation of premium leg

- k^{th} to default swap, maturity T
  - $t_1, \ldots, t_{l-1}, t_l, \ldots, T$ *premium payment dates*
  - *Periodic premium p is paid until* τ^k
- l^{th} premium payment
  - $\tau^k > t_l$ *payment of p at date* t_l
  - *Present value:* $pB(t_l)S^k(t_l)$
  - $t_{l-1} \leq \tau^k \leq t_l$ *accrued premium of* $(\tau^k - t_{l-1})p$ *at* τ^k
  - *Present value:* $\displaystyle\int_{t_{l-1}}^{t_l} pB(t)(t - t_{l-1})dS^k(t)$
- PV of premium leg given by summation over l

Non homogeneous baskets

- $i = 1, \ldots, n$ *names*
- $M_i = (1 - \delta_i) N_i$ *loss given default for i*

- Payment at *k*th default of M_i if *i* is in default
  - *No simultaneous defaults*
  - *Otherwise, payoff is not defined*
- *i k*th default iff *k*-1 defaults before τ_i
  - $N^{(-i)}(\tau_i)$ *number of defaults (i excluded) at* τ_i
  - *k-1 defaults before* τ_i *iff* $N^{(-i)}(\tau_i) = k - 1$

Non homogeneous baskets

- (discounted) Payoff $\sum_{i=1}^{n} M_i B(\tau_i) 1_{\{N^{(-i)}(\tau_i)=k-1\}} 1_{\{\tau_i \le T\}}$
- Upfront Premium
  - ... by iterated expectations theorem

$$\sum_{i=1}^{n} M_i E\left[E\left[B(\tau_i) 1_{\{N^{(-i)}(\tau_i)=k-1\}} 1_{\{\tau_i \le T\}} \mid V\right]\right]$$

  - ... by Fubini + conditional independence

$$\int_0^T B(t) Q(N^{(-i)}(t) = k-1 \mid V) dp_t^{i|V}$$

  - where $p_t^{i|V} = Q(\tau_i \le t \mid V)$
  - $Q(N^{(-i)}(t) = k-1 \mid V)$: formal expansion of $\prod_{j \ne i} \left(1 - p_t^{j|V} + p_t^{j|V} u\right)$

Non homogeneous baskets

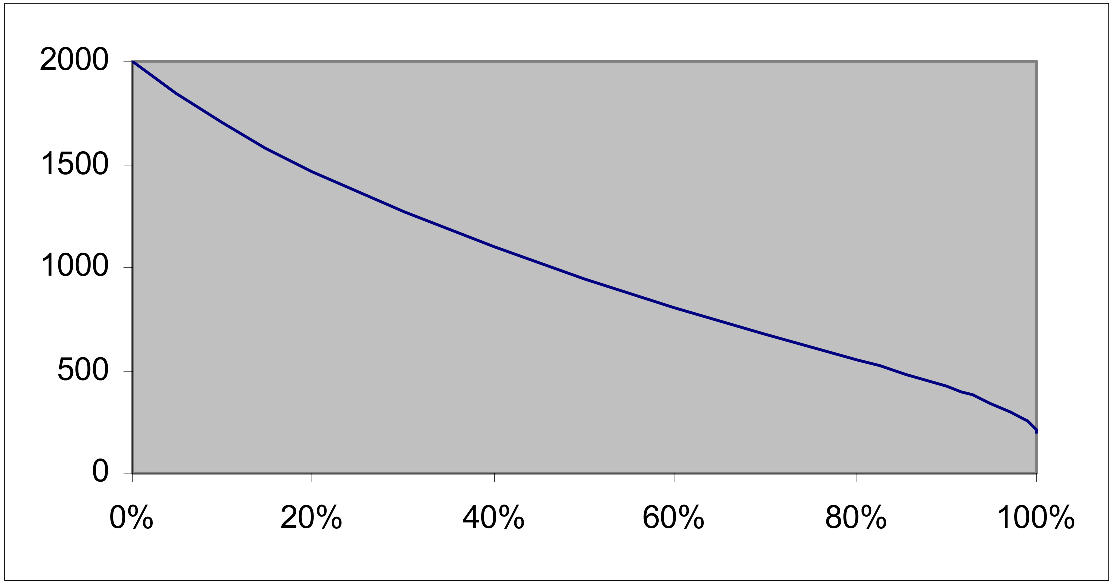
- *Guido Fubini*

First to default swap

- *Case where* $k = 1$
- $Q\left(N^{(-i)}(t) = 0 \mid V\right) = \prod_{j \neq i}\left(1 - p_t^{j|V}\right)$ *no defaults for* $j \neq i$
- *premium* $= \sum_{i=1}^{n} M_i E\left[\int_0^T B(t) \prod_{j \neq i}\left(1 - p_t^{j|V}\right) dp_t^{i|V}\right]$
- $= \int_0^T \sum_{i=1}^{n} M_i B(t) E\left[\prod_{j \neq i}\left(1 - p_t^{j|V}\right) \frac{dp_t^{i|V}}{dt}\right] dt$ *(regular case)*
- *One factor Gaussian* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$
- *Archimedean* $p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$

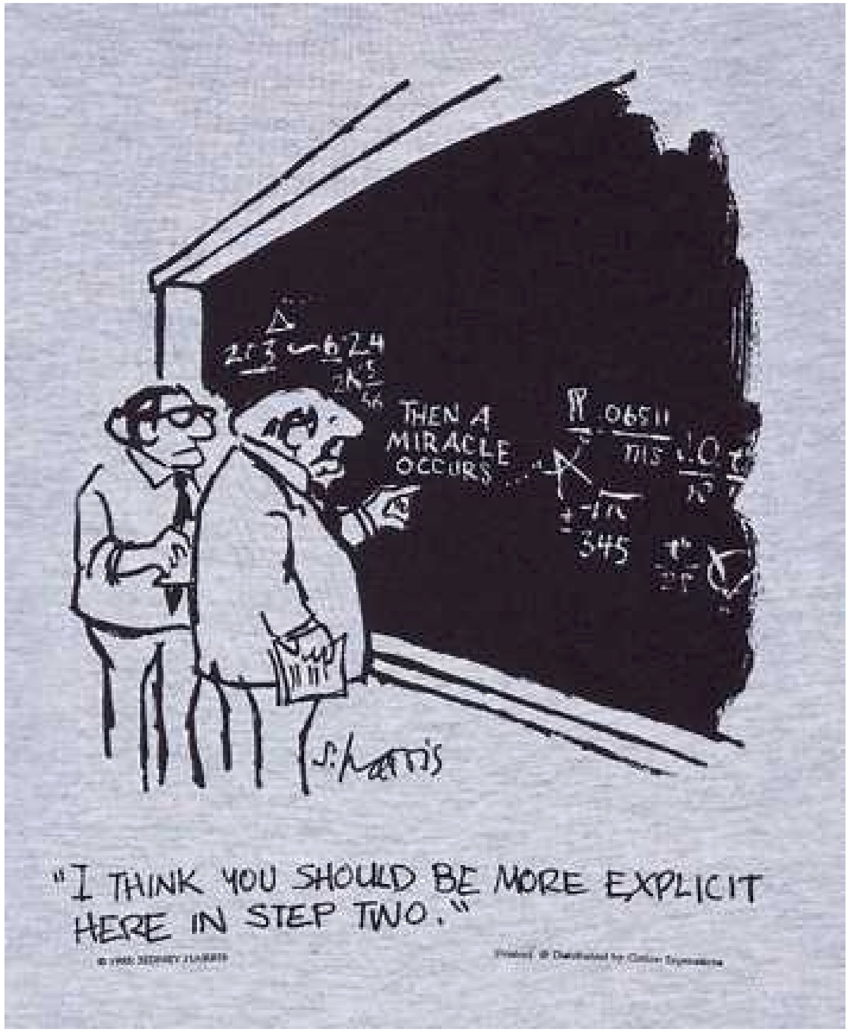
First to default swap

- *One factor Gaussian copula*
- *n=10 names, recovery rate = 40%*
- *5 spreads at 50 bps, 5 spreads at 350 bps*
- *maturity = 5 years*
- *x axis: correlation parameter, y axis: annual premium*



Valuation of CDO's

«Everything should be made as simple as possible, not simpler»



- *Explicit premium computations for tranches*
- *Use of loss distributions over different time horizons*
- *Computation of loss distributions from FFT*
- *Involves integration par parts and Stieltjes integrals*

Valuation of CDO's

- *Tranches with thresholds* $0 \leq A \leq B \leq \sum N_j$
- *Mezzanine: pays whenever losses are between A and B*
- *Cumulated payments at time t: M(t)*

$$M(t) = (L(t) - A))\,1_{[A,B]}(L(t)) + (B - A)1_{]B,\infty[}(L(t))$$

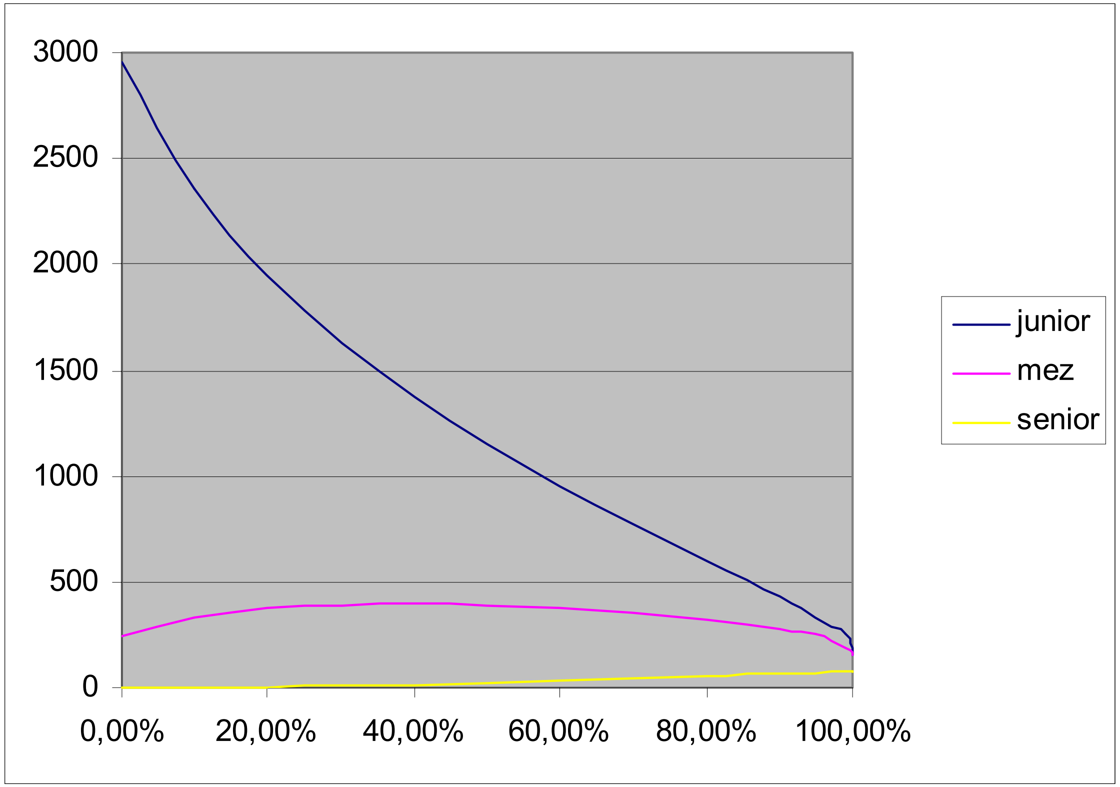
- *Upfront premium:* $E\left[\int_0^T B(t)dM(t)\right]$
  - *B(t) discount factor, T maturity of CDO*
- *Stieltjes integration by parts* $B(T)E[M(T)] + \int_0^T E[M(t)]dB(t)$
- *where* $E[M(t)] = (B - A)Q(L(t) > B) + \int_A^B (x - A)dF_{L(t)}(x)$

Valuation of CDO's

- Thomas Stieltjes

Valuation of CDO's

- One factor Gaussian copula
- n=50 names, all at 100 bps, recovery = 40%
- maturity = 5 years, x axis: correlation parameter
- 0-4%, junior, 4-15% mezzanine, 15-100% senior



Contingent default swaps

- Credit protection on OTC derivatives
  - $C(t)$ *market value of a (portfolio of) OTC derivatives*
  - *credit protection payment of* $C(\tau)$ *at default*
  - *T maturity of the contract,* $B(t)$ *discount factor*
  - *Discounted payment on default leg*

$$C(\tau)1_{\{\tau\leq T\}}B(\tau)$$

- Premium of default protection

$$\int_0^T E\left(C(\tau)B(\tau) \mid \tau = t\right) dQ(\tau = t)$$

  - *requires the computation of conditional distribution of market value at default time*

Counterparty risk on credit default swaps

- Let us consider a defaultable CDS
  - τ_1 *: default date of underlying name*
  - τ_2 *: default date of CDS counterparty*
  - *Default payment if* $\tau_1 < \tau_2 \wedge T$
- Discounted default payment

$$N_1(1-\delta_1)1_{\{\tau_1<\tau_2\wedge T\}}B(\tau_1)$$

- Present Value of default payment

$$\int_{\mathbb{R}^2} N_1(1-\delta_1)B(t_1)1_{\{t_1<t_2\wedge T\}}dQ(\tau_1 \le t_1 \mid \tau_2 = t_2)dQ(\tau_2 = t_2)$$

  - *Can be computed explicitely in the factor model*
  - *One also needs to take into account the case* $\tau_2 < \tau_1$

Conclusion

- Factor models of default times:
  - *Very simple computation of basket credit derivatives and CDO's*
  - *One can deal easily with a large range of correlations*
  - *Computation of loss distributions with credit migration often requires Monte Carlo approaches*
  - *Computation of risk measures*
    - Explicit for corporate portfolios
    - Monte Carlo for retail portfolios
  - *Integration of market and credit risk*
    - Difficult in default time models
    - Difficult either in structural models