A comparative analysis of CDO pricing models

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Abstract

We compare some popular CDO pricing models. Dependence between default times is modelled through Gaussian, stochastic correlation, Student t, double t, Clayton and Marshall-Olkin copulas. We detail the model properties and compare the semi-analytic pricing approach with large portfolio approximation techniques. The ability of the models to fit the correlation skew observed in CDO market quotes is also assessed. Eventually, we relate CDO premiums and the distribution of conditional default probabilities which appears as a key input in the copula specification.

Introduction

This paper provides a comparison of some popular CDO pricing models. We use a factor approach leading to semi-analytic pricing expressions that ease model risk assessment. We focus on “copula models” since there are predominantly used in the credit derivatives markets, though the factor approach also applies to various intensity models (see Mortensen [2006] for an example). The pricing of synthetic CDOs involves the computation of aggregate loss distributions over different time horizons. In our “bottom-up” approach, CDO tranche premiums depend upon the individual credit risk of names in the underlying portfolio and the dependence structure between default times.

There are currently several approaches to CDO pricing. One may start from a specification of dependent default intensities. A typical example is Duffie and Gârleanu [2001]. An alternative route is the structural approach, corresponding to a multivariate hitting time model, as illustrated by Hull *et al.* [2005]. The previous approaches involve a calibration to marginal default distributions. On the other hand, the copula approach directly specifies the dependence structure, though in a somehow ad-hoc way. While the Gaussian copula model, introduced to the credit field by Li [2000] has become an industry standard, its theoretical foundations, such as credit spread dynamics may be questioned. For this purpose, copulas such as Clayton, Student *t*, double *t*, or Marshall-Olkin copulas have been proposed.

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The factor approach is quite standard in credit risk modelling (see for instance Crouhy *et al*. [2000], Merino and Nyfeler [2002], Pykhtin and Dev [2002], Gordy [2003] and Frey and McNeil [2003]). In the case of homogeneous portfolios, it is often coupled with large sample approximation techniques. In such a framework, Gordy and Jones [2003] analyse the risks within CDO tranches. In order to deal with numerical issues, Gregory and Laurent [2003] and Laurent and Gregory [2005] have described a semi-analytical approach, based on factor models, for the pricing of basket credit derivatives and CDOs. This topic is also discussed by, among others, Andersen *et al*. [2003] and Hull and White [2004]. We will further rely on this factor approach, which also provides an easy to deal framework for model comparisons. Other contributions dedicated to comparing various copulas in the credit field are Das and Deng [2004] or the book by Cherubini *et al*. [2004]. The models studied here are the following:

- The Gaussian copula, more precisely, its one factor sub-case. This model is widely used by the financial industry.
- A stochastic correlation extension of the Gaussian copula.
- The Student *t* extension of the Gaussian copula with six and twelve degrees of freedom.
- A double *t* one factor model as introduced by Hull and White [2004].
- The Clayton copula model, that can also be seen as a frailty model with a Gamma distribution.
- A multivariate exponential model associated with multiple defaults. The associated copula is the Marshall-Olkin copula.

We refer to Andersen [2007] within this book for a discussion of other recent extensions of the factor copula approach.

For simplicity and to ease model comparisons, we will thereafter restrict to cases where the copula of default times is a symmetric function with respect to its coordinates. For instance, in the Gaussian copula case, this means that the correlation parameter is constant, whatever the couples of names[4]. Comparing CDO pricing models is easier due to the small number of parameters involved. We study the dependence of CDO tranche premiums with respect to the choice of dependence parameter. This involves some results in the theory of stochastic orders. For example, we can show that first to default swaps or base correlation CDO tranche premiums are monotonic with respect to the relevant dependence parameter. We also discuss some extreme cases such as independence and comonotonicity (or "perfect positive dependence") between default times. The theory of stochastic orders also provides some comparison results between CDO tranche premiums depending on the granularity of the reference credit portfolio.

We then compare CDO pricing models based under different copula assumptions. We show that popular indicators such as Kendall's τ or the tail dependence parameter poorly explain the differences between CDO tranche premiums. On the other hand, the distribution of the conditional default probabilities appears as the key input. This explains for instance that, for a given time horizon, the Clayton copula and the one factor Gaussian copula almost lead to the same CDO tranche premiums. The conditional default probabilities are also of first importance in large portfolio approximations that dramatically simplify the computation of CDO tranche premiums.

Eventually, we study the ability of the studied models to fit market quotes. Double *t* and stochastic correlation models appear to provide the better fits, while for instance the Clayton and the Gaussian copula provide some strikingly similar CDO tranche premiums.

The paper is organized as follows: we firstly recall the semi-analytical pricing approach of basket credit derivatives or CDO tranches in a factor framework. The second section reviews the models under study. The third section is devoted to applications of the theory of stochastic orders to the pricing of CDO tranches. Though the third section is more theoretical in nature, it has quite important practical implications: we are able to show the existence of a unique implied dependence correlation parameter in most cases. For instance, we give a formal proof of the uniqueness of implied base correlations, a result of importance for practitioners. Some comparison results between large portfolio approximations

[4] Practitioners then talk of "flat correlation". The symmetry assumption does not preclude the case of heterogeneous credit spreads for different names.

and semi-analytic approaches are provided and granularity issues are discussed. The fourth section contains empirical investigations. Our comparison methodology relies on the uniqueness of implied dependence parameters for base correlation tranches. We firstly study how the different models at hand differ as far as the pricing of basket default swaps and CDO tranches is concerned. We then discuss the ability of the different models to reproduce market quotes on standardized CDO tranches based on the iTraxx index. Eventually, we provide an analysis of the differences between the studied models based on the distribution of conditional default probabilities.

I) Semi-analytical pricing of basket default swaps and CDOs

In this section, we recall how the factor or conditional independence approach can be associated with tractable computations for basket default swaps and CDO tranches (see Laurent and Gregory [2005]).

Throughout the paper, we will consider n obligors and denote the random vector of default times as $\left(\tau_1,\ldots,\tau_n\right)$. We will denote by F and S respectively the joint distribution and survival functions such that for all $\left(t_1,\ldots,t_n\right)\in\mathbb{R}^n$, $F\left(t_1,\ldots,t_n\right)=Q\left(\tau_1\le t_1,\ldots,\tau_n\le t_n\right)$ and $S\left(t_1,\ldots,t_n\right)=Q\left(\tau_1> t_1,\ldots,\tau_n> t_n\right)$ where Q represents some pricing probability measure. $F_1,\ldots,F_n$ represent the marginal distribution functions and $S_1,\ldots,S_n$ the corresponding survival functions. C denotes the copula of default times[5] which is such that $F(t_1,\ldots,t_n)=C\left(F_1(t_1),\ldots,F_n(t_n)\right)$. We denote by E_i, $i=1,\ldots,n$ the nominals associated with n credits, with δ_i being the corresponding recovery rates and by $M_i=E_i\left(1-\delta_i\right)$ the loss given default for name i. We will thereafter assume that recovery rates are deterministic and concentrate upon the dependence of default times.

We will consider a latent factor V such that conditionally on V, the default times are independent. The factor approach makes it simple to deal with a large number of names and leads to very tractable pricing results. We will denote by $p_t^{i|V}=Q\left(\tau_i\le t\middle|V\right)$ and $q_t^{i|V}=Q\left(\tau_i> t\middle|V\right)$ the conditional default and survival probabilities. Conditionally on V, the joint survival function is:

$$S(t_1,\cdots,t_n|V)=\prod_{1\le i\le n}q_{t_i}^{i|V}$$

Basket Default Swaps and CDO tranches are now standardized products. As for the pricing of the CDO tranche, we need to consider the aggregated loss process defined as $L(t)=\sum_{i=1}^{n}M_iN_i(t)$, where $N_i(t)$ are the default indicators processes associated with the different names and M_i the corresponding losses given default. It can be shown that we only need the marginal distributions of $L(t)$ up to maturity in order to price the default and the premium leg of a CDO tranche. The computation of the default payment leg involves $E\left[\left(L(t)-K\right)^+\right]$ where K are the attachment points of the tranches. Semi-analytical techniques allow for quick computation of the aggregated loss distribution. This is usually done by considering its characteristic function. Thanks to the conditional independence assumption, and since recovery rates are deterministic, the characteristic function of the aggregated loss can be written as: $\varphi_{L(t)}(u)=E\left[e^{iuL(t)}\right]=E\left[\prod_{1\le j\le n}\left(q_t^{j|V}+p_t^{j|V}e^{iuM_j}\right)\right]$. The computation of the expectation involves a numerical integration over the distribution of the factor V, which can be easily achieved

[5] Let F be a joint distribution function defined on $\mathbb{R}^n$ and $F_1,\ldots,F_n$ be the corresponding marginal distribution functions. Then, there exists a distribution function C over $\left[0,1\right]^n$ such that for all $x=\left(x_1,\ldots,x_n\right)\in\mathbb{R}^n$, $F(x)=C\left(F_1(x_1),\ldots,F_n(x_n)\right)$. If $F_1,\ldots,F_n$ are all continuous, then C is uniquely defined. Conversely, if C is an n-copula and $F_1,\ldots,F_n$ are univariate distribution functions, $x\to C\left(F_1(x_1),\ldots,F_n(x_n)\right)$ defines a joint distribution function.

numerically provided that the dimension of V is small[6]. Eventually, the distribution of the aggregated loss is provided by inversion of the characteristic function or recursion techniques. For more details about these approaches, we refer to Laurent and Gregory [2005], Andersen *et al*. [2003], Hull and White [2004]. Jackson *et al*. [2007] discuss the efficiency of different methods for the computation of loss distributions.

For modelling purpose, it is important to notice that the only inputs to the model are the conditional default probabilities $p_t^{i|V}$, which include all model specification.

II) The models under study

There are now a number of books dedicated to copulas such as Joe [1997], Nelsen [1999] or Cherubini *et al*. [2004]. As for the insurance case, we can also refer to the paper by Frees and Valdez [1998]. We detail below some "factor copulas" that are useful in the pricing of basket credit derivatives and CDOs. We will thereafter restrict ourselves to one parameter copulas to ease comparisons. The symmetry assumption is made about the copula of default times and not about the joint distribution of default times. This assumption can be related but is weaker than the exchangeability assumption. For instance, we may have constant correlations in a Gaussian copula but different marginal default probabilities and recovery rates. For an analysis of heterogeneity effects within the Gaussian copula, we refer to Gregory and Laurent [2004].

II.1 One factor Gaussian copula

The default times are modelled from a Gaussian vector $(V_1,\ldots,V_n)$. As in Li [2000], the default times are given by: $\tau_i = F_i^{-1}(\Phi(V_i))$ for $i=1,\ldots,n$ where F_i^{-1} denotes the generalized inverse of F_i and Φ is the Gaussian cdf. In the one factor case, $V_i = \rho V + \sqrt{1-\rho^2}\bar{V}_i$ where $V,\bar{V}_i$ are independent Gaussian random variables and $0 \le \rho \le 1$[7]. Then:

$$p_t^{i|V} = \Phi\left(\frac{-\rho V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho^2}}\right).$$

$\rho = 0$ corresponds to independent default times while $\rho = 1$ is associated with the comonotonic case[8]. When $\rho = 1$, we simply have $p_t^{i|V} = 1_{\left\{V \le \Phi^{-1}(F_i(t))\right\}}$.

There is no upper or lower tail dependence when $\rho < 1$ while the coefficient of tail dependence is equal to 1 when $\rho = 1$[9]. The relation between Kendall's τ [10] and linear correlation parameter ρ^2 is

[6] In the examples below, the dimension of V will be equal to one or two. Gössl [2007] considers some factor reduction techniques in a Gaussian copula framework.

[7] As a consequence, the correlation between V_i and V_j is equal to ρ^2. Let us remark that some papers rather write the latent variables as $V_i = \sqrt{\rho}V + \sqrt{1-\rho}\bar{V}_i$.

[8] Let $X = (X_1,\ldots,X_n)$ be a random vector with marginal distribution functions $F_1,\ldots,F_n$. X is said to be comonotonic if it has the same distribution as $\left(F_1^{-1}(U),\ldots,F_n^{-1}(U)\right)$ where U is a $[0,1]$ uniform random variable and F_i^{-1} is the generalized inverse of F_i. Moreover, a random vector is comonotonic if and only if the associated copula is the upper Fréchet copula, such that for all $u = (u_1,\ldots,u_n) \in [0,1]^n$, $C^+(u_1,\ldots,u_n) = \min(u_1,\ldots,u_n)$. The Fréchet copula acts as an upper bound, since for any copula C, we have $C(u) \le C^+(u)$ for all $u \in [0,1]^n$.

[9] Let X and Y be two random variables, with distribution functions F_X, F_Y, and let C denote the copula associated with (X,Y). The coefficient of upper tail dependence is such that:

given by: $\rho_K = \frac{2}{\pi}\arcsin\rho^2$. An important result is that the one factor Gaussian copula is increasing in the supermodular order[11] with respect to the correlation parameter ρ. This result was proved by Bäuerle and Müller [1998] and further generalized by Müller and Scarsini [2000], Müller [2001]. Since default times are increasing functions of the V_i's, the default times do also increase, with respect to the supermodular order, as the correlation parameter increases. Loosely speaking, default times are more dependent when the correlation parameter increases, which is rather intuitive, though the formal proofs are quite involved. The notion of dependence with respect to the supermodular order makes sense especially for non Gaussian vectors, such as default times, as will be detailed below. We refer to the books by Müller and Stoyan [2002] or Denuit *et al*. [2005] for detailed comments about stochastic orders.

II.2 Stochastic Correlation

There has been much interest in simple extensions of the Gaussian copula model (see Andersen and Sidenius [2005], Schloegl [2005]) in order to match "correlation smiles" in the CDO market. Let us present the simplest version of such a model. The latent variables are given by:

$$V_i = B_i\left(\rho V + \sqrt{1-\rho^2}\bar{V}_i\right) + (1-B_i)\left(\beta V + \sqrt{1-\beta^2}\bar{V}_i\right),$$

for $i = 1,\ldots,n$, where B_i are Bernoulli random variables, $V, \bar{V}_i$ are standard Gaussian random variables, all these being jointly independent and ρ, β are some correlation parameters, $0 \le \beta \le \rho \le 1$. We denote by $p = Q(B_i = 1)$. The above model is a convex sum of one factor Gaussian copulas, involving a mixing distribution over factor exposure. In our examples, there are here two states for each name, one corresponding to a high correlation and the other to a low correlation. We could equivalently write the latent variables as:

$$V_i = \left(B_i\rho + (1-B_i)\beta\right)V + \sqrt{1-\left(B_i\rho + (1-B_i)\beta\right)^2}\bar{V}_i,$$

This makes clear that we deal with a stochastic correlation Gaussian model. We have a factor exposure ρ with probability p and β with correlation $1-p$. It can be easily checked that the marginal distributions of the V_i's are Gaussian. As above, we define the default dates as $\tau_i = F_i^{-1}(\Phi(V_i))$ for $i = 1,\ldots,n$.

$$\lim_{u\to1} Q\left(X > F_X^{-1}(u) \middle| Y > F_Y^{-1}(u)\right) = \lim_{u\to1}\frac{C(u,u)+1-2u}{1-u},$$

whenever the limit exists. We say that there is upper tail dependence if the coefficient is positive. From the definition, it can be seen that the coefficient of upper tail dependence is always less or equal to 1. It is equal to 1 for the upper Fréchet copula C^+. We can also consider the coefficient of lower tail dependence defined as:

$$\lim_{u\to0} Q\left(X \le F_X^{-1}(u) \middle| Y \le F_Y^{-1}(u)\right) = \lim_{u\to0}\frac{C(u,u)}{u}.$$

This coefficient is also less or equal to 1 and is equal to one for the upper Fréchet copula C^+.

[10] Given a bivariate copula C, Kendall's τ is given by $\rho_K = 4\iint_{[0,1]^2} C(u,v)dC(u,v) - 1$.

[11] Let $f:\mathbb{R}^n \to \mathbb{R}$. We consider the difference operators $\Delta_i^\varepsilon f(x) = f(x+\varepsilon e_i) - f(x)$, where e_i is the i-th unit vector and $\varepsilon > 0$. f is said to be supermodular, if $\Delta_i^\varepsilon \Delta_j^\delta f(x) \ge 0$ holds for all $x \in \mathbb{R}^n, 1 \le i \le j \le n$ and $\varepsilon, \delta > 0$. A random vector $X = (X_1,\ldots,X_n)$ is said to be smaller than the random vector $Y = (Y_1,\ldots,Y_n)$, with respect to the supermodular order, if $E[f(X)] \le E[f(Y)]$ for all supermodular functions such that the expectation exists. This means that the coordinates of Y are more dependent in a rather strong mathematical sense than the coordinates of X.

The default times are independent conditionally on V and we can write the conditional default probabilities

$$p_t^{i|V} = p\Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right) + (1-p)\Phi\left(\frac{-\beta V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\beta^2}}\right).$$

We denote by C_γ^G the bivariate Gaussian copula with covariance term γ. We can check that the bivariate copula of default times can be written as:

$$p^2 C_{\rho^2}^G\left(u,v\right) + 2p(1-p)C_{\rho\beta}^G\left(u,v\right) + (1-p)^2 C_{\beta^2}^G\left(u,v\right),$$

for u,v in $\left[0,1\right]$. As a consequence, the previous model might be seen as a mixture of Gaussian copulas, involving all combinations of correlations. The tail dependence coefficient is equal to zero if $\beta \le \rho < 1$, to p^2 if $\beta < \rho = 1$ and to 1 if $\beta = \rho = 1$. It is also possible to provide an analytical though lengthy expression for Kendall's τ as:

$$\frac{2}{\pi}\times\left(\begin{array}{l} p^4 \arcsin\left(\rho^2\right) + 2p^2(1-p)^2 \arcsin\left(\rho\beta\right) + (1-p)^4 \arcsin\left(\beta^2\right) \\ \quad + 4p^3(1-p)\arcsin\left(\frac{\rho^2+\rho\beta}{2}\right) + 2p^2(1-p)^2 \arcsin\left(\frac{\rho^2+\beta^2}{2}\right) + 4p(1-p)^3 \arcsin\left(\frac{\beta^2+\rho\beta}{2}\right)\end{array}\right)$$

Since the supermodular order is closed under mixtures, it can be proved that increasing $\left(\rho,\beta,p\right)$ leads to an increase in dependence in the supermodular sense. The proof is postponed in the appendix.

The previous two state model can be easily generalized. Let us consider the following modelling of latent variables V_i:

$$V_i = \tilde{\rho}_i V + \sqrt{1-\tilde{\rho}_i^{\,2}}\,\overline{V}_i\ ,\ i = 1,\ldots,n\ ,$$

where $\tilde{\rho}_1,\ldots,\tilde{\rho}_n$ are independent stochastic correlations with distribution function *F*. We still have independent default times conditionally on *V* and:

$$p_t^{i|V} = \int_0^1 \Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right) dF(\rho)$$

We can rather easily compare two stochastic correlation models in a fairly general framework. Let us consider another stochastic correlation model associated with distribution function *G*. We denote by $\tilde{\beta}_1,\ldots,\tilde{\beta}_n$ the corresponding stochastic correlation parameters:

$$W_i = \tilde{\beta}_i V + \sqrt{1-\tilde{\beta}_i^{\,2}}\,\overline{V}_i\ ,\ i = 1,\ldots,n$$

Let us assume that $G(u) \le F(u), \forall u \in \left[0,1\right]$. This means that $\tilde{\rho}_1 \le \tilde{\beta}_1,\ldots,\tilde{\rho}_n \le \tilde{\beta}_n$ with respect to first order stochastic dominance. As a consequence, there exists non-negative random variables $v_1,\ldots,v_n$ independent from $V,\overline{V}_1,\ldots,\overline{V}_n$ such that: $\tilde{\beta}_1 = \tilde{\rho}_1 + v_1,\ldots,\tilde{\beta}_n = \tilde{\rho}_n + v_n$ [12], where the previous equalities hold in distribution. $\left(W_1,\ldots,W_n\right)\left|\tilde{\rho}_1,\ldots,\tilde{\rho}_n,v_1,\ldots,v_n\right.$ and $\left(V_1,\ldots,V_n\right)\left|\tilde{\rho}_1,\ldots,\tilde{\rho}_n,v_1,\ldots,v_n\right.$ are Gaussian with correlation parameter respectively equal to $\tilde{\beta}_1 = \tilde{\rho}_1 + v_1,\ldots,\tilde{\beta}_n = \tilde{\rho}_n + v_n$ and $\tilde{\rho}_1,\ldots,\tilde{\rho}_n$. This ensures that:

$$\left(V_1,\ldots,V_n\right)\left|\tilde{\rho}_1,\ldots\tilde{\rho}_n,v_1,\ldots,v_n\right. \le_{\mathrm{sm}} \left(W_1,\ldots,W_n\right)\left|\tilde{\rho}_1,\ldots\tilde{\rho}_n,v_1,\ldots,v_n\right.,$$

and eventually $\left(V_1,\ldots,V_n\right) \le_{\mathrm{sm}} \left(W_1,\ldots,W_n\right)$. Ordering of stochastic correlation models is related to the first order stochastic dominance of the mixing correlation parameter.

The reader can find further examples of the stochastic correlation approach in Burtschell *et al*. [2007].

II.3 Student *t* copula

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[12] We simply set $v_i = G^{-1}\left(F\left(\tilde{\rho}_i\right)\right) - \tilde{\rho}_i$, $i = 1,\ldots,n$.

The Student *t* copula is a simple extension of the Gaussian copula. It has been considered for credit and risk issues by a number of authors, including Andersen *et al*. [2003], Demarta and McNeil [2005], Embrechts *et al*. [2003], Frey and McNeil [2003], Greenberg *et al*. [2004], Mashal and Zeevi [2003], Mashal *et al*. [2003], Schloegl and O'Kane [2005].

In the Student *t* approach, the underlying vector $(V_1,\ldots,V_n)$ follows a Student *t* distribution with ν degrees of freedom. In the symmetric case which we are going to consider, we have $V_i=\sqrt{W}X_i$ where $X_i=\rho V+\sqrt{1-\rho^2}\bar{V}_i$, $V,\bar{V}_i$ are independent Gaussian random variables, W is independent from $(X_1,\ldots,X_n)$ and follows an inverse Gamma distribution with parameters equal to $\frac{\nu}{2}$ (or equivalently $\frac{\nu}{W}$ follows a χ_ν^2 distribution). Let us remark that the covariance between V_i and V_j, $i\neq j$ is equal to $\frac{\nu}{\nu-2}\rho^2$ for $\nu>2$. We further denote by t_ν the distribution function of the standard univariate Student *t*, that is the univariate cdf of the V_i's. We then have $\tau_i=F_i^{-1}\left(t_\nu\left(V_i\right)\right)$. It can be seen that conditionally on (V,W) default times are independent and:

$$p_t^{i|V,W}=\Phi\left(\frac{-\rho V+W^{-1/2}t_\nu^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right).$$

Thus we deal with a two factor model. As for the Gaussian copula, we have Kendall's τ expressed as: $\rho_K=\frac{2}{\pi}\arcsin\rho^2$. The Student *t* copula has upper and lower tail dependence with equal coefficients, being equal to $2t_{\nu+1}\left(-\sqrt{\nu+1}\times\sqrt{\frac{1-\rho^2}{1+\rho^2}}\right)$. Let us remark that even for $\rho=0$, we still have tail dependence. Thus, $\rho=0$ does not correspond to the independence case. In fact, there is always tail dependence whatever the parameters ρ and ν. Thus, we cannot match the product copula[13] by using the Student *t* copula. However, when $\rho=1$, all the V_i's are equal and this corresponds to the comonotonic case. Since the supermodular order is closed under mixtures and using the supermodular order of Gaussian copulas, we readily obtain that the Student *t* copula is positively ordered with respect to the parameter ρ in the supermodular sense.

II.4 Double *t* copula

This model is also a simple extension of the one factor Gaussian copula. It has been considered for the pricing of CDOs by Hull and White [2004]. As for the Gaussian copula, it belongs to the class of additive factor copulas. We refer to Cousin and Laurent [2007] and the references therein for further examples and discussion.

The default times are modelled from a latent random vector $(V_1,\ldots,V_n)$. The latent variables are such that $V_i=\rho\left(\frac{\nu-2}{\nu}\right)^{1/2}V+\sqrt{1-\rho^2}\left(\frac{\bar{\nu}-2}{\bar{\nu}}\right)^{1/2}\bar{V}_i$ where $V,\bar{V}_i$ are independent random variables following Student *t* distributions with ν and $\bar{\nu}$ degrees of freedom and $\rho\geq 0$. Since the Student distribution is not stable under convolution, the V_i's do not follow Student distributions; the copula associated with $(V_1,\ldots,V_n)$ is not a Student copula. Thus, this model differs from the previous one. As

[13] Random variables $X_1,\ldots,X_n$ are independent if and only if the associated copula is the product copula $C^\perp$ such that: $\forall\left(u_1,\ldots,u_n\right)\in[0,1]^n$, $C^\perp\left(u_1,\ldots,u_n\right)=u_1\times\ldots\times u_n$.

for the one factor Gaussian copula model, $\rho = 0$ is associated with independent default times and $\rho = 1$ with comonotonic default times.

The default times are then given by: $\tau_i = F_i^{-1}\left(H_i\left(V_i\right)\right)$ for $i = 1, \ldots, n$ where H_i is the distribution function of V_i [14]. Then:

$$p_t^{i|V} = t_{\bar{\nu}}\left(\left(\frac{\bar{\nu}}{\bar{\nu}-2}\right)^{1/2} \frac{H_i^{-1}\left(F_i(t)\right) - \rho\left(\frac{\nu-2}{\nu}\right)^{1/2} V}{\sqrt{1-\rho^2}}\right).$$

It is possible to derive some tail dependence parameters in the double *t* model. Using Malevergne and Sornette [2004], we can express the coefficient of tail dependence (the coefficients of upper and lower tail dependence are equal) as:

$$\lambda = \frac{1}{1 + \left(\frac{\sqrt{1-\rho^2}}{\rho}\right)^{\nu}},$$

when $\nu = \bar{\nu}$. If $\nu < \bar{\nu}$, then the tail of the factor V is bigger than the tail of the idiosyncratic risk $\bar{V}_i$. As a consequence, the coefficient of tail dependence is equal to one. In the tails, the idiosyncratic risk can be neglected, and extreme movements are driven solely by the factor. On the other hand, if $\nu > \bar{\nu}$, then the tail of the factor is smaller than the tail of the idiosyncratic risk and there is no tail dependence between the default times.

II.5 Clayton copula

Schönbucher and Schubert [2001], Schönbucher [2002], Gregory and Laurent [2003], Rogge and Schönbucher [2003], Madan *et al.* [2004], Laurent and Gregory [2005], Schloegl and O'Kane [2005], Friend and Rogge [2005] have been considering this model in a credit risk context.

Let us proceed to a formal description of the model. We consider a positive random variable V, which is called a frailty, following a standard Gamma distribution with shape parameter $1/\theta$ where $\theta > 0$. Its probability density is given by $f(x) = \frac{1}{\Gamma\left(1/\theta\right)} e^{-x} x^{(1-\theta)/\theta}$ for $x > 0$. We denote by Ψ the Laplace transform of f. We get $\psi(s) = \int_0^\infty f(x) e^{-sx} dx = \left(1+s\right)^{-1/\theta}$. We then define some latent variables V_i's as:

$$V_i = \psi\left(-\frac{\ln U_i}{V}\right),$$

where $U_1, \ldots U_n$ are independent uniform random variables also independent from V. Eventually, the default times are such that:

$$\tau_i = F_i^{-1}\left(V_i\right), \quad i = 1, \ldots, n$$

The previous equations imply a one factor representation where V is the factor. The conditional default probabilities can be expressed as:

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

Low levels of the latent variable are associated with shorter survival default times. For this reason, V is called a "frailty".

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[14] H_i must be computed numerically and depends upon ρ.

Let us remark that the V_i 's have uniform marginal distributions. Since the default times are increasing functions of these V_i 's, the copula of default times is the joint distribution of the V_i 's. We readily check that $Q(V_1 < u_1, \dots, V_n < u_n) = \psi\left(\psi^{-1}(u_1) + \cdots \psi^{-1}(u_n)\right) = \left(u_1^{-\theta} + \cdots + u_n^{-\theta} - n + 1\right)^{-1/\theta}$, for any $(u_1, \dots, u_n) \in [0,1]^n$. The distribution function of the V_i 's is known as the Clayton copula. The Clayton copula is Archimedean and the generator of the copula is $\varphi(t) = t^{-\theta} - 1$, i.e.

$$C_\theta(u_1, \dots, u_n) = \varphi^{-1}\left(\varphi(u_1) + \cdots + \varphi(u_n)\right).$$

From Embrechts *et al*. [2003], we obtain Kendall's τ for a Clayton copula as: $\rho_K = \dfrac{\theta}{\theta + 2}$ where $\theta \in [-1, \infty) \setminus \{0\}$. The Clayton copula exhibits lower tail dependence for $\theta > 0$, $\lambda_L = 2^{-1/\theta}$ and no upper tail dependence i.e. $\lambda_U = 0$. When $\theta = 0$, we obtain the product copula, i.e. default times are independent. When $\theta = +\infty$, the Clayton copula turns out to be the upper Fréchet bound corresponding to the case where default times are comonotonic.

As the parameter θ increases, the Clayton copula increases with respect to the supermodular order (Wei and Hu [2002]).

II.6 Multivariate exponential models and the Marshall-Olkin copula

The reliability theory denotes these as "shock models". There are also known as multivariate exponential models as in Marshall and Olkin [1967]. They were introduced to the credit domain by Duffie and Singleton [1998] and also discussed by Li [2000], Wong [2000]. More recently, Elouerkhaoui [2003a,b], Giesecke [2003], Lindskog and McNeil [2003] considered the use of such models.

We present here the simplest form of the model corresponding to a single fatal shock[15]. We consider some latent variables $V_i = \min\left(V, \bar{V}_i\right)$, $i = 1, \dots, n$ where $V, \bar{V}_i$, $i = 1, \dots, n$ are independent exponentially distributed random variables with parameters $\alpha, 1-\alpha$, $\alpha \in]0,1[$. The corresponding survival copula[16] belongs to the Marshall-Olkin family (see Nelsen [1999], pages 46-49) and can be expressed as:

$$\hat{C}(u_1, \dots, u_n) = \min\left(u_1^\alpha, \dots, u_n^\alpha\right) \prod_{i=1}^{n} u_i^{1-\alpha}$$

The default times are then defined as:

$$\tau_i = S_i^{-1}\left(\exp\left(-\min\left(V, \bar{V}_i\right)\right)\right)$$

Since $t \to S_i^{-1}\left(\exp(-t)\right)$ are increasing functions, the copula of default times is the same as the copula of $\min\left(V, \bar{V}_i\right)$. We can also check that the survival function of τ_i is indeed S_i. From the definition of default times, we readily see that default times are conditionally independent upon V and the conditional survival probabilities are given by:

$$q_t^{i|V} = 1_{V > -\ln S_i(t)} S_i(t)^{1-\alpha}.$$

There is upper and lower tail dependence with the same coefficient equal to α. It can be shown (see Embrechts *et al*. [2003]) that Kendall's τ is given by: $\rho_K = \dfrac{\alpha}{2-\alpha}$. $\alpha = 0$ corresponds to the independence and $\alpha = 1$, implies that $\tau_i = S_i^{-1}(V)$ i.e. default dates are comonotonic.

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[15] The reader can find some extensions to the case of non fatal shocks in Cousin and Laurent [2007].

[16] The survival copula of default times, $\hat{C}$ is such that $S(t_1, ..., t_n) = \hat{C}\left(S_1(t_1), \dots, S(t_n)\right)$.

Let us consider the case of equal marginal distributions of default times. Then, $Q\left(\tau_i = \tau_j\right) \geq Q\left(V < \min\left(\bar{V}_i, \bar{V}_j\right)\right) > 0$. Thus the model allows for simultaneous defaults with positive probability.

It can be proved that increasing α leads to an increase in the dependence between default dates with respect to the supermodular order. The proof is postponed in the appendix.

III Ordering of CDO tranche premiums

III.1 Monotonic CDO premiums with respect to dependence parameters

Increasing the correlation parameter ρ within Gaussian and Student *t* copula, increasing the parameter θ in the Clayton copula or increasing the parameter α (that represents the relative magnitude of the common shock) in the exponential model leads to an increase in dependence between default times. As a consequence, it can be proven that CDO tranche premiums of equity or senior type, i.e. either with an attachment point equal to zero or a detachment point equal to 100% are monotonic with respect to the dependence parameter. We will thereafter concentrate on equity tranches (i.e. first loss tranches) that are usually associated with the base correlation approach. In the Gaussian copula case, we can formally prove that equity tranche premiums are decreasing with respect to the correlation parameter. This has a great practical importance, since it guarantees the uniqueness of base correlations whatever the maturity of the CDO or the marginal distributions of default times.

Let us consider the Gaussian copula case. To emphasize the dependence of the aggregate loss distributions upon the correlation parameter, let us denote by $L_\rho(t)$ the aggregate loss for time *t*, associated with some correlation parameter ρ. Then, for all time horizons *t*, and attachment points *K*, we have:

$$\rho \leq \rho' \Rightarrow E\left[\left(L_\rho(t) - K\right)^+\right] \leq E\left[\left(L_{\rho'}(t) - K\right)^+\right]$$ [17].

Let us also remark that in all the studied models, $E\left[L(t)\right] = \sum_{i=1}^{n} M_i F_i(t)$. Thus, the expected loss on the reference portfolio is the sum of the expected losses on the names and is invariant with respect to the correlation structure. From call-put parity, we have:

$$\rho \leq \rho' \Rightarrow E\left[\min\left(K, L_\rho(t)\right)\right] \geq E\left[\min\left(L_{\rho'}(t), K\right)\right]$$

Since the present value of the default leg of an equity tranche involves a discounted average of such expectations (see Laurent and Gregory [2005]), we conclude that the value of the default leg of an equity tranche decreases when the correlation parameter increases. Such a result also holds for the Student *t*, Clayton and Marshall-Olkin copulas with respect to the corresponding dependence parameter. We could not yet prove such a result for the double t model, though numerical results show that the value of the default leg of an equity tranche decreases with respect to the dependence parameter ρ.

[17] An important result from actuarial theory states that if two sets of default times are ordered with respect to the supermodular order then the corresponding aggregate losses are ordered with respect to the stop-loss order. Let X and Y be two scalar positive random variables with finite mean. We say that X precedes Y in stop-loss order if $E\left[\left(X-K\right)^+\right] \leq E\left[\left(Y-K\right)^+\right]$ for all $K \geq 0$. It can readily be shown that if two random vectors with positive coordinates $X = \left(X_1, \ldots, X_n\right)$ and $Y = \left(Y_1, \ldots, Y_n\right)$ are ordered for the supermodular order, then $\sum_{i=1}^{n} X_i$ is smaller than $\sum_{i=1}^{n} Y_i$ for the stop-loss order. We refer to Müller [1997], Dhaene and Goovaerts [1997], Hu and Wu [1999], Denuit *et al*. [2001] for some discussion about this topic. The usefulness of the supermodular order is made clear from the above discussion: it provides some monotonicity results on CDO tranche premiums with respect to the copula dependence parameter.

To complete the analysis, we also need to consider the behaviour of the premium leg of a CDO tranche with respect to the dependence parameter. As above, to ease the exposition and notations, we detail the Gaussian copula case, though the analysis is exactly the same for the Student *t*, Clayton and Marshall-Olkin copulas. We recall that the premium paid is proportional to the outstanding nominal of the tranche, that is $\left(K - L_\rho(t)\right)^+$ in the case of an equity tranche with detachment point K. Using the same line of reasoning as above, we have:

$$\rho \le \rho' \Rightarrow E\left[\left(K - L_\rho(t)\right)^+\right] \le E\left[\left(K - L_{\rho'}(t)\right)^+\right].$$

We conclude that the value of the premium leg increases with the correlation parameter. Since meanwhile, the value of the default leg decreases, the equity tranche premium actually decreases when the correlation parameter increases.

III.2 Comonotonic case

We study possible bounds on CDO tranche premiums. Tchen [1980] proved that the random vector of default times $(\tau_1,\ldots,\tau_n)$ is always smaller, with respect to the supermodular order, that the comonotonic vector of default times $\left(F_1^{-1}(U),\ldots,F_n^{-1}(U)\right)$. As a consequence, the case of perfect dependence or "comonotonicity" actually provides a model free lower bound on equity tranche premiums.

III.3 Independence case

The dual case of independence case leads to upper bounds on equity tranche premiums in the studied models. For all models at hand, except for Student *t* (see below), this is a consequence of corollary (3.5) in Bäuerle and Müller [1998]. The Student *t* copula must be treated slightly differently (see Appendix).

Moreover, the independence bound is reached respectively for $\rho = 0$ (Gaussian and double *t*), $\theta = 0$ (Clayton) and for $\alpha = 0$ (Marshall-Olkin).

It is well known that the factor structure and the exchangeability property lead to "positive dependence". For instance, one can easily state that the covariances between default indicators $\mathrm{cov}\left(1_{\{\tau_i \le t\}}, 1_{\{\tau_j \le t\}}\right) = \mathrm{var}\left(p_t^{\cdot|V}\right) \ge 0$. Thus, it is not surprising that equity tranche premiums computed in factor models are lower than those computed under the assumption of independent default times.

One issue is whether the independence case is associated with a *model free* upper bound on equity tranche premiums. The answer is negative. For instance, let us consider a Gaussian copula with constant negative correlation equal to $-\frac{1}{n-1}$. This leads to admissible correlation matrix; as a consequence of Müller and Scarsini [2000], the corresponding copula is smaller than the product copula with respect to the supermodular order. Thus the equity tranche premium will be greater than when computed under the independence assumption.

From the previous remarks, we can state some important properties of base correlations. Whenever it exists, the base correlation is unique. This results from the monotonicity of equity tranche premiums with respect to the Gaussian correlation parameter stated in subsection III.1. However, in the case of negative association between default times, it may be that no base correlation can be found[18]. Since

[18] Let us consider the following counterexample involving three names with equal credit curves. We consider a Gaussian copula model such that the correlation between the first two names is equal to -100%. One could think of two competitors, only one could survive. Thus, $V_1 = V, V_2 = -V$. If we assume that marginal default probabilities $F_1(t), F_2(t)$ are less than 0.5, we can indeed check that only

base correlation may not exist, even for arbitrage-free CDO tranche premiums, it differs from the implied volatility in the Black-Scholes model.

III.4 Large portfolio approximations

Large portfolio approximations are well known in the credit portfolio field (see Vasicek [2002], Schönbucher [2002] or Schloegl and O'Kane [2005]). The Basel II agreement talks about "infinitely granular" portfolios. In this subsection, we show that true equity tranche premiums are smaller that those computed under a large portfolio approximation.

We now recall a useful result from Dhaene *et al*. [2002]. Let $Z=(Z_1,\ldots,Z_n)$ be a random vector and V a random variable. Then:

$$E\left[Z_1\left|V\right.\right]+\ldots E\left[Z_n\left|V\right.\right]\leq_{cx} Z_1+\ldots+Z_n ,$$

where $\leq_{cx}$ is the convex order[19].

Let us apply this result to the credit case. Here, $Z_i=M_i 1_{\{\tau_i\leq t\}}$ and $L(t)=Z_1+\ldots+Z_n$ are respectively the individual loss on name i and the aggregated loss at time t. As above M_i denotes the deterministic loss given default on name i. We have: $E\left[Z_i\left|V\right.\right]=M_i p_t^{i|V}$ where $p_t^{i|V}=Q\left(\tau_i\leq t\left|V\right.\right)$ denotes the conditional default probability of name i. Then, the approximation of the loss is provided by $E\left[L(t)\left|V\right.\right]=\sum_{i=1}^{n}M_i p_t^{i|V}$ which is a deterministic function of the factor V [20]. As a consequence the computation of the expected loss on an equity tranche $E\left[\min\left(K,\sum_{i=1}^{n}M_i p_t^{i|V}\right)\right]$ can be done by a simple quadrature without any inversion of the characteristic function or recursion techniques. Moreover, since $\sum_{i=1}^{n}M_i p_t^{i|V}\leq_{cx} L(t)$, we have $E\left[\min\left(K,L(t)\right)\right]\leq E\left[\min\left(K,\sum_{i=1}^{n}M_i p_t^{i|V}\right)\right]$. Thus, the true value of the

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one of the first two names can default: $\tau_1\leq t\Leftrightarrow V\leq\Phi^{-1}\left(F_1(t)\right)<0$ and $\tau_2\leq t\Leftrightarrow -V\leq\Phi^{-1}\left(F_2(t)\right)<0$. This implies that $\{\tau_1\leq t\}\cap\{\tau_2\leq t\}=\varnothing$. The third name is uncorrelated with the first two names: $V_3=\overline{V}_3$. The nominals are equal to 1 for the first two names and 0.5 for the third name. We assume zero recoveries. Let us consider a $[1.5-3]$ senior tranche. Since names 1 and 2 cannot default altogether, the maximal loss on the credit portfolio is equal to 1.5. Thus, the premium associated with the previous tranche is equal to zero. On the other hand, the lowest admissible flat correlation is -50%. For smaller values, the covariance matrix would not be semi-definite positive. Thanks to the previous ordering results on Gaussian vectors, such a correlation structure leads to the lowest senior tranche premium consistent with a flat correlation matrix. Let us remark that there is a positive probability that names 1 and 2 default altogether leading to a loss of at least 0.5 on the $[1.5-3]$ tranche. As a consequence, the senior tranche premium is positive for any base correlation. Since the arbitrage free premium of the senior tranche is equal to zero, it is not possible to find a base correlation (even allowing for negative base correlations) that matches this premium. Of course, this case is rather unlikely, but it shows that base correlation cannot be assimilated to implied volatility which is always defined.

[19] Let X and Y be two random variables. We say that X is smaller than Y with respect to the convex order and we denote $X\leq_{cx}Y$ if $E\left[f(X)\right]\leq E\left[f(Y)\right]$, for all convex functions such that the expectations are well defined.

[20] If we assume that the Z_i's are independent conditionally upon V and identically distributed, we have: $\frac{1}{n}\sum_{i=1}^{n}Z_i\underset{a.s.}{\rightarrow}E\left[Z_i\left|V\right.\right]$. The right-hand term is known as the large homogeneous portfolio approximation.

default leg of an equity tranche is smaller than the one computed under the large portfolio approximation. Using the same reasoning, we also have $E\left[\left(K-\sum_{i=1}^{n}M_i p_t^{i|V}\right)^+\right]\le E\left[\left(K-L(t)\right)^+\right]$.
Therefore, the true value of the premium leg on an equity tranche is larger than the one computed under the large portfolio approximation. We conclude that true equity tranche premiums are smaller that those computed under a large portfolio approximation. Clearly, this is a model dependent upper bound.

III.5 The case of basket default swaps

Let us consider the case of a homogeneous first to default swap, i.e. all names have the same nominal and recovery rate[21]. It can be treated as a homogeneous CDO equity tranche with detachment point equal to the common loss given default. Thus, the previous results stated for CDO tranches apply. For instance, increasing the correlation parameter in the one factor Gaussian copula model always leads to a decrease in the first to default swap premium.

IV) Comparing Basket Default Swaps and CDO premiums

In order to conduct model comparisons, we proceeded the following way. Since the studied copulas depend upon a one dimensional parameter, we have chosen that parameter so that either first to default (for basket default swaps) or equity tranches premiums (for CDO tranches) are equal. Such a correspondence between parameters is meaningful since equity tranche premiums are monotonic with respect to the relevant dependence parameter (see previous section). We then compute the premiums of basket default swaps and various CDO tranches and study the departures between the different models and also between model and market quotes.

IV.1 First to default swaps with respect to the number of names

We firstly computed first to default swap premiums under different models as a function of the number of names in the basket, from 1 to 50. We assumed flat and equal CDS premiums of 80 bps, recovery rates of 40% and 5 year maturity. The default free rates are provided in the appendix. The dependence parameters were set to get equal premiums for 25 names. They are reported in Table 1.

	Gaussian	Student (6)	Student (12)	Clayton	MO
dependence	$\rho^2=30\%$	$\rho^2=11.9\%$	$\rho^2=21.6\%$	$\theta=0.173$	$\alpha=49\%$

Table 1: dependence parameters for the pricing of first to default swaps.

Table 2 reports the first to default premiums. Let us remark that Gaussian, Student *t* and Clayton copulas lead to quite similar premiums, while the Marshall-Olkin deviate quite significantly. The second line in the table corresponds to a plain CDS on a single name and thus all models provide the same input premium of 80 bps. We can also notice that the premiums always increase with the number of names[22].

Names	Gaussian	Student (6)	Student (12)	Clayton	MO
1	80	80	80	80	80
5	332	339	335	336	244
10	567	578	572	574	448
15	756	766	760	762	652
20	917	924	920	921	856
25	1060	1060	1060	1060	1060
30	1189	1179	1185	1183	1264

[21] We refer to Laurent and Gregory [2005], for a general analysis of basket default swaps.
[22] This feature is model independent: the survival function of first to default time in a homogeneous basket is given by: $S_n^1(t)=Q\left(\tau_1>t,\ldots,\tau_n>t\right)\ge Q\left(\tau_1>t,\ldots,\tau_n>t,\tau_{n+1}\ge t\right)=S_{n+1}^1(t)$.

35	1307	1287	1298	1294	1468
40	1417	1385	1403	1397	1672
45	1521	1475	1500	1492	1875
50	1618	1559	1591	1580	2079

Table 2: First to default premiums with respect to the number of names (bps pa).

Table 3 provides Kendall's τ for the different models. As can be seen, even once the models have been calibrated on a first to default swap premium with 25 names, the non linear correlations are quite different.

	Gaussian	Student (6)	Student (12)	Clayton	MO
ρ_K	19%	8%	14%	8%	32%

Table 3: Kendall's τ for the studied models.

IV.2 *k*-th to default swaps

We then considered 10 names with credit spreads evenly distributed between 60 bps and 150 bps, a constant recovery rate of 40% and maturity still equal to 5 years. Table 4 reports the dependence parameters. They are set so that the first to default premiums are equal for all models.

Gaussian	Clayton	Student (6)	Student (12)	MO
$\rho^2 = 30\%$	$\theta = 0.1938$	$\rho^2 = 16.5\%$	$\rho^2 = 23.6\%$	$\alpha = 36\%$

Table 4: dependence parameters for the pricing of *k*-th to default swaps.

The columns of Table 5 provides first, second… until last to default premiums. As in the previous example, the differences between Gaussian, Student *t* and Clayton copulas are minor while the Marshall-Olkin copula leads to strikingly different results for higher order basket default swaps.

Rank	Gaussian	Clayton	Student (6)	Student (12)	MO
1	723	723	723	723	723
2	275	274	278	276	173
3	122	123	122	122	71
4	55	56	55	55	56
5	24	25	24	25	55
6	11	11	10	10	55
7	4.7	4.3	3.5	4.0	55
8	1.5	1.5	1.1	1.3	55
9	0.39	0.39	0.25	0.35	55
10	0.06	0.06	0.04	0.06	55

Table 5: First to last to default swap premiums (bps pa) for different models.

Once again, Kendall's τ is poorly related to the premium structure (see Table 6).

	Gaussian	Clayton	Student (6)	Student (12)	MO
ρ_K	19%	9%	11%	15%	22%

Table 6: Kendall's τ for the studied models.

IV.3 CDO tranche premiums under different models

As a practical example, we considered 100 names, all with a recovery rate of $\delta = 40\%$ and equal unit nominal. The credit spreads are all equal to 100 bps. They are assumed to be constant until the maturity of the CDO. The attachment points of the tranches are $A = 3\%$ and $B = 10\%$. The CDO maturity is equal to five years. The default free rates are provided in the appendix.

We considered CDO margins for equity, mezzanine and senior tranches[23] for the different models. We firstly considered the Gaussian model and computed the margins with respect to the correlation parameter ρ^2. These results show a strong negative dependence of the equity tranche with respect to the correlation parameter, a positive dependence of the senior tranche and a bumped curve for the mezzanine, which is not as sensitive to the correlation parameter.

ρ^2	equity	mezzanine	Senior
0 %	5341	560	0.03
10 %	3779	632	4.6
30 %	2298	612	20
50 %	1491	539	36
70 %	937	443	52
100%	167	167	91

Table 7: CDO margins (bp pa) Gaussian copula with respect to the correlation parameter.

In order to compare the different pricing models, we set the dependence parameters to get the same equity tranche premiums. This gives the following correspondence table:

ρ^2	0%	10%	30%	50%	70%	100%
θ	0	0.05	0.18	0.36	0.66	∞
ρ_6^2			14%	39%	63%	100%
ρ_{12}^2			22%	45%	67%	100%
ρ^2 $t(4)$-$t(4)$	0%	12%	34%	55%	73%	100%
ρ^2 $t(5)$-$t(4)$	0%	13%	36%	56%	74%	100%
ρ^2 $t(4)$-$t(5)$	0%	12%	34%	54%	73%	100%
ρ^2 $t(3)$-$t(4)$	0%	10%	32%	53%	75%	100%
ρ^2 $t(4)$-$t(3)$	0%	11%	33%	54%	73%	100%
α	0	27%	53%	68%	80%	100%

Table 8: correspondence between parameters for the pricing of CDO tranches.

For instance, when the Gaussian copula parameter is equal to 30%, we must set the Clayton copula parameter to 0.18 in order to get the same equity tranche premium[24].

Once the equity tranches were matched, we computed the premiums of the mezzanine and senior tranche with the different models. It can be seen that Clayton and Student *t* provide results that are close to the Gaussian case. For instance, for a Gaussian correlation of 30%, the senior tranche premium computed under the Gaussian assumption is equal to 20bps, while we obtained 18 bps under the Clayton assumption and 19 bps with a Student *t* with 12 degrees of freedom.

ρ	0%	10%	30%	50%	70%	100%
Gaussian	560	633	612	539	443	167
Clayton	560	637	628	560	464	167
Student (6)			637	550	447	167
Student (12)			621	543	445	167
$t(4)$-$t(4)$	560	527	435	369	313	167

[23] Corresponding to $[0-3\%]$, $[3-10\%]$ and $[10-100\%]$ tranches.

[24] We could not match the independence case with the Student *t* copula. Even for a zero correlation parameter, there is still tail dependence. As a consequence, no correlation parameter in the Student *t* copula allows a fit to the equity tranche premium computed under Gaussian copula and correlation equal to 0 or 10%.

t(5)-t(4)	560	545	454	385	323	167
t(4)-t(5)	560	538	451	385	326	167
t(3)-t(4)	560	495	397	339	316	167
t(4)-t(3)	560	508	406	342	291	167
MO	560	284	144	125	134	167

Table 9: mezzanine tranche premiums (bps pa) computed under the various models for different levels of Gaussian copula correlation.

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0.03	4.6	20	36	52	91
Clayton	0.03	4.0	18	33	50	91
Student (6)			17	34	51	91
Student (12)			19	35	52	91
t(4)-t(4)	0.03	11	30	45	60	91
t(5)-t(4)	0.03	10	29	45	59	91
t(4)-t(5)	0.03	10	29	44	59	91
t(3)-t(4)	0.03	12	32	47	71	91
t(4)-t(3)	0.03	12	32	47	61	91
MO	0.03	25	49	62	73	91

Table 10: senior tranche premimus (bps pa) computed under the various models for different levels of Gaussian copula correlation.

As for the basket default swap premiums, the premiums computed under the Marshall-Olkin copula are fairly different, except of course for the extreme cases of independence and comonotonicity. The double t model lies between these two extremes, i.e. Gaussian and Marshall-Olkin copulas.

Let us now consider a non-parametric measure of dependence such as Kendall's τ. We used the analytical formulas for the Gaussian, Clayton, Student and Marshall-Olkin copulas. Table 11 shows that the level of dependence associated with the Marshall-Olkin copula is bigger than in the Gaussian, Clayton or Student t copulas. Though Gaussian and Clayton copulas lead to similar CDO premiums, Kendall's τ are quite different.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	0%	6%	19%	33%	49%	100%
Clayton	0%	3%	8%	15%	25%	100%
Student (6)			9%	25%	44%	100%
Student (12)			14%	30%	47%	100%
MO	0%	16%	36%	52%	67%	100%

Table 11: Kendall's τ (%) for the studied models and for different levels of Gaussian copula correlation.

Let us remark that Kendall's τ increases with the correlation parameter. Since the copulas are positively ordered with respect to the dependence parameter, $\theta_1 \leq \theta_2$ implies that $\rho_{K,C_{\theta_1}} \leq \rho_{K,C_{\theta_2}}$ where $\rho_{K,C}$ denotes Kendall's τ associated with copula C. Moreover, $\rho_{K,C^+} = 1$.

Table 12 provides the tail dependence coefficients associated with the different models. The different columns in the table correspond to the different Gaussian correlation coefficients involved in the previous tables, i.e. 0%, 10%, 30%, 50%, 70% and 100%. Since the copulas are positively ordered with respect to the dependence parameter (as a consequence of the supermodular order), $\theta_1 \leq \theta_2$ implies $C_{\theta_1} \prec C_{\theta_2}$ which in turn implies that the tail dependence coefficients are positively ordered with respect to the relevant dependence parameter. We can check the increase of the tail dependence coefficients from 0 to 100% on each row.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	0%	0%	0%	0%	0%	100%
Clayton	0%	0%	2%	15%	35%	100%
Student (6)			5%	12%	25%	100%
Student (12)			1%	4%	13%	100%
t(4)-*t*(4)	0%	0%	1%	10%	48%	100%
t(5)-*t*(4)	0%	0%	0%	0%	0%	100%
t(4)-*t*(5)	0%	100%	100%	100%	100%	100%
t(3)-*t*(4)	0%	100%	100%	100%	100%	100%
t(4)-*t*(3)	0%	0%	0%	0%	0%	100%
MO	0%	27%	53%	68%	80%	100%

Table 12: coefficient of lower tail dependence (%) for the studied models and for different levels of Gaussian copula correlation.

It can be noticed that for a typical 30% Gaussian correlation, the level of tail dependence is rather small for Gaussian, Clayton and Student *t* copulas. This is also the case for the *t*(4)-*t*(4) model which however leads to quite different senior tranche premiums. The tail dependence is much bigger for the Marshall-Olkin copula. The previous table shows no obvious link between tail dependence and the price of the senior tranche. The reason for this is rather simple. It can be seen that the probability of a default payment occurring on the senior tranche over a 5 year time horizon, $Q\left(L(5) \geq 10\%\right) \simeq 30\%$. Thus, we are still far way from the tail of the loss distribution.

We also considered the bivariate default probabilities corresponding to the CDO maturity, $Q\left(\tau_i \leq 5, \tau_j \leq 5\right)$ for $i \neq j$. From the symmetry of the distributions, these do not depend of the chosen couple of names. The univariate default probability for a five years horizon is $Q\left(\tau_i \leq 5\right) = 8.1\%$. In the independence case, the bivariate default probability is $\left(8.1\%\right)^2 = 0.66\%$. The bivariate default probabilities are very close for the Gaussian, Clayton and Student *t* copulas. We have stronger bivariate default probabilities for the double *t* models and even larger for the Marshall-Olkin copula. Let us remark that since the marginal default probabilities are given, the variance of the loss distribution and the linear correlation between default indicators only involve the bivariate default probability. The larger the bivariate default probabilities, the larger will be the variance of the loss distribution and the linear correlation between default indicators.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	0.66%	0.91%	1.54%	2.41%	3.59%	8.1%
Clayton	0.66%	0.88%	1.45%	2.24%	3.31%	8.1%
Student (6)			1.41%	2.31%	3.52%	8.1%
Student (12)			1.49%	2.36%	3.56%	8.1%
t(4)-*t*(4)	0.66%	1.22%	2.38%	3.49%	4.67%	8.1%
t(5)-*t*(4)	0.66%	1.16%	2.27%	3.38%	4.57%	8.1%
t(4)-*t*(5)	0.66%	1.18%	2.28%	3.37%	4.54%	8.1%
t(3)-*t*(4)	0.66%	1.34%	2.57%	3.69%	5.02%	8.1%
t(4)-*t*(3)	0.66%	1.31%	2.55%	3.70%	4.87%	8.1%
MO	0.66%	2.63%	4.53%	5.65%	6.53%	8.1%

Table 13: bivariate default probabilities (5 year time horizon) for the studied models and for different levels of Gaussian copula correlation.

To further study some possible discrepancies between Gaussian, Clayton and Student *t* copulas, we kept the previous correspondence table between parameters and recomputed the tranche premiums for different input credit spreads. We want here to check whether the Gaussian copula can provide a good fit to Clayton and Student *t* copula premiums uniformly over credit spread curves. In tables 14, 15, 16 below, credit spreads have been shifted from 100 bps to 120 bps.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	6476	4530	2695	1731	1085	200
Clayton	6476	4565	2748	1781	1132	200
Student (6)			2765	1765	1104	200
Student (12)			2730	1748	1093	200

Table 14: equity tranche premiums (bps pa) after a shift of credit spreads.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	853	857	765	652	527	200
Clayton	853	867	794	687	564	200
Student (6)			807	672	537	200
Student (12)			782	661	531	200

Table 15: mezzanine tranche premiums (bps pa) after a shift of credit spreads.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	0.2	8	28	46	64	109
Clayton	0.2	7	25	42	60	109
Student (6)			23	44	63	109
Student (12)			26	45	64	109

Table 16: senior tranche premiums (bps pa) after a shift of credit spreads.

We can see that the same set of parameters still enables to provide quite similar premiums for the different models, especially for the senior tranche. These overall results are not surprising keeping in mind the results in Greenberg *et al*. [2004]. Demarta and McNeil [2005] also use some proximity between the *t*-EV copula and Gumbel or Galambos copulas for suitable choices of parameters. Breymann *et al*. [2003] show some similarity between Student *t* and Clayton copulas as far as extreme returns are concerned[25].

IV.4 Market and model CDO tranche premiums

While the previous results relied on constant credit spreads, we now consider another example related to the Dow Jones iTraxx Europe index. The CDO maturity is equal to five years. The attachment detachment points correspond to the standard iTraxx CDO tranches, i.e. 3%, 6%, 9%, 12% and 22%. The index is based on 125 names. The 5 year credit spreads of the names lie in between 9 bps and 120 bps with an average of 29 bps and a median of 26 bps. The credit spreads and the default free rates are detailed in the appendix. To ease comparisons, we assumed constant credit spreads with respect to maturity.

We discuss the ability of each copula to produce a smile on pricing tranches on this index as is observed in the market. We calibrated the different models on the market quote for the [0-3%] equity tranche. The parameters used for the stochastic correlation model were $\gamma^2 = 6.6\%$ with probability 0.66, $\beta^2 = 20\%$ with probability 0.1 and $\rho^2 = 80\%$ with probability 0.24. Better fits are presumably possible as we did not perform an optimization to match the market prices. Let us remark that we could not fit a Student *t* model with 6 degrees of freedom on the equity tranche market quote. We provide results both for tranches as quoted in the market and for “equity type” tranches.

Tranches	Market	Gaussian	Clayton	Student (12)	*t*(4)-*t*(4)	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[3-6%]	101	163	163	164	82	122	14
[6-9%]	33	48	47	47	34	53	11

[25] This also shows that the dynamics of the credit spreads implied by the copula is not relevant for the pricing of CDOs. From Schönbucher and Schubert [2001], we know that Gaussian and Clayton copulas differ quite significantly from that point of view.

[9-12%]	16	17	16	15	22	29	11
[12-22%]	9	3	2	2	13	8	11

Table 17: iTraxx CDO tranche premiums (bps pa) using market and model quotes.

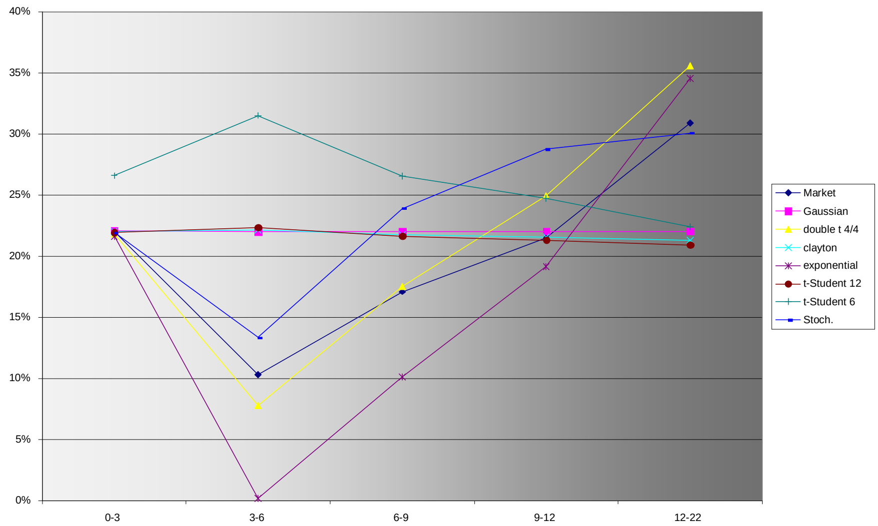
Tranches	Market	Gaussian	Clayton	Student (12)	*t*(4)-*t*(4)	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[0-6%]	466	503	504	504	456	479	418
[0-9%]	311	339	339	340	305	327	272
[0-12%]	233	253	253	254	230	248	203
[0-22%]	128	135	135	135	128	135	113

Table 18: iTraxx "equity tranche" CDO premiums (bps pa) using market and model quotes.

Most practitioners deal with implied Gaussian correlation, that is the flat correlation in the one factor Gaussian copula model associated with a given premium. Table 19 and Graph 1 show that correlation parameters are smaller for mezzanine tranches leading to a so called "correlation smile". Friend and Rogge [2005], Greenberg *et al*. [2004], Finger [2005] also report such an effect meaning that the Gaussian copula fails to price exactly the observed prices of iTraxx tranches. It can be seen that Clayton or Student *t* copulas are still close to Gaussian and thus do not create any correlation smile. This is consistent with previous empirical studies (see also Schönbucher [2002], Schloegl and O'Kane [2005]). The Marshall-Olkin model underestimates the prices of the mezzanine tranches and overestimates the super senior. The double *t* model provides a better overall fit but overestimates the senior tranches. The stochastic correlation model fits reasonably to the market prices, in particular the equity and junior super senior. It overestimated the mezzanine tranche premiums and would therefore underestimate the super senior [22-100%] region. This could be associated to the lack of extreme or a fat tail risk on the loss distribution.

Tranches	Market	Gaussian	Clayton	Student (12)	*t*(4)-*t*(4)	Stoch.	MO
[0-3%]	22%	22%	22%	22%	22%	22%	22%
[3-6%]	10%	22%	22%	22%	8%	13%	0%
[6-9%]	17%	22%	22%	22%	18%	24%	10%
[9-12%]	22%	22%	23%	21%	25%	29%	19%
[12-22%]	31%	22%	21%	21%	36%	30%	35%

Table 19: implied compound correlation for iTraxx tranches.

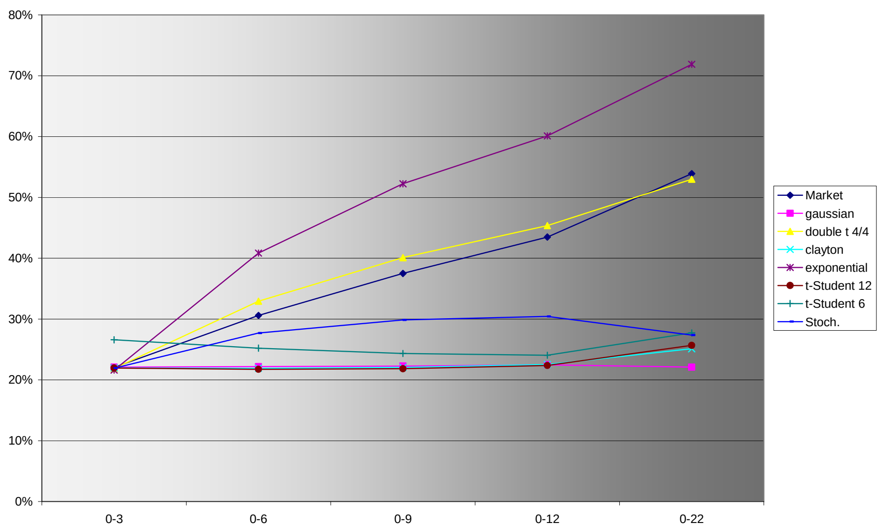


Graph 1: implied compound correlation for iTraxx CDO tranches based on market and model quotes. Tranches are on the x - axis, compound correlations on the y - axis.

Table 20 and Graph 2 show the “equity-type” implied correlations or “base correlations”. We believe the best criteria to assess the ability of a model to fit the market is the difference in compound correlation. The relative pricing error on each tranche should be reasonably close to this although there can be problems for tranches that are rather insensitive to correlation. Base correlation may not be appropriate because small mispricings lower on the capital structure cause dramatic deviations on high base correlation tranches. This can be seen in Graph 2 where reasonable fits to compound can be seen to look extremely poor in terms of their implied base correlations. For example in the stochastic correlation model, the [0-22%] mispricing on base correlation is 27% whereas the [12-22%] tranche is priced within 1bp.

Tranches	Market	Gaussian	Clayton	Student (12)	$t(4)$-$t(4)$	Stoch.	MO
[0-3%]	22%	22%	22%	22%	22%	22%	22%
[0-6%]	31%	22%	22%	22%	33%	28%	41%
[0-9%]	37%	22%	22%	22%	40%	30%	52%
[0-12%]	43%	22%	23%	23%	45%	30%	60%
[0-22%]	54%	22%	25%	26%	53%	27%	72%

Table 20: implied base correlation for iTraxx tranches.



Graph 2: implied base correlation for iTraxx CDO tranches computed from market and model quotes. Tranches are on the x - axis, base correlations on the y - axis.

IV.5 Conditional default probability distributions drive CDO tranche premiums

The pricing of basket default swaps or CDOs only involve loss distributions over different time horizons. The characteristic function of the aggregate loss only involves the conditional default probabilities $p_t^{i|V}$. When these are identically distributed, the characteristic function can be written as:

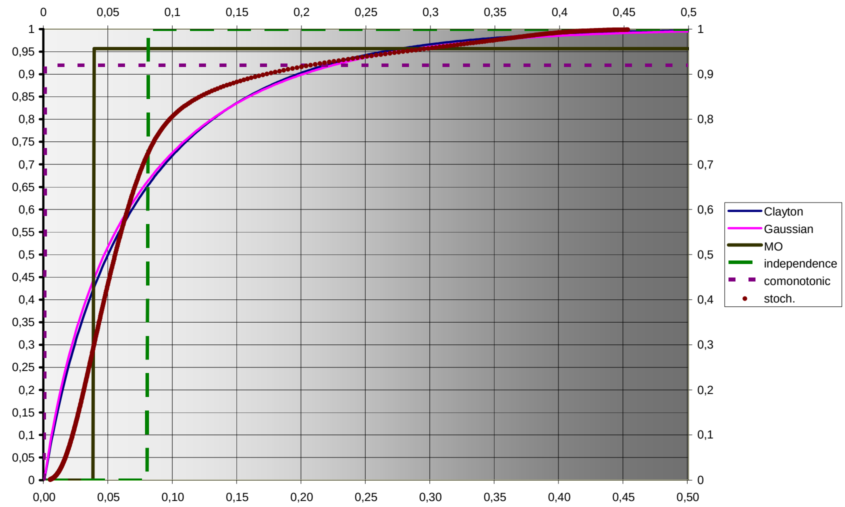
$$\varphi_{L(t)}(u) = \int \prod_{1 \le j \le n} \left(1 - p + pe^{iuM_j}\right) G(dp),$$

where G is the distribution function of the conditional default probabilities[26]. In other words, two models associated with the same distributions of conditional default probabilities will lead to the same joint distribution of default indicators and eventually to the same CDO premiums. As an example, let us consider Gaussian, stochastic correlation, Clayton and Marshall-Olkin copulas. We have

[26] $G(p) = Q\left(p_t^{i|V} \le p\right)$ for $0 \le p \le 1$.

$$p_t^{i|V} = \Phi\left(\frac{-\rho V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho^2}}\right), \qquad p_t^{i|V} = p\Phi\left(\frac{-\rho V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho^2}}\right) + (1-p)\Phi\left(\frac{-\beta V + \Phi^{-1}(F_i(t))}{\sqrt{1-\beta^2}}\right), \qquad V$$

Gaussian for the Gaussian and stochastic correlation copulas, $p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$, V standard Gamma for the Clayton copula and $p_t^{i|V} = 1 - 1_{V > -\ln S_i(t)} S_i(t)^{1-\alpha}$, V exponential for the Marshall-Olkin copula.



Graph 3: distribution functions of conditional default probabilities for different models.

Let us go back to the previous CDO example with flat credit curves of 100bps. For a Gaussian correlation of 30%, the correspondence table gives $\theta = 0.18$ and $\alpha = 53\%$. Graph 3 provides the distribution functions of the 5 year conditional default probabilities. It can be seen that the distribution functions are almost identical in the Gaussian and Clayton copula cases. For the Marshall-Olkin copula, the conditional default probability only takes values 1 and $1 - S_i(t)^{1-\alpha}$ which leads to a step distribution function. The independence case is associated with a Dirac mass at the marginal default probability while the conditional default probability is a Bernoulli variable in the comonotonic case. It is quite clear that the differences between Marshall-Olkin copula on one hand, Gaussian and Clayton copulas on the other hand are quite substantial. We also provide the distribution of conditional default probabilities for a stochastic correlation model. Here, $\beta^2 = 10\%$ with probability 0.8 and $\rho^2 = 90\%$ with probability 0.2. We can see that the stochastic correlation model lies in between Marshall-Olkin and Gaussian. An interesting area of research consists in building the distribution of $p_t^{i|V}$ from the market prices which could give some insight on choice of model. Such construction can be found in Hull and White [2006]. The practical relevance of conditional default probabilities is also emphasized in Burtschell *et al.* [2007]. A general investigation of the use of conditional default probabilities in the pricing of CDOs and connexions with the theory of stochastic orders is done in Cousin and Laurent [2007].

Conclusion

We discussed the choice of dependence structure in basket default swap and CDO modelling. We compared some popular copula models against the one factor Gaussian copula that is currently the industry standard. We considered an assessment methodology based on the matching of basket default swap premiums and CDO tranches:

- The results show that for pricing purposes, and once correctly calibrated, Student t and Clayton copula models provide rather similar results, close to the Gaussian copula.

- The Marshall-Olkin copula associated with large probabilities of simultaneous jumps leads to strikingly different results and a dramatic fattening of the tail of the loss distributions.
- The double *t* model lies in between and provides a better fit to market quotes. We found that related models such as the random factor loadings model of Andersen and Sidenius [2005] led to similar correlation smiles.
- The stochastic correlation copula can also achieve a reasonable skew, close to that observed in the market.
- Non parametric measures of dependence, such as Kendall's τ or the tail dependence coefficient are of little help for explaining model quotes.
- The distribution of the conditional default probability is the key input when pricing CDO tranche premiums and when comparing different models.

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Appendix

1) Data for the Basket default swaps and CDO examples

Basket default swaps and homogeneous CDO examples

1D	1W	1M	2M	3M	6M	9M	1Y	2Y	3Y	4Y	5Y
2.02	2.05	2.06	2.07	2.08	2.14	2.23	2.37	2.80	3.17	3.47	3.71

Table a: default free yield curve (continuous rates)

iTraxx example

9	14	17	20	21	23	25	28	31	34	37	45	68
10	14	18	20	21	23	25	28	31	35	37	45	72
10	15	18	20	21	23	26	28	32	35	37	46	73
10	15	18	20	21	23	26	28	33	35	38	47	106
10	15	18	20	22	24	26	29	33	35	38	48	120
10	15	18	20	22	24	26	30	33	36	40	51	
10	16	18	21	22	24	27	30	34	36	43	52	
10	17	18	21	22	45	27	30	34	36	44	53	
13	17	19	21	23	25	27	31	34	37	44	56	
13	17	19	21	23	25	27	31	34	37	44	58	

Table b: 5 year credit spreads iTraxx Europe

The default free rates were obtained from the swap market in Euros on the 08/02/2005.

1D	1W	1M	2M	3M	6M	9M	1Y	2Y	3Y	4Y	5Y
2.07	2.09	2.10	2.12	2.14	2.18	2.24	2.34	2.59	2.78	2.93	3.06

Table c: default free yield curve (continuous rates)

2) Supermodular ordering and stochastic correlation model

Let $p \le p'$ and consider the following model:

$$V_i = \min(C_i, D_i)\left(\rho V + \sqrt{1-\rho^2}\bar{V}_i\right) + \left(1-\min(C_i, D_i)\right)\left(\beta V + \sqrt{1-\beta^2}\bar{V}_i\right)$$

where $C_1,\ldots,C_n,D_1,\ldots,D_n,V,\bar{V}_1,\ldots,\bar{V}_n$ are all independent, $C_1,\ldots,C_n$ are Bernoulli variables with parameter $\frac{p}{p'}$ and $D_1,\ldots,D_n$ are Bernoulli variables with parameter p'. $\min(C_i,D_i)$ is a Bernoulli variable with parameter p. As a consequence, $(V_1,\ldots,V_n)$ follow a stochastic correlation model with parameters (ρ,β,p). We now compare with:

$$W_i = D_i\left(\rho' V + \sqrt{1-\rho'^2}\bar{V}_i\right) + (1-D_i)\left(\beta' V + \sqrt{1-\beta'^2}\bar{V}_i\right),$$

where $\rho \le \rho' \le 1, \beta \le \beta' \le 1$. $(W_1,\ldots,W_n)$ follows a stochastic correlation model with parameters (ρ',β',p'). $(W_1,\ldots,W_n)|C_1,\ldots,C_n,D_1,\ldots,D_n$ is Gaussian with correlation parameter $\rho' D_i + \beta'(1-D_i)$, while $(V_1,\ldots,V_n)|C_1,\ldots,C_n,D_1,\ldots,D_n$ is Gaussian with correlation parameter $\rho\min(C_i,D_i)+\beta(1-\min(C_i,D_i))$. Since $\rho\min(C_i,D_i)+\beta(1-\min(C_i,D_i)) \le \rho' D_i + \beta'(1-D_i)$, we have: $(V_1,\ldots,V_n)|C_1,\ldots,C_n,D_1,\ldots,D_n \le_{\text{sm}} (W_1,\ldots,W_n)|C_1,\ldots,C_n,D_1,\ldots,D_n$. From the invariance of supermodular order under mixing: $(V_1,\ldots,V_n) \le_{\text{sm}} (W_1,\ldots,W_n)$. Thus increasing the probability of being in the high correlation state p, or increasing any of the two correlation parameters ρ,β leads to an increase in dependence with respect to the supermodular order.

3) Supermodular ordering and Marshall-Olkin copula

Since the supermodular ordering is invariant under increasing transforms, we will consider the latent variables V_i. When $\alpha = 0$, these are independent and when $\alpha = 1$, there are comonotonic. We want to address the dependence of the vector of default times $(V_1,\ldots,V_n)$ with respect to α. Intuitively, increasing α gives more relative importance to the common shock V and should be associated with an increased dependence.

We set $\beta \ge \alpha$. We denote by $(V_1',\ldots,V_n')$ the latent variables associated with parameter β. In a distributional sense, we can equivalently write:

$$(V_1',\ldots,V_n') \equiv \left(\min\left(V,\hat{V},\bar{V}_1'\right),\ldots,\min\left(V,\hat{V},\bar{V}_n'\right)\right),$$

Where $V,\hat{V},\bar{V}_1',\ldots,\bar{V}_n'$ are independent exponential random variables with parameters equal to: $\alpha,\beta-\alpha,1-\beta,\ldots,1-\beta$. Let us remark that $\left(\min\left(t,\hat{V},\bar{V}_1'\right),\ldots,\min\left(t,\hat{V},\bar{V}_n'\right)\right)$ and $\left(\min\left(t,\bar{V}_1\right),\ldots,\min\left(t,\bar{V}_n\right)\right)$ have the same marginal distributions for all *t*, since $\min\left(\hat{V},\bar{V}_i'\right)$ are independent exponential random variables with parameter $1-\alpha$ and $\min\left(t,\hat{V},\bar{V}_i'\right) = \min\left(t,\min\left(\hat{V},\bar{V}_i'\right)\right)$. Morevover $\min\left(t,\hat{V},\bar{V}_i'\right)$ is increasing in $\hat{V}$. Thus, this corresponds to model 3.2 in Bäuerle and Müller [1998]. We can then conclude that:

$$\left(\min\left(V,\bar{V}_1\right),\ldots,\min\left(V,\bar{V}_n\right)\right) \le_{sm} \left(\min\left(V,\hat{V},\bar{V}_1'\right),\ldots,\min\left(V,\hat{V},\bar{V}_n'\right)\right),$$

which means that increasing the dependence parameter α does indeed lead to an increase in the dependence between default times with respect to the supermodular order.

4) Premiums computed under the Student *t* copula and under the independence assumption.

As noticed before, a zero linear correlation ($\rho = 0$) in the Student *t* copula is not associated with the independence case. From the stated results on stochastic orders, we just know that it acts as a lower bound on first to default swaps or equity tranche premiums. Comparing with the independence case is a bit more involved. We remark that $\tau_i \le t \Leftrightarrow V_i \le t_v^{-1}(F(t)) = z$, where $F = F_i$ denotes the common marginal distribution of default times. For notational simplicity we have omitted the dependence of z with respect to t. Let us remark that for practical purpose, $z < 0$, corresponds to default probabilities smaller than 0.5. We concentrate on the zero correlation case. Since $V_i = \sqrt{W} \times \overline{V}_i$, we have $V_i \leq z \Leftrightarrow \frac{-z}{\sqrt{W}} + \overline{V}_i \leq 0$. By using Theorem 3.4 in Bäuerle and Müller [1998], we conclude that the default indicators are greater, with respect to the supermodular order than their independent counterparts.