VARIANCE OPTIMAL CAP PRICING MODELS

J.P. LAURENT[1] and O. SCAILLET[2]

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Abstract

We propose new closed-form pricing formulas for interest rate options which guarantee perfect compatibility with volatility smiles. These cap pricing formulas are computed under variance optimal measures in the framework of the market model or the Gaussian model and achieve an exact calibration of observed market prices. They are presented in a general setting allowing to study model and num¶raire choice e®ects on the computed prices. We show that price sensitivities of pricing formulas with respect to observed option prices are equal to mean-variance hedging portfolio holdings. We also show that the theoretical cap prices are equal to the cost of the hedging portfolio, thus closing the gap between pricing and hedging. A numerical example and an empirical application on market data are given to illustrate the practical use of the calibration procedure.

keywords : discount bond option, cap pricing formula, volatility smile, variance optimal measure, implied pricing model.

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[1]ISFA Actuarial School, University of Lyon and CREST. E-mail: laurent.jeanpaul@online.fr

[2]Institut d'Administration et de Gestion and D¶partement des Sciences Economiques, Universit¶ Catholique de Louvain. E-mail: scaillet@ires.ucl.ac.be

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Introduction

For accounting and regulatory purposes, assessing the value of a portfolio of interest rate derivative assets is one of the tasks carried out by ¯nancial institutions. Such ¯nancial assets are routinely used to tailor the interest rate risk exposure induced by existing deposits and loans, and are now part of standard ¯nancial products such as capped mortgage loans. Financial institutions such as investment banks must evaluate them in accordance with observed prices of liquid ¯nancial products, following the \mark-to-market" principle. For banks managing large portfolios, only a few option prices are observed and unobserved prices must thus be inferred in order to get the current market value of the book. Indeed the relevant information is di±cult and costly to obtain and only a small part of portfolios corresponds to assets whose prices are easily available. The unobserved option prices are in general rebuilt through the use of pricing models.

It is a well known feature that the standard Black-Scholes model is not consistent with observed option prices of di®erent exercise prices and leads to the presence of volatility smiles (see e.g. Bates (1996), Bakshi, Cao and Chen (1997)). As we show below, similar departures occur between standard interest rate option pricing models and quoted market prices of interest rate options. Recently some authors have raised the problem of calibrating pricing models to volatility smiles. In the case of stocks or exchange rates (Madan and Milne (1994), Rubinstein (1994), Buchen and Kelly (1996), Jackwerth and Rubinstein (1996), Magnien, Prigent and Trannoy (1996), Avellaneda et al. (1997), Laurent and Leisen (1998)), risk neutral densities can be extracted from a ¯nite set of observed option prices.

The bayesian implied approach initiated by Buchen and Kelly (1996), Jackwerth and Rubinstein (1996), consists in slightly modifying an a priori structural option pricing model so that a perfect calibration to all observed option prices can be achieved. The modi¯ed pricing model will thus be exactly calibrated and remain economically sound. The di®erence between the a priori and a posteriori models is similar to an error term due to omission of various e®ects and variables in the initial structural model. Unlike practices relying on interpolating volatility smiles, this approach is aimed to provide proper risk-neutral densities, i.e. positive and integrating to one. Once the risk-neutral densities have been obtained, the option prices are then computed by some numerical integration procedure. Thus, the bayesian implied approach provides a consistent pricing scheme.

Other implied approaches, commonly used by practitioners to build option pricing models face various di±culties. These are based upon Breeden and Litzenberger (1978) idea that a complete set of observed option prices fully determines the pricing kernel. Derman and Kani (1994), Rubinstein (1994), Dupire (1996) extend these approaches to the dynamic case. Since there are usually less traded options than states of the world, various artificial methods are used to fill in missing option prices. Shimko (1993) shows that interpolation of implied volatilities is likely to provide negative risk-neutral densities or densities which do not integrate to one. Derman, Kani and Chriss (1996) show that the existence of implied option pricing models cannot be guaranteed. Because of the artificial information put in the model, the hedging performance and out of sample consistency of these models are usually poor as shown in Dumas, Fleming and Whaley (1998).

The aim of this paper is to present an implied bayesian approach in the framework of interest rate options. Unlike previous studies, we consider both pricing and hedging issues.

1. As shown by Dumas, Fleming and Whaley (1998), the main weakness of the implied approach lies in its hedging performance. Either it is not considered at all, either the building of some implied risk-neutral density is related to some unrealistic assumption of complete markets. Our implied approach can be used to build hedging portfolios based on all traded options. The derivatives of the pricing formulas w.r.t. observed option prices are in fact equal to the portfolio holdings in the optimal mean-variance hedging portfolio. The caplet prices provided by the explicit pricing formula are shown to be equal to the cost of the hedging portfolio, thus closing the gap between pricing and hedging and providing a comprehensive model.

2. A characteristic of interest rate modeling is the importance of the change of numéraire technique (El Karoui and Rochet (1989), Geman, El Karoui and Rochet (1995), Jamshidian (1993, 1997)). We show here, by suitable changes of numéraires, that it is possible to directly derive explicit option pricing formulas consistent with observed volatility smiles. The retained a priori models are the well known market and Gaussian models.

3. Our implied approach is essentially static, i.e. does not allow for dynamic trading. The space of attainable claims is made of linear combinations of traded interest rate derivatives and is thus of finite dimension. The market is here incomplete unlike the standard case where prices of interest rate derivatives are uniquely determined and equal to the prices of dynamic self-financing replicating portfolios. There are both technical, economic and practical advantages to this static approach. First, our pricing formulas are consistent with volatility smiles by construction. In complete markets, the deterministic relationships between derivative and underlying asset prices are rejected by the data leading to the immediate conclusion that pricing models are misspecified. As far as hedging is concerned, dynamic portfolios are more sensitive to transaction costs than static ones. Moreover, the hedging properties of our static approach are

likely to be better since the hedge involves holding a portfolio of all observed liquid caplets of the same maturity. In practice it is very easy to implement and only requires to solve linear equations. Lastly, the static approach only relies on measure theory and does not require the sophisticated apparatus of semimartingale theory.

In Section 1, we recall market practice regarding Libor rates and present payoffs and numéraires that are commonly used in interest rate modeling such as FRAs, caps, ... It provides the basic notations and paves the way towards Sections 2 to 3.

In Section 2 we describe the implied model approach for the pricing and hedging of interest rate derivatives. We focus on the variance optimal measures of Schweizer (1992). We link the induced price with the approximation price, i.e. the price of the best mean-variance hedging portfolio, and give its sensitivities (deltas) w.r.t. observed prices.

In Section 3, we consider closed-form pricing formulas for interest rate options, based on variance optimal measures, that are consistent with observed volatility smiles. These pricing formulas are simple and easy to compute.

Eventually, Section 4 contains a numerical example and an empirical application based on real market data (DEM caplet prices). This section aims to show that the calibration procedure is straightforward to implement and can be used in real time applications to provide prices and hedging portfolios.

Concluding remarks are given in Section 5.

1 The market

We begin with the description of what forms the ground of our analysis. We focus on Libor rates (i.e. money market rates quoted among banks in London) and later consider asset payoffs that depend on these Libor rates. FRAs, interest rate swaps and caps on Libor rates are the main products on the OTC interest rate derivatives market. A typical plain vanilla cap on three month Libor will pay every three months the positive part of the difference between the current three month Libor and a predetermined rate called the exercise (or strike) rate until the expiration of the cap. Usual cap maturities vary from six months to ten years. The reference Libor is often a three month Libor, although other references are not scarce. This means that caps are long term options based on short term money market rates. The Libor may correspond to USD rates but other active markets exist in other currencies (for example, DEM). Each individual payment of a cap is a separate financial contract, named a caplet (see e.g. Brace, Gatarek and Musiela (1997), Jamshidian (1997), Musiela and Rutkowski (1997) for illustrations). Thus a cap is a collection of caplets. If one is able to price a caplet (the building block of a cap) the valuation of the cap is straightforward. Let us remark that caplets have to be distinguished from options on continuously compounded discount bond yields since the caplet payoffs are based on money market conventions (Longstaff (1990), Leblanc and Scaillet (1998)).

1.1 Libor rates and forward prices

We adopt the usual international conventions in terms of interest rate fixing and delivery dates for interbank deposits, three-month Libor swaps, and three-month Libor caps and swaptions. Typically, this time period has three dates, the first date τ_0 corresponding to the fixing of the Libor which prevails between the dates τ_1 and τ_2 ($\tau_0 < \tau_1 < \tau_2$). The date τ_2, i.e. the payment date of the Libor. In London, date τ_1 is two trading days after date τ_0. Time spaces between these dates may vary because of the presence of nontrading days and generally differ when one shifts to another underlying interest rate reference for the contracts.

Along the paper, we assume that discount bonds maturing at dates τ_2 and τ_1 are traded assets and that their prices are observed at date 0 and date τ_0. Libor rates and forward prices are defined in Table 1 where $B(\tau_0;\tau_2)$ is the price at date τ_0 of the discount bond delivering one money unit, say a Deutsche Mark (DEM) or a Euro, at date τ_2, $\delta = J(\tau_1;\tau_2)/360$ and $J(\tau_1;\tau_2)$ are the number of years and the number of days between τ_1 and τ_2, respectively.

Table 1 : Libor rates and forward prices

Libor at date τ_0	$x = x(\tau_0)$	$\delta^{-1}\dfrac{B(\tau_0;\tau_1) - B(\tau_0;\tau_2)}{B(\tau_0;\tau_2)}$
Forward Libor (date 0)	$x(0)$	$\delta^{-1}\dfrac{B(0;\tau_1) - B(0;\tau_2)}{B(0;\tau_2)}$
Forward price at date τ_0	$z^{-1} = B(\tau_0;\tau_1;\tau_2)$	$\dfrac{B(\tau_0;\tau_2)}{B(\tau_0;\tau_1)}$
Forward price at date 0	$B(0;\tau_1;\tau_2)$	$\dfrac{B(0;\tau_2)}{B(0;\tau_1)}$

The forward Libor $x(0)$ is the Libor which one is able to lock at date 0. The forward Libor $x(0)$ is thus equivalent to the fixed rate of an FRA (forward rate agreement) on the Libor with maturity date τ_0. The forward price at date τ_0 of a discount bond with maturity $\tau_2 - \tau_1$ at date τ_1 is given by the ratio : $B(\tau_0;\tau_1;\tau_2) = B(\tau_0;\tau_2)/B(\tau_0;\tau_1)$, and we deduce the relation between the inverse of the forward price z and Libor rate x :

$$z = 1 + \delta x. \tag{1.1}$$

In some cases writing numéraires, asset payoffs or risk neutral measures in terms of the inverse of the forward price will lead to simpler computations.

1.2 Numéraires and asset payoffs

An asset payoff is characterized by an amount paid in a given numéraire at a given date. For obvious practical purposes, there is often a delay between the time when the asset payoff is determined (the so-called fixing date) and its payment date. This can be viewed (through a standard discounting argument) as receiving at the fixing date a given amount of discount bonds maturing at the payment date. This means that the numéraire can precisely be this discount bond. In our context the fixing date is τ_0, and we consider amounts known at that time which depend on the Libor rate x. These amounts can be expressed in one of the following three useful numéraires in interest rate modeling:

- the discount bond maturing at τ_1 [the first numéraire U_1].
- the discount bond maturing at τ_2 [the second numéraire U_2].
- the "exchange principal asset" : one discount bond maturing at τ_1 in long (buy) position and one discount bond maturing at τ_2 in short (sell) position [the third numéraire U_3].

We denote by $U_i(\tau)$, the price at time τ; $\tau = 0; \tau_0$, of numéraire U_i :

$$\begin{cases} U_1(\tau) = B(\tau;\tau_1); \\ U_2(\tau) = B(\tau;\tau_2); \\ U_3(\tau) = B(\tau;\tau_1) - B(\tau;\tau_2). \end{cases}$$

Each numéraire will be used later in the derivation of the cap pricing formulas. In general they play a crucial role when considering various interest rate derivatives. Forward measures, Libor and dual swap measures (each of them related to a different numéraire) prove to be a key tool for computing pricing formulas for interest rate options such as caps and swaptions. A suitable choice of numéraire and clever use of exchange rates between numéraires often ease price computations (see Section 3.1). We also denote by $U_{i/j}(\tau)$ the exchange rate between numéraires U_i and U_j (the relative price of U_i w.r.t. U_j) at date τ :

$$U_{i/j}(\tau) = \frac{U_i(\tau)}{U_j(\tau)}; \ \tau = 0; \tau_0; \ i; j = 1; 2; 3. \tag{1.2}$$

From Table 1 we can directly deduce the exchange rates between the numéraires at date τ_0 (see Table 2) as functions of x, the Libor rate at date τ_0.

Table 2 : Exchange rates between numéraires (Libor)

	numéraire 1	numéraire 2	numéraire 3
numéraire 1	$U_{1/1}(\tau_0) = 1$	$U_{1/2}(\tau_0) = 1 + \delta x$	$U_{1/3}(\tau_0) = \dfrac{1+\delta x}{\delta x}$
numéraire 2	$U_{2/1}(\tau_0) = \dfrac{1}{1+\delta x}$	$U_{2/2}(\tau_0) = 1$	$U_{2/3}(\tau_0) = \dfrac{1}{\delta x}$
numéraire 3	$U_{3/1}(\tau_0) = \dfrac{\delta x}{1+\delta x}$	$U_{3/2}(\tau_0) = \delta x$	$U_{3/3}(\tau_0) = 1$

We are thus able to express the payoffs of standard financial contracts, such as caplets and digital caplets, in units of these three numéraires. Note that digital caplets are also traded on interest rate markets though less frequently than caplets. They are often used when customizing financial asset payoffs involving an interest rate guarantee. The payoff expressions are gathered in Table 3 where $(x - c)^+ = \max(0, x - c)$, $\mathbf{1}_{x(\tau_0) \geq c} = 1$ if $x(\tau_0) \geq c$ and 0 otherwise, g_i $(i = 1, 2, 3)$ is a real function, and c is the exercise rate.

Table 3 : Numéraires and asset payoffs (Libor)

	numéraire 1	numéraire 2	numéraire 3
Discount bond τ_1	1	$1 + \delta x$	$\dfrac{1+\delta x}{\delta x}$
Discount bond τ_2	$\dfrac{1}{1+\delta x}$	1	$\dfrac{1}{\delta x}$
FRA	$\dfrac{(x-c)\delta}{1+\delta x}$	$(x-c)\delta$	$\dfrac{(x-c)}{x}$
Caplet	$\dfrac{(x-c)^+\delta}{1+\delta x}$	$(x-c)^+\delta$	$\dfrac{(x-c)^+}{x}$
Digital Caplet	$\dfrac{\mathbf{1}_{x \geq c}}{1+\delta x}$	$\mathbf{1}_{x \geq c}$	$\dfrac{\mathbf{1}_{x \geq c}}{\delta x}$
General asset	$g_1(x)$	$g_2(x) = (1+\delta x) g_1(x)$	$g_3(x) = \dfrac{1+\delta x}{\delta x} g_1(x)$

These payoffs share the remarkable property that they only depend on the Libor rate x. It is clear that whatever the numéraire, the same amount of cash will be received at date τ_2. Let us also remark that for any given asset, the following numéraire invariance property holds :

$$g_i = U_{j/i}(\tau_0) g_j, \quad \forall i, j = 1, 2, 3. \tag{1.3}$$

From equation (1.1), we can express the exchange rates between the three numéraires at date τ_0 as functions of the inverse of the forward price z (see Table 4).

Table 4 : Exchange rates between numéraires (inverse of forward price)

	numéraire 1	numéraire 2	numéraire 3
numéraire 1	1	z	$\frac{z}{z-1}$
numéraire 2	$\frac{1}{z}$	1	$\frac{1}{z-1}$
numéraire 3	$\frac{z-1}{z}$	$z-1$	1

Similarly payoffs may be rewritten using the inverse of the forward discount bond price instead of the Libor. They are shown in Table 5 where :

$$\tilde{g}_i(z) = g_i((z-1)/\delta); \quad i = 1, 2, 3. \tag{1.4}$$

Table 5 : Numéraires and asset payoffs (inverse of forward price)

	numéraire 1	numéraire 2	numéraire 3
Discount bond τ_1	1	z	$\frac{z}{z-1}$
Discount bond τ_2	$\frac{1}{z}$	1	$\frac{1}{z-1}$
FRA	$\frac{z-(1+\delta c)}{z}$	$z-(1+c\delta)$	$\frac{z-(1+c\delta)}{z-1}$
Caplet	$\frac{(z-(1+\delta c))^+}{z}$	$(z-(1+c\delta))^+$	$\frac{(z-(1+c\delta))^+}{z-1}$
Digital Caplet	$\frac{\mathbf{1}_{z\geq 1+\delta c}}{z}$	$\mathbf{1}_{z\geq 1+\delta c}$	$\frac{\mathbf{1}_{z\geq 1+\delta c}}{z-1}$
General asset	$\tilde{g}_1(z)$	$\tilde{g}_2(z) = z\tilde{g}_1(z)$	$\tilde{g}_3(z) = \frac{z}{z-1}\tilde{g}_1(z)$

Let us remark that for any given asset, the following numéraire invariance property holds :

$$\tilde{g}_i = U_{j/i}(\tau_0)\tilde{g}_j; \ \forall i, j = 1, 2, 3. \tag{1.5}$$

2 Pricing and hedging interest rate derivatives in an implied approach

We now consider the procedure for building viable pricing and hedging models from an a priori pricing model and a set of observed prices in the framework of interest rate derivatives.

2.1 A priori pricing models

The a priori measure is usually associated with a caplet pricing model derived either in discrete or continuous time, from equilibrium or arbitrage arguments. In other words the a priori model is a kind of benchmark or structural model. A characteristic of interest rate pricing models is their multiplicity. The setting of Heath, Jarrow and Morton (hereafter HJM) (1992) with its applications in the Gaussian case and in the lognormal case provides an illustration. For a detailed presentation of these two widely used and documentated models we refer to Miltersen, Sandmann and Sonderman (1997), Jamshidian (1997), Musiela and Rutkowski (1997) for the lognormal or market model, and to El Karoui and Rochet (1989), El Karoui, Myneni and Viswanathan (1992), Jamshidian (1993), Brace and Musiela (1994) for the Gaussian model. Since these models are adopted by market practitionners, we will consider one or another as an a priori model. We will show thereafter that both a priori models lead to tractable implied models.

Before recalling the caplet pricing formulas in these two models, let us introduce a probability measure μ for the Libor rate x on $B_{[a;b]}$, the Borel algebra for the interval $[a;b] \subset \mathbb{R}$. This probability measure can be discrete or continuous without loss of generality. The bounds a and b may be seen as lower and upper bounds for Libor rates and b may be infinite. The measure μ may be viewed as a technical reference measure as in Madan and Milne (1994) and Elliott and Madan (1998). It may be taken equal to the Lebesgue measure or a lognormal measure in order to ease computations. It may also be chosen in order to reflect expectations based on historical observations, and will then be related to the so-called historical measure.

In Section 1, we have defined some exchange rates between the numéraires : $1 + x\delta$, $x\delta$, $1/(1 + x\delta)$, ... For these exchange rates to be well behaved we assume that they are strictly positive, μ a.s.. Hence we will not necessarily consider all numéraires when considering pricing models and we will restrict ourselves in practical applications to a subset of numéraires whose exchange rates are strictly positive. Furthermore we assume that the numéraires have finite moments of order 2 under μ. This guarantees that all caplet and digital caplet payoffs are in $L^2(\mu)$. If μ is the historical measure on Libor rates, the tail index tells us whether Libor rates exhibit too heavy tails (Levy distributions) to ensure that caplet payoffs are in $L^2(\mu)$. If μ is another reference measure (such as a lognormal one), it usually involves finite moments of any order.

Finally, since $z = 1 + \delta x$ (equation (1.1)), the probability measure μ on $B_{[1+\delta a;1+\delta b]}$ related to the inverse of the forward discount bond price z can be readily derived from the knowledge of μ.

A. Market model

In the market model the Libor rate is assumed to be lognormal under μ, and is equal to :

$$x = \frac{U_3(0)}{U_2(0)\delta} \exp\left(\sigma \sqrt{\tau_0}\, \varepsilon - \frac{\sigma^2 \tau_0}{2} \right); \tag{2.6}$$

where ε is a standard Gaussian variable (with cdf denoted by Φ) under the pricing measure associated with numéraire U_2. This leads to the following pricing formula :

$$\left\{\begin{array}{lll} Cap(c) & = & U_3(0)\Phi(d_1) - c\delta U_2(0)\Phi(d_2); \\ \text{with :} & & \\ & d_1 = & \frac{1}{\sigma\sqrt{\tau_0}}\log\left(\frac{U_3(0)}{U_2(0)c\delta}\right) + \frac{\sigma}{2}\sqrt{\tau_0}; \\ \text{and :} & & \\ & d_2 = & d_1 - \sigma\sqrt{\tau_0}. \end{array}\right. \tag{2.7}$$

The formula (2.7) is parametrized through the volatility σ of the forward Libor, and is a Black-Scholes formula on interest rate. Thanks to this example we can notice that our framework also covers the standard stock and exchange rate cases after adequate substitution for the underlying asset.

B. Gaussian model

In this model, the inverse of the forward price is assumed to be lognormal under μ, and is equal to :

$$z = \frac{U_1(0)}{U_2(0)}\exp\left(\tilde\sigma\sqrt{\tau_0}\,\varepsilon - \frac{\tilde\sigma^2\tau_0}{2}\right); \tag{2.8}$$

where ε is a standard Gaussian variable under the pricing measure associated with numéraire U_2. This gives :

$$\left\{\begin{array}{lll} Cap(c) & = & U_1(0)\Phi\left(\tilde d_1\right) - (1 + c\delta)U_2(0)\Phi\left(\tilde d_2\right); \\ \text{with :} & & \\ & \tilde d_1 = & \frac{1}{\tilde\sigma\sqrt{\tau_0}}\log\left(\frac{U_1(0)}{U_2(0)(1 + c\delta)}\right) + \frac{\tilde\sigma}{2}\sqrt{\tau_0}; \\ \text{and :} & & \\ & \tilde d_2 = & \tilde d_1 - \tilde\sigma\sqrt{\tau_0}. \end{array}\right. \tag{2.9}$$

The parameter in the caplet pricing formula (2.9) is $\tilde\sigma$. In the Vasicek model (Vasicek (1977)), $\tilde\sigma^2\tau_0$ specializes to :

$$\tilde\sigma^2\tau_0 = \frac{\sigma^2}{2\lambda^3}\left(\left(e^{-\lambda(\tau_2-\tau_0)} - e^{-\lambda(\tau_1-\tau_0)}\right)^2 - \left(e^{-\lambda\tau_2} - e^{-\lambda\tau_1}\right)^2\right); \tag{2.10}$$

which involves two parameters σ and λ, the volatility and the mean reversion coefficient of the instantaneous interest rate, respectively.

2.2 Variance optimal pricing models

We introduce a sequence of observed prices, i.e. a finite sequence of pairs : $(g_{i;j}; P_j)$, $j \in J$; $i = 1; 2; 3$, where P_j stands for the observed price of the asset delivering $g_{i;j}$ units of U_i at date τ_2. These payoffs $g_{i;j}$ are related by the numéraire invariance property (1.3) and have finite moments of order 2 under μ, i.e. $g_{i;j} \in L^2(\mu)$. The sequence of observed prices may in particular include observed caplet prices with different exercise rates. We will further assume that the sequence of observed prices includes two out of the three numéraires (the third one beeing deduced by linear combination). We denote by G_i, the subspace of $L^2(\mu)$ spanned by $(g_{i;j})$; $j \in J$; it is the (static) investment opportunity set.

Let us take the discount bond U_2 for numéraire (as in the market and Gaussian model), and introduce different sets of risk neutral densities. By taking any element of such sets, we can build a pricing model consistent with observed prices. The set F_2^e of equivalent risk-neutral probability densities associated to numéraire U_2 is defined by :

$$F_2^e = \left\{ f_2 \in L^2(\mu);\ f_2 > 0;\ U_2(0) \int g_{2;j} f_2 d\mu = P_j;\ \forall j \in J \right\}:$$

The other spaces involved in the choice among risk neutral measures will be F_2 the set of risk-neutral probability densities associated to numéraire U_2 absolutely continuous w.r.t. μ :

$$F_2 = \left\{ f_2 \in L^2(\mu);\ f_2 \geq 0;\ U_2(0) \int g_{2;j} f_2 d\mu = P_j;\ \forall j \in J \right\};$$

and F_2^s the set of risk-neutral signed-measure densities associated to numéraire U_2 :

$$F_2^s = \left\{ f_2 \in L^2(\mu);\ U_2(0) \int g_{2;j} f_2 d\mu = P_j;\ \forall j \in J \right\}:$$

Since the numéraires are assumed to be observed, $(g_{2;j} = 1; U_2(0))$ belongs to the sequence of observed prices which implies : $\int f_2 d\mu = 1$.

Let us now consider an a priori pricing model, say the market model. We denote by $f_2^{0\sigma}$, the density of this a priori model w.r.t μ, associated to numéraire U_2. It is indexed σ in order to explicitly show its dependence on the volatility parameter. If the a priori model is not consistent with observed option prices, $f_2^{0\sigma} \notin F_2^e$, the basic idea in the implied approach is to find a probability measure as close as possible to an a priori probability measure among all risk neutral probability measures. The a priori probability measure is used as a starting point and then modified to achieve an exact calibration of the observed prices. We examine the two minimization problems which differ in their optimization sets :

$$\min_{f \in F_2^s} \int \left(f - f_2^{0\sigma} \right)^2 d\mu;$$

$$\min_{f \in F_2} \int \left(f - f_2^{0\sigma} \right)^2 d\mu:$$

Various criteria have been proposed in order to measure the proximity between probability measures such as the quadratic, cross-entropy (Kullback-Leibler) or goodness-of-fit criteria. On economic grounds, we show in Section 2.3 that the quadratic criterion is related to the standard mean-variance hedging problem. Indeed the induced option price appears to be equal to the price of the hedging portfolio which minimizes the quadratic residual risk. The sensitivities of the option price w.r.t. to observed option prices are equal to the holdings of the optimal hedging portfolio. This economic interpretation prompts to use a quadratic criterion.

On practical grounds, since tractability also matters, the quadratic criterion reveals to be very attractive. It is by far the most simple one and leads to explicit expressions for the a posteriori pricing measure and for caplet price forecasts thanks to the use of convenient numéraires. Furthermore the robustness of the predicted option prices w.r.t. to choices concerning numéraires, proximity criteria, or a priori models is another very convincing argument. This appears in several empirical papers (Jackwerth and Rubinstein (1996), Jondeau and Rockinger (1997), Frachot, Laurent and Pichot (1999)) and is confirmed here in the empirical section. For these various reasons we prefer to rely on an $L^2(\mu)$-approach.

Since the two previous minimization sets, F_2^s, F_2 are non-empty, closed and convex subsets of $L^2(\mu)$, there exists a unique minimization element (by projection theorem) for each minimization problem. The solutions are called the variance optimal signed measure density associated to numéraire U_2 and a priori model $f_2^{0\sigma}$ (and denoted by $\tilde{f}_2^{2\sigma}$) and variance optimal probability measure density associated to numéraire U_2 and a priori model $f_2^{0\sigma}$ (and denoted by $f_2^{2\sigma}$). $\tilde{f}_2^{2\sigma}$ and $f_2^{2\sigma}$ mirror some risk premium updated thanks to information provided by observed prices.

Of course, if the a priori pricing model is already consistent with observed prices (no volatility smile), then $\tilde{f}_2^{2\sigma} = f_2^{0\sigma}$.

Let us notice that there does not always exist a minimal distance element between $f_2^{0\sigma}$ and F_2^e, since the later set is not closed. However if $\tilde{f}_2^{2\sigma}$ happens to be in F_2^e, it is clearly a minimal distance element between $f_2^{0\sigma}$ and F_2^e. In a continuous-time framework and when asset prices are continuous semimartingales, it has been proven by Delbaen and Schachermayer (1996) that the variance optimal signed measure density is always strictly positive and thus equivalent to μ (see also Gouriéroux, Laurent and Pham (1998) for a discussion, and Laurent and Pham (1999) for applications).

The solutions of these standard convex optimization problems can be written through the first order conditions as :

$$\begin{aligned} \tilde{f}_2^{2\sigma} &= f_2^{0\sigma} + \sum_{j\in J} \tilde{\lambda}^{\sigma}_{2,j}\, g_{2,j}, \\ f_2^{2\sigma} &= \left(f_2^{0\sigma} + \sum_{j\in J} \lambda^{\sigma}_{2,j}\, g_{2,j} \right)^+ ; \quad \mu \text{ a.s.} \end{aligned} \tag{2.11}$$

where $\tilde{\lambda}^{\sigma}_{2,j}, \lambda^{\sigma}_{2,j}$ are real numbers uniquely determined by the price constraints. These forms are obtained by applying the Lagrange Multiplier Theorem (see Luttmer (1996), Proposition 2 and Hansen and Jagannathan (1997), Proposition A.2).

The variance optimal measures depend on the a priori density $f_2^{0\sigma}$ (and on the volatility parameter σ). The variance optimal signed measure of Schweizer (1992) corresponds to the case where $f_2^{0\sigma} = 1$, and will lead to a nice interpretation of option prices in the next subsection. This variance optimal signed measure also appears in a portfolio context in Hansen and Richard (1987), Hansen and Jagannathan (1991) and Bansal, Hsieh and Viswanathan (1993).

Since risk-neutral densities are closely related to second order derivatives of option prices w.r.t. to exercise price (Breeden and Litzenberger (1978)), the previous criteria can also be seen as smoothness criteria. This guarantees that option prices will be smooth functions of the exercise price and thus the implied approach may also be seen as a numerical approach to properly interpolate option prices.

The variance optimal measures also depend on the choice of μ but only through the Lagrange multipliers $\tilde{\lambda}^{\sigma}_{2,j}, \lambda^{\sigma}_{2,j}$ as can be seen in equation (2.11). Magnien, Prigent and Trannoy (1996) use the Lebesgue measure on a finite length interval $[a, b]$. In Section 3, we will use lognormal measures (i.e. x (resp. z) will be lognormal under μ (resp. μ)).

When μ is the Lebesgue measure on some finite interval $[a, b]$, Michelli, Smith, Swetits and Ward (1985), Irvine, Marin and Smith (1986)[1], Magnien, Prigent and Trannoy (1996) characterize and compute the variance optimal probability measure when observed prices are call option prices. In that framework, the variance optimal probability measure is related to B-splines.

2.3 Variance optimal signed measures and mean-variance hedging

In this section we show that option prices computed under the variance optimal signed measure correspond to the “approximation price” of the option introduced by Schweizer (1992) and that the price sensitivities of option prices w.r.t. to observed option prices are the portfolio holdings in the optimal mean-variance hedging portfolio. Thus, we can link a dual optimization problem in the space of risk-neutral densities to a primal portfolio choice problem.

Let us consider the mean-variance hedging problem :

$$\min_{\alpha_i} \int \Big(\sum_{j \in J} \alpha_{i,j} g_{i,j} - g_i\Big)^2 d\mu.$$

where g_i is a square integrable payoff (for $i = 1, 2, 3$). This problem consists in finding a (static) portfolio $\sum_{j \in J} \alpha_{i,j} g_{i,j}$ which minimizes the square of the hedging residual :

[1] We are grateful to F. Magnien for providing the two last references.

$\sum_{j \in J} \alpha_{i,j} g_{i,j} - g_i$, or equivalently the $L^2(\mu)$ distance to payoff g_i. This linear regression problem has been introduced in a dynamic framework by Duffie and Richardson (1991) and further studied among others by Schweizer (1992) and Gouriéroux, Laurent and Pham (1998).

A direct application of the projection theorem guarantees that under non redundancy and no arbitrage, there exists a unique α_i^* to the previous minimization problem.

The mapping : $g_i \longrightarrow P_i^*[g_i] = \sum_{j \in J} \alpha_{i,j}^* P_j$ is a continuous linear functional on $L^2(\mu)$ consistent with observed prices and $\sum_{j \in J} \alpha_{i,j}^* P_j$ is called the approximation price of g_i. The price of the approximating portfolio $\sum_{j \in J} \alpha_{i,j}^* g_{i,j}$ is thus equal to a linear combination of the asset prices. As is well known in international portfolio management (Solnik (1974)), the approximating portfolio and thus its price are numéraire dependent (since $P_i^*[g_i]$ is obtained by taking as reference numéraire U_i). We are now able to state the following :

Property 1 (approximation price)
The approximation price $P_i^*[g_i]$ of some payoff g_i is equal to the price of this payoff under the variance optimal signed measure associated to numéraire U_i, $U_i(0) \int \tilde{f}_i g_i \, d\mu$, where $\tilde{f}_i$ is the variance optimal signed measure density associated to numéraire U_i and to $f_i'^{\sigma} = 1$.

Proof : see Appendix.

The variance optimal densities depend on observed option prices, $P_j, j \in J$, through the Lagrange multipliers. As a consequence, prices of payoffs g_i under these variance optimal measures do also depend on observed prices. One may think of computing the price sensitivities w.r.t. to all these observed prices and not only w.r.t. to the underlying price, since all observed payoffs are treated symmetrically in our approach.

The following property relates these price sensitivities or deltas to the holdings in the optimal mean-variance hedging portfolio :

Property 2 (sensitivities)
Let us consider an arbitrary square integrable payoff g_i. The sensitivities of the price of g_i under the variance optimal signed measure (associated to numéraire U_i) w.r.t. to observed option prices P_j are equal to the regression coefficients of the linear regression of g_i on the observed option payoffs $g_{i,j}$ under the measure μ, or equivalently to the holdings α_i^* in the mean-variance hedging portfolio. The second order derivatives are equal to zero.

This statement is a direct consequence of the previous property relating approximation prices (obtained from solutions of least square problems involving hedging residuals), and prices computed under variance optimal measures. Since we deal with multiple underlying assets (the observed option payoffs), there are multiple deltas. However, since we remain in a static framework, the second order derivatives are equal to zero. Such deltas can obviously be used to build static hedges of a given payoff.

3 Explicit cap pricing formulas under the variance optimal measure

3.1 Modified market and Gaussian models

In order to shed light on the model building procedure, some of the previous points are now illustrated on caplet pricing. We will see that it is easy to derive closed-form caplet pricing formulas consistent with observed option prices in the variance optimal measure setting.

Let us consider a sequence of observed prices corresponding to numéraires and caplets, namely $(g_{2;0}(x) = 1; U_2(0))$, $(g_{2;j}(x) = (x - c_j)^+\delta; P_j)$; $c_1 = 0$; $j \geq 1; j \in J$. From Equation (2.11), the variance optimal signed measure density associated to the a priori model $f_2^{0\sigma} = 1$ (which leads to the interpretation in terms of approximation prices) and to numéraire U_2 can be written as :

$$\tilde{f}_2^{2}(x) = 1 + \lambda_0 + \sum_{j \geq 1; j \in J} \lambda_j (x - c_j)^+\delta; \quad \mu \text{ a.s.}; \tag{3.1}$$

which leads to :

$$Cap(c) = U_2(0) \int (x - c)^+\delta \left(1 + \lambda_0 + \sum_{j \geq 1; j \in J} \lambda_j (x - c_j)^+\delta\right) d\mu(x): \tag{3.2}$$

Let us introduce the density function f_4^0 defined by :

$$U_2(0) = \frac{U_4(0) f_4^0(x)}{x^2\delta^2}; \quad \mu \text{ a.s.};$$

where $U_4(0) = U_2(0) \int x^2\delta^2 d\mu(x)$. Since x is in $L^2(\mu)$ this density is well defined. Straightforward computations give the following property, where Q_i^0 denotes the probability measure on Libor rate whose density w.r.t. μ is equal to f_i^0 :

$$Q_i^0(E) = \int_E f_i^0 d\mu; \tag{3.3}$$

where E is some μ-measurable set.

Property 3 (caplet price forecast)
A caplet pricing formula consistent with observed numéraires and caplet prices in the variance optimal signed measure is given by :

$$Cap(c) = (1 + \lambda_0)Cap^0(c) + \sum_{j \geq 1; j \in J} \lambda_j Cap^j(c); \tag{3.4}$$

where $Cap^0(c)$ is the a priori pricing formula (i.e. $U_2(0)E^{Q_2^0}[(x-c)^+\delta]$) and $Cap^j(c)$ is equal to :

$$Cap^j(c) = U_4(0)Q_4^0(E_j) - U_3(0)\,\delta\,(c+c_j)Q_3^0(E_j) - U_2(0)\,c\,c_j\,\delta^2Q_2^0(E_j); \qquad (3.5)$$

E_j the exercise region being equal to $\{x \geq c \vee c_j\}$ and $c \vee c_j = \sup(c; c_j)$. The Lagrange multipliers λ_j are determined by the linear equations :

$$\begin{aligned} P_j &= (1+\lambda_0)Cap^0(c_j) + \sum_{i\geq 1; i\in J} \lambda_i Cap^i(c_j); \quad j \geq 1; j \in J; \\ U_2(0) &= U_2(0)(1+\lambda_0) + \sum_{i\geq 1; i\in J} \lambda_i Cap^0(c_i). \end{aligned}$$

Hence we get an explicit formula which only requires solving linear equations for its practical implementation. This explicit formula is particularized in the next two examples based on lognormal assumptions on x under μ (market model), or on z under μ (Gaussian model).

A. Modified market model

When x is lognormal (equation (2.6)), the a priori caplet pricing formula $Cap^0(c)$ is given by (2.7), while the other formula used in equation (3.5) specializes to :

$$Cap^j(c) = U_2(0)\left[x^2(0)\delta^2\exp(\sigma^2\tau_0)\phi(d_0^j) - x(0)\delta^2(c+c_j)\phi(d_1^j) + c\,c_j\,\delta^2\phi(d_2^j)\right]; \qquad (3.6)$$

with :

$$\left\{\begin{aligned} d_1^j &= \frac{1}{\sigma\sqrt{\tau_0}}\log\frac{x(0)}{c\vee c_j} + \frac{\sigma}{2}\sqrt{\tau_0}; \\ d_2^j &= d_1^j - \sigma\sqrt{\tau_0}; \\ d_0^j &= d_1^j + \sigma\sqrt{\tau_0}. \end{aligned}\right.$$

This simple expression is due to the lognormality of the Libor rate under Q_2^0, Q_3^0 and Q_4^0.

B. Modified Gaussian model

When z is lognormal (equation (2.8)), the same kind of explicit formulas can be derived under the measure that minimizes the $L^2(\mu)$-distance between $f_2^{\tilde{0};\sigma} = 1$ and the set of densities for z compatible with observed prices. This measure takes the form :

$$\tilde{f}_2^{2}(z) = 1 + \tilde{\lambda}_0 + \sum_{j\geq 1; j\in J} \tilde{\lambda}_j (z - (1+c_j\delta))^+; \quad \mu \text{ a.s.};$$

which gives :

$$\tilde{C}ap(c) = (1+\tilde{\lambda}_0)\tilde{C}ap^0(c) + \sum_{j\geq 1; j\in J} \tilde{\lambda}_j \tilde{C}ap^j(c);$$

with :

$$\tilde{C}ap^0(c) = U_1(0)\phi(\tilde{d}_1) - \tilde{c}U_2(0)\phi(\tilde{d}_2); \qquad (3.7)$$

$$\hat{C}ap^j(c) = U_2(0)\left[z^2(0)\exp(\sigma^2\tau_0)\Phi(\hat{d}_0^j) - z(0)(\hat{c} + \hat{c}_j)\Phi(\hat{d}_1^j) + \hat{c}\hat{c}_j\,\Phi(\hat{d}_2^j)\right]; \tag{3.8}$$

with :

$$\begin{cases} \hat{c} &= 1 + c\delta; \\ \hat{d}_1 &= \frac{1}{\sigma\sqrt{\tau_0}}\log\frac{z(0)}{\hat{c}} + \frac{\sigma}{2}\sqrt{\tau_0} \\ \hat{d}_2 &= \hat{d}_1 - \sigma\sqrt{\tau_0}; \\ \hat{c}_j &= 1 + c_j\delta; \\ \hat{d}_1^j &= \frac{1}{\sigma\sqrt{\tau_0}}\log\frac{z(0)}{\hat{c}\vee\hat{c}_j} + \frac{\sigma}{2}\sqrt{\tau_0}; \\ \hat{d}_2^j &= \hat{d}_1^j - \sigma\sqrt{\tau_0}; \\ \hat{d}_0^j &= \hat{d}_1^j + \sigma\sqrt{\tau_0}. \end{cases}$$

Finally similar explicit caplet pricing formulas are also available under the variance optimal probability measure. This is due to the fact that the positive part of a piecewise linear function is still piecewise linear. Thus the densities w.r.t. μ (resp. μ) are piecewise linear functions of x (resp. z) and the computations go along the same lines as in the signed case.

3.2 Further use of change of numéraire

When introducing variance-optimal measures, we took as benchmark the numéraire U_2, and we were able to state explicit caplet formulas with such measures. However, variance optimal measure densities give rise to different pricing models when we change our reference numéraire from U_2 to U_i. This is not surprising since we know that the option price is equal to an approximation price for a mean-variance hedging problem, which is not numéraire invariant.

Let us consider the density $\tilde{f}_i^j$ (resp. f_i^j) will be the variance optimal signed (resp. probability) measure densities associated with numéraire U_i when the numéraire U_j is used as reference (for notational simplification, we drop here the dependence on σ).

In order to make a comparison, let us consider some other numéraire, U_i, $i \neq 1$ while keeping U_2 as reference numéraire for the moment. By the state price invariance property (see e.g. Duffie (1992), Geman, El Karoui and Rochet (1995), Bajeux and Portait (1997), Gouriéroux, Laurent and Pham (1998)), the density associated to numéraire U_i can be written as :

$$\tilde{f}_i^2 = \frac{U_{i/2}}{U_{i/2}(0)}\tilde{f}_2^2; \tag{3.9}$$

where $U_{i/2}$ is the exchange rate between numéraires U_i and U_2 at time τ_0 (and is a μ-measurable function). Let us remark that $\tilde{f}_i^2$ is in $L^2(\mu)$ under the standing assumption that the Libor rate is strictly positive (which in turn implies that $U_{i/2} \in L^\infty(\mu)$). The density f_i of the a priori pricing model, associated to numéraire U_i can also be obtained by state price invariance property :

$$f_i^0 = \frac{U_{i/2}}{U_{i/2}(0)} f_2^0. \tag{3.10}$$

From the characterization of $\tilde{f}_2^2$ in equation (2.11) and from the relations between payoffs under different numéraires, $g_{2,j} = U_{i/2} g_{i,j}$, we obtain by (3.9) and (3.10) :

$$\tilde{f}_i^2 = f_i^0 + \frac{U_{i/2}}{U_{i/2}(0)} \sum_{j \in J} \tilde{\lambda}_{2,j} U_{i/2} g_{i,j}. \tag{3.11}$$

On the other hand, we can directly compute the variance optimal signed measure density associated with numéraire U_i while choosing as reference numéraire U_i. By adapting the characterization result (2.11) to numéraire U_i instead of U_2 there exist some real numbers $\tilde{\lambda}_{i,j}$, $j \in J$ such that :

$$\tilde{f}_i^i = f_i^0 + \sum_{j \in J} \tilde{\lambda}_{i,j} g_{i,j}. \tag{3.12}$$

Now, since in the usual cases $U_{i/2}$ is a non constant random variable and the a posteriori model differs from the a priori model (some of the Lagrange multipliers are different from zero), we clearly see from (3.11) and (3.12) that :

$$\tilde{f}_i^2 \neq \tilde{f}_i^i,$$

which states that the variance optimal measures will differ if we start with numéraire U_i ($\tilde{f}_i^i$) instead of U_2 ($\tilde{f}_i^2$).

Further explicit, but more complicated, pricing formulas for caplets can be derived under some of these new measures associated with other numéraires. However, we will not further consider these additional caplet pricing formulas. Indeed, numéraire dependencies have been empirically investigated by Frachot, Laurent and Pichot (1999) and appear to be second order effects.

4 Numerical example and empirical application

In this last section, we begin by checking the practical relevance of our approach on a numerical example based on simulated data. The example is designed to obtain prices similar to those currently traded on the DEM Libor market. We then proceed further on real caplet data.[2]

[2] The Gauss programs developed for this section are available on request.

For the numerical example, ten caplet prices have been generated with the Gaussian model (eq. (2.9)). These simulated prices are one year caplet prices on three month Libor ($\tau_0 = \tau_1 = 1$, $\tau_2 = 1.25$, $\delta = 0.25$) with equally spaced exercise rates from 2.5% to 7%, and will act as observed option prices. The volatility parameter of the Gaussian model is set equal to 0.242% (eq. (2.10) : $\bar{\sigma} = 0.242\%$, $\sigma = 1\%$, $\theta = 5\%$). The yield curve is taken 4% flat ($U_1(0) = 0.9608$, $U_2(0) = 0.9512$, $U_3(0) = 0.0096$, $x(0) = 4.02\%$). These data are used as input data for a calibration procedure based on the modified market model (eq. (2.7), (3.4), (3.6)). Hence we take the market model as our a priori model in this example. The volatility parameter of the market model corresponds to the implied volatility of the observed at-the-money caplet price ($\sigma = 24.478\%$). In Table 6 the price forecasts are compared with the (unobserved) true prices. The strike rates of the caplet prices to be inferred are the intermediate rates from 2.75% to 6.75%.

Table 6 : True prices and price forecasts (in percent)

strike	true	forecast
2.75	0.31263	0.31234
3.25	0.21172	0.21181
3.75	0.12848	0.12845
4.25	0.06813	0.06814
4.75	0.03085	0.03084
5.25	0.01169	0.01170
5.75	0.00365	0.00365
6.25	0.00093	0.00093
6.75	0.00019	0.00019

The results show that our simple procedure is very successful in rebuilding the unobserved data while matching exactly (by construction) the available market prices. The difference is not visible to the naked eye if the true prices are plotted on a graph together with their forecasts for each exercise rate. The absolute errors are of orders 10^{-6} to 10^{-8} while the relative errors are of orders 10^{-2} to 10^{-4}. Reversing the role of the Gaussian model and the market model in such an example leads to similar results.

Let us now apply the calibration approach to real market data. The collected data are one year caplet prices for the three month DEM Libor. The quotes (Tue. 06/10/1998 around 4 pm) take the form of a lognormal volatility smile which can be translated into caplet prices. The data (implied volatilities and caplet prices) are presented in Table 7. The discount bond prices are equal to 0.9665 and 0.9582 for the one year and fifteen month maturities, respectively.

Table 7 : Implied volatilities and observed caplet prices (in percent)

strike	volatility	price
2.5	24.841	0.23747
3	24.321	0.14219
3.5	24.249	0.07558
4	24.464	0.03695
4.5	24.857	0.01739
5	25.331	0.00814
5.5	25.799	0.00384
6	26.197	0.00182
6.5	26.500	0.00086
7	26.712	0.00040

From these observed data, we get the following price forecasts taking for the rebuilding procedure either the modified market model or the modified Gaussian model (Table 8). The volatility parameters are taken equal to their respective at-the-money implied volatility ($\sigma = 24.464\%$, $\bar{\sigma} = 0.225\%$). The price forecasts made by the two modified models do not differ very much from each other. For the strike rates : 2.75%, 3.75%, 4.75%, 5.75%, 6.75%, the Gaussian model forecasts are slightly higher while the reverse holds for the other exercise rates. However we do not see a particular reason for this special alternate ordering.

Table 8 : Price forecasts with modified market and Gaussian models (in percent)

strike	market	Gaussian
2.75	0.18671	0.18707
3.25	0.10506	0.10496
3.75	0.05325	0.05327
4.25	0.02541	0.02539
4.75	0.01189	0.01189
5.25	0.00558	0.00558
5.75	0.00264	0.00265
6.25	0.00125	0.00122
6.75	0.00059	0.00062

The sensitivities (or deltas) w.r.t. observed caplet prices are gathered in Table 9 and 10 for the modified market and Gaussian models, respectively. These sensitivities may be used to build a static hedge since they correspond to the holdings of the approximating portfolio as discussed in Section 2.3. A line (column) in Tables 9 and 10 gives the sensitivities of a (each) forecast caplet price w.r.t. each (a particular) observed caplet price. The sensitivities are of course higher near the central diagonal of the Tables, i.e. for forecast and observed caplet prices with close strike rates. Moreover the sensitivities are relatively similar in both models and have always the same sign. The static hedges under the two models thus involve the same type of short and long positions in the observed caplet prices.

Table 9 : Sensitivities in the modified market model (in percent)

strike	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7
2.75	47.727	62.498	-13.312	4.135	-1.434	0.534	-0.208	0.083	-0.033	0.009
3.25	-9.412	58.629	62.289	-15.420	5.350	-1.992	0.777	-0.312	0.123	-0.037
3.75	2.168	-10.689	55.370	66.582	-18.364	6.839	-2.667	1.070	-0.424	0.127
4.25	-0.442	2.181	-8.991	52.357	69.793	-20.655	8.057	-3.234	1.281	-0.384
4.75	0.083	-0.410	1.691	-7.830	50.409	72.005	-22.321	8.961	-3.549	1.064
5.25	-0.014	0.073	-0.300	1.393	-7.129	49.143	73.516	-23.455	9.289	-2.786
5.75	0.002	-0.012	0.051	-0.238	1.221	-6.693	48.328	74.413	-23.411	7.023
6.25	-0.001	0.002	-0.008	0.040	-0.205	1.127	-6.471	48.126	72.962	-17.254
6.75	0.000	-0.001	0.002	-0.007	0.038	-0.210	1.206	-7.147	52.962	57.217

Table 10 : Sensitivities in the modified Gaussian model (in percent)

strike	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7
2.75	45.970	64.105	-12.838	3.639	-1.199	0.456	-0.199	0.099	-0.056	0.030
3.25	-9.552	60.094	59.705	-13.504	4.450	-1.694	0.741	-0.370	0.207	-0.114
3.75	2.436	-12.053	58.035	63.839	-16.750	6.378	-2.790	1.393	-0.782	0.429
4.25	-0.531	2.628	-10.058	53.807	68.877	-20.907	9.146	-4.567	2.564	-1.409
4.75	0.098	-0.487	1.866	-7.942	49.734	74.278	-25.951	12.960	-7.276	3.998
5.25	-0.015	0.076	-0.293	1.248	-6.224	45.942	80.006	-31.987	17.954	-9.868
5.75	0.002	-0.010	0.038	-0.165	0.825	-4.857	42.424	86.025	-38.768	21.308
6.25	-0.001	0.001	-0.004	0.018	-0.092	0.542	-3.784	39.231	91.604	-40.475
6.75	0.000	-0.001	0.001	-0.001	0.008	-0.052	0.369	-3.066	37.544	86.974

Finally, it is interesting to have a look at the a posteriori densities underlying the price forecasts, and to see how the priors on the Libor rate x are affected by the information contained in observed prices. Figures 1 and 2 are designed for such a purpose. They correspond to the modified market and Gaussian models, respectively. The solid line in Figure 1 is the lognormal density of x associated with equation (2.6), while in Figure 2 it is obtained from the lognormal density of z (equation (2.8)) and the link (1.1) between x and z. The plots show that the shift in the shape of the Libor rate density is quite large in the Gaussian case. The informational updating is much less pronounced in the modified market model, reflecting that the lognormal prior on x, and thus the market model is closer to what translates in the observed option prices. In fact our procedure is also aimed to be an investigation tool to uncover which type of pricing model a trader uses when quoting derivative prices.

Please insert Figure 1 : Densities in the modified market model

Please insert Figure 2 : Densities in the modified Gaussian model

5 Concluding remarks

We have presented a general approach for the valuation and the hedging of a book of interest rate derivative products, such as caplets. The proposed valuation is different from the usual approach based on continuously adjusted self-financing portfolios. In particular, we do not rely on a particular time evolution of the state variables and institutional assumptions such as frictionless trading. The usual dynamic approach makes it difficult to take into account observed prices, and is mainly aimed at providing a structural model that will be modified according to the information contained in observed prices. We conjecture that our static approach will tend to outperform the dynamic ones regarding the hedging performance. The hedge here directly involves short and long positions in assets with similar payoffs instead of a dynamic trading in the underlying asset. Of course, the quality of the static approach will depend on the one hand on the number of currently traded assets and observed prices and on the other hand on the quality of the a priori model.

Our main focus was interest rate products such as caplets. This is a rich framework to introduce different numéraires and provide simple general formulas for interest rate options. Our framework ought to be extended in order to take into account more general assets such as swaptions. However, this work involves technical difficulties since swaption payoffs depend on the whole forward curve and not only on a single Libor rate.

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Appendix : Proof of Property 1

From Riesz-Fréchet Representation Theorem (see Rudin (1974), Theorem 6.16), there exists a unique function $f_i^* \in L^2(\mu)$ such that $P_i^*[g_i] = U_i(0) \int g_i f_i^* d\mu;\ \forall g_i \in L^2(\mu)$. We also have $\int f_i^* d\mu = 1$. Indeed, the numéraire U_i is a traded asset and its approximation price is equal to $U_i(0)$. Thus, f_i^* belongs to the set F_i^s of risk-neutral signed density measures.

Let us now take some g_i orthogonal to the investment opportunity set $G_i = \{g_{i;j}\}$. Its approximation price is zero and thus we have the implication :

$$\forall j \in J; \quad \int g_i g_{i;j} d\mu = 0 \Rightarrow \int g_i f_i^* d\mu = 0:$$

Therefrom we deduce that f_i^* belongs to the investment opportunity set G_i which means that f_i^* has the interpretation of a portfolio. Let us denote A_i, the subspace of G_i spanned by the zero price portfolios :

$$A_i = \{\sum_{j \in J} \alpha_j g_{i;j} \mid \sum_{j \in J} \alpha_j P_j = 0\}:$$

Since the approximation price of any arbitrage portfolio is equal to zero, we deduce that the portfolio f_i^* is orthogonal to A_i.

We now have to show that $f_i^- = f_i^*$. Since f_i^- is the unique $L^2(\mu)$-norm minimum element of F_i^s, we simply have to show that :

$$\int (f_i^-)^2 d\mu = \int (f_i^*)^2 d\mu:$$

or equivalently that :

$$\int f_i^-(f_i^- + f_i^*) d\mu = \int f_i^*(f_i^- + f_i^*) d\mu:$$

$f_i^- + f_i^*$ is the payoff of a portfolio since we know from our previous results that f_i^- and f_i^* are two portfolios. Since the expectations of a given portfolio under any measure consistent with observed prices are the same, the last equality is true.

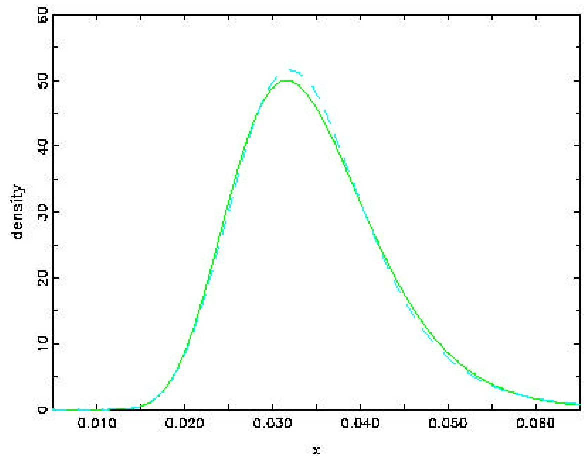

Figure 1 : **Densities in the modified market model**

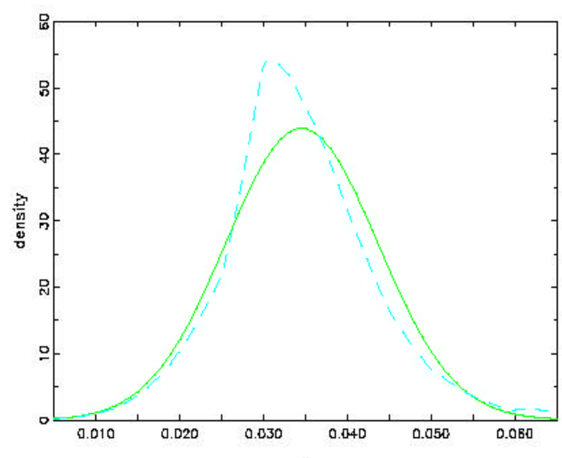

Figure 2 : **Densities in the modified Gaussian model**