Basket Default Swaps, CDO's and Factor Copulas

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This version : September 2003

Abstract

We consider a factor approach to the pricing of basket credit derivatives and synthetic CDO tranches. Our purpose is to deal in a convenient way with dependent defaults for a large number of names. We provide semi-explicit expressions of the stochastic intensities of default times and pricing formulae for basket default swaps and CDO tranches. Two cases are studied in detail: mean-variance mixture models and frailty models. We also compare prices under Gaussian and Clayton copulas.

Introduction

This paper addresses the pricing of multiname credit derivatives and CDO tranches. The purpose is to describe a widely applicable technique for efficiently pricing these products when a large number of credits are involved. The curse of dimensionality is solved thanks to the use of a small number of latent factors that drive the dependence between default times. Thanks to the factor approach, we can provide semi-explicit expressions for basket credit derivatives and synthetic CDO's. This results in a substantial reduction in computational time compared with Monte Carlo simulation techniques. Moreover, this approach is parsimonious with respect to the number of parameters, thus easing calibration.

The pricing of first to default swaps has initially relied on reduced form models (see Duffie [1998]) and leads to simple expressions of prices. However, no such simple results can be derived for more general basket credit derivatives or CDO's. In Duffie and Gârleanu [2001], dependence of default times is

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JEL Classification: G 13

Key words: default risk, copulas, factor models, basket default swaps, synthetic CDO's.

The first version circulated as in internal BNP Paribas working paper in may 2001; a second version has been disclosed on the web in october 2002. The authors thank D. Duffie, C. Gouriéroux, M. Musiela, members of the credit derivative research team at BNP Paribas and participants at the Quantitative Finance RISK conference, the Credit Derivatives RISK seminar, the Bachelier seminar on default risk held at University of Evry, the French Finance Association and Quant'03 conferences. We also thank X. Burtschell for computational assistance. The usual disclaimer applies.

also through correlated stochastic risk intensities. To achieve reasonable levels of dependencies, one must introduce jumps and the pricing of CDO's is achieved through Monte Carlo simulations. Another approach is based on a multivariate extension of the Cox process approach that was introduced by LANDO [1998]. This results in a series of models such as the Gaussian copula approach introduced for the pricing of basket credit derivatives by LI [1999, 2000]. The multivariate exponential copula of MARSHALL and OLKIN [1967] (see DUFFIE and SINGLETON [1998], KIJIMA [2000], LI [2000]) provides another framework which allows for simultaneous defaults and is associated with non smooth joint distribution functions. SCHÖNBUCHER and SCHUBERT [2001] study the dynamics of default intensities and show that Clayton copulas, a member of the Archimedean copula family, are related to the dependent intensities approaches of KUSUOKA [1999], DAVIS and LO [2001], JARROW and YU [2001], GIESECKE [2001]). BOUYÉ et al [2000], SCHMIDT and WARD [2002], GREGORY and LAURENT [2003] also consider some applications of copulas for the pricing of basket credit derivatives.

On the other hand, latent factor models have been widely used for the computation of default events and loan loss distributions (see CROUHY, GALAI and MARK [2000], BELKIN, SUCHOVER and FOREST [1998], FINGER [1999], KOYLUOGLU and HICKMAN [1998], GORDY [2002], MERINO AND NYFELER [2002], SCHÖNBUCHER [2002]). VASICEK [1997] noticed that the one factor Gaussian model was well suited for analytical computation of loss distributions. The new Basel agreement relies on such approaches. Latent factor models have been thoroughly studied in the statistical literature[1]. In the credit area, FREY, MCNEIL and NYFELER [2001] relate factor and copula approaches. The main feature of these models is that default events, conditionally on some latent state variables are independent. This eases the computation of aggregate loss distributions through dimensionality reduction. This factor approach is nicely suited for large dimensional problems. Since semi-explicit expressions of most relevant quantities can be obtained, it provides an alternative route to Monte Carlo approaches, while we can still rely on the latter when useful.

We propose to use this latent factor approach to the modelling of multiname credit derivatives and synthetic CDO's. The dimension of the problem is not anymore the number of names but the dimension of the factor. For the sake of simplicity, we thereafter present one factor models, though the technique applies to multiple factors. The prices are eventually obtained by quadrature techniques. The premises of this technique seem to have been known among major participants of the credit default swap market until the late 90's. This concerns mainly the analytical valuation of homogeneous first to default swaps in the one factor Gaussian model. We show here that most of complex multiname credit derivatives such as non homogeneous basket default swaps, CDO tranches can be priced in a semi-analytical way. This results in dramatically quicker computations compared with Monte Carlo simulations especially with respect to sensitivity analysis (see GREGORY & LAURENT [2003]). Moreover, factor models appear to be parsimonious with respect to the number of parameters, which should ease model calibration.

The paper is organized as follows: The first section recalls a simple multivariate Cox process framework for the modelling of default times. The stated results will provide some building blocks for the pricing of basket credit derivatives and synthetic CDO's. The second section introduces some factor structure in the modelling. We consider Gaussian cases, mean-variance mixtures and Archimedean copulas (or frailty

[1]See JUNKER and ELLIS [1998] for some characterizations of one factor models and GOURIÉROUX and MONFORT [2002] for some application to credit risk.

models). The third section deals with the computation of the various basket default swaps premiums. The fourth section considers the pricing of synthetic CDO tranches.

1 Modelling of default times

In the following, we will consider n names, with associated random default times $\tau_1, \dots, \tau_n$ defined on a common probability space $(\Omega, \mathcal{G}, Q)$[2]. We will denote by $t \in \mathbb{R}^+ \to N_i(t) = 1_{\{\tau_i \le t\}}$, for $i = 1, \dots, n$ the indicator of default processes. We will denote by $\mathcal{H}_{i,t} = \sigma(N_i(s), s \le t)$ and $\mathbb{H}_i$ will be the natural filtration of default time τ_i, with $\mathbb{H} = \vee_{i=1}^n \mathbb{H}_i$. In the following, we will consider reduced-form models of default times defined by :

$$\tau_i = \inf\{u \in \mathbb{R}^+, \int_0^u h_i(v)dv \ge -\ln U_i\},\ i = 1, \dots, n,$$

where the h_i's are some given $\mathcal{F}_t$ - adapted processes, where $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}^+}$ is some given filtration, and the U_i are some uniform random variables. In the following, we will assume independence between $\mathbb{F}$ and $\sigma(U_1, \dots, U_i)$. This Cox modelling framework has been introduced for defaults by Lando [1998]. The h_i's represent some marginal default intensities, while the $-\log U_i$ can be seen as stochastic barriers[3]. A nice feature of that Cox modelling is that filtration $\mathbb{F}$ possesses the martingale invariance property with respect to $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. We can check that for all $t \ge 0$, for all $\mathcal{F}_\infty$ -measurable random variable H, $E[H \mid \mathcal{G}_t] = E[H \mid \mathcal{F}_t]$ where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$[4] and conclude from Dellacherie & Meyer [1975]. This so-called (H) Hypothesis has been investigated in detail by Jeanblanc & Rutkowski [2000] for default modelling. The martingale invariance property is aimed at precluding arbitrage opportunities after defaultable securities are introduced in the market place. Another assessment of the martingale invariance property is :

$$Q(\tau_1 > t_1, \dots, \tau_n > t_n \mid \mathcal{F}_\infty) = Q(\tau_1 > t_1, \dots, \tau_n > t_n \mid \mathcal{F}_T),$$

for any $T > 0$ and $t_1, \dots, t_n \in [0, T]$.

[2] We assume here the use of some pricing measure Q and do not discuss the existence or uniqueness of such a measure.

[3] In Jarrow & Yu [2001], a Cox process type framework is also discussed. There, there is usually some dependence between the stochastic barriers $U_1, \dots, U_n$ and the h_i's. In this framework, the filtration $\mathbb{F}$ thus includes some information about default events and cannot be interpreted as a pre-default filtration. We can also remark that the h_i's cannot be interpreted as short-term credit spreads since they are not intensities with respect to $\mathcal{G}_t$. See Kusuoka [1999] and Bielecki & Rutkowski [2002] for further material and discussion.

[4] We remark that $\mathcal{G}_t \subset \mathcal{F}_t \vee \sigma(U_1, \dots, U_n)$. From iterated expectations theorem, we get $E[H \mid \mathcal{G}_t] = E[E[H \mid \mathcal{F}_t \vee \sigma(U_1, \dots, U_n)] \mid \mathcal{G}_t]$. In a second step, we use independence between H and the U_i's to show $E[H \mid \mathcal{F}_t \vee \sigma(U_1, \dots, U_n)] = E[H \mid \mathcal{F}_t]$. Let $A \in \mathcal{F}_t \vee \sigma(U_1, \dots, U_n)$. We want to check that:

$$E[1_A H] = E[1_A E[H \mid \mathcal{F}_t]]. \quad (1.1)$$

Let $A = B \cap C$ where $B \in \mathcal{F}_t$ and $C \in \sigma(U_1, \dots, U_n)$. We readily see that equation (1.1) is satisfied. We remark that $\mathcal{F}_t \vee \sigma(U_1, \dots, U_n)$ is generated by sets $B \cap C$ where $B \in \mathcal{F}_t$ and $C \in \sigma(U_1, \dots, U_n)$. Moreover, the set $\{A \in \mathcal{F}_t \vee \sigma(U_1, \dots, U_n),\ E[1_A H] = E[1_A E[H \mid \mathcal{F}_t]]\}$ is a monotone class. We can conclude from monotone class theorem.

Some of the models in the literature (see Kijima [2000], Kijima & Muromachi [2000], Bielecki & Rutkowski [2001] and the discussion about affine models below) make the following additional assumption of conditional independence (with respect to filtration $\mathbb{F}$) :

$$Q(\tau_1 > t_1, \dots, \tau_n > t_n \mid \mathcal{F}_T) = \prod_{i=1}^{n} Q(\tau_i > t_i \mid \mathcal{F}_T),$$

for any $T > 0$ and $t_1, \dots, t_n \in [0, T]$. Obviously, such assumption is fulfilled when the U_i, $i = 1, \dots, n$ are independent and then $Q(\tau_i > t_i \mid \mathcal{F}_T) = \exp\left(-\int_0^{t_i} h_i(s)ds\right)$. The Affine Jump Diffusion setting of Duffie, Pan & Singleton [2000] and the discrete affine framework of Gouriéroux, Monfort & Polimenis [2002] provide some convenient specification of the h_i's that account for a variety of dynamics for the credit spreads. The previous assumption implies that the compensator of the counting process $N_i(t)$ remains the same under the filtrations $\mathcal{F}_t \vee \mathcal{H}_{i,t}$ and $\mathcal{F}_t \vee \mathcal{H}_t$. Similarly, the stochastic intensities of default $h_i(t)1_{\{\tau_i \le t\}}$ and the hazard processes $\int_0^{t\wedge\tau_i} h_i(s)ds$ remain unchanged. However, the conditional independence assumption between default times precludes simultaneous defaults and jumps in credit spreads at default times of other names, which can be a highly desirable feature. Thus, a variety of models where this assumption is relaxed have been proposed ; see Li [2000], Giesecke [2001], Schönbucher & Schubert [2001], Jouanin et al [2001], Schmidt & Ward [2002].

In order to get tractable expressions of basket credit derivatives premiums and CDO tranches, we will make the assumption that the default times are conditionally independent upon some enlarged filtration $\mathbb{F} \vee \sigma(V)$ where V is $\mathcal{G}$-measurable random variable. When V is not $\mathcal{F}_\infty$ measurable, it can be see as a latent mixing variable corresponding to unobserved random effects. V will thereafter be called the factor. The previous assumption can be written:

$$Q(\tau_1 > t_1, \dots, \tau_n > t_n \mid \mathcal{F}_T \vee \sigma(V)) = \prod_{i=1}^{n} Q(\tau_i > t_i \mid \mathcal{F}_T \vee \sigma(V)),$$

for any $T > 0$ and $t_1, \dots, t_n \in [0, T]$. For simplicity, we will assume that the factor has a density, that we denote f.

2 Factor approaches

For simplicity, we will assume that the h_i's are deterministic and continuous[5], though this assumption can be relaxed and thus concentrate on the dependence upon the latent factor. $S_i(t) = Q(\tau_i > t) = \exp\left(-\int_0^t h_i(v)dv\right)$ and $F_i(t) = Q(\tau_i \le t)$ denote the marginal survival and distribution function of τ_i respectively. The associated density f_i is given by $f_i(t) = h_i(t)S_i(t)$, for all $t \ge 0$. We also notice that $\tau_i = S_i^{-1}(U_i)$ where $S_i^{-1}(u) = \{\inf v, S_i(v) \le u\}$ defines the generalized inverse of S_i. As a consequence the thresholds U_i are also independent conditionally on V. We will denote by $q_t^{i|V} = Q(\tau_i > t \mid V)$ and $p_t^{i|V} = Q(\tau_i \le t \mid V) = 1 - q_t^{i|V}$, the conditional survival and conditional default probabilities respectively. As a direct consequence of the iterated expectations theorem, we can write the joint survival and joint

[5] Or equivalently that $\mathcal{F}_T$ is degenerated.

distribution functions respectively as: $S(t_1, \dots, t_n) = Q(\tau_1 > t_1, \dots, \tau_n > t_n) = \int \prod_{i=1}^n q_{t_i}^{i|v} f(v)dv$, and:

$$F(t_1, \dots, t_n) = Q(\tau_1 \le t_1, \dots, \tau_n \le t_n) = \int \prod_{i=1}^n p_{t_i}^{i|v} f(v)dv.$$

We can also notice that the distribution function of the thresholds U_i's is equal to survival copula of default times.

2.1 One factor Gaussian copulas

The *Gaussian Copula* has been introduced by Li [1999, 2000] for the pricing of basket credit derivatives and corresponds to the dependence structure underlying CreditMetrics (see Gupton, Finger & Bhatia [1997], *CreditMetrics - Technical Document*) and the New Basel Agreement. We consider thereafter a special case that is associated with a one factor representation: let $(X_1, \dots, X_n)$ be a Gaussian vector, where $X_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$ and $V, \bar{V}_i,\ i = 1, \dots, n$ are independent standard Gaussian random variables[6]. We define the stochastic thresholds by $U_i = 1 - \Phi(X_i)$, for $i = 1, \dots, n$, where Φ stands for the cdf of a standard Gaussian variable. We then obtain default times as $\tau_i = S_i^{-1}(U_i)$ or equivalently as $\tau_i = F_i^{-1}(\Phi(X_i)$, for $i = 1, \dots, n$. We then readily get:

$$p_t^{i|V} = \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho_i V}{\sqrt{1-\rho_i^2}}\right),$$

which allows to compute the joint distribution function by:

$$F(t_1, \dots, t_n) = \int \prod_{i=1}^n p_{t_i}^{i|v} f(v)dv = \int \left(\prod_{i=1}^n \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i v}{\sqrt{1-\rho_i^2}}\right)\right) f(v)dv,$$

where here, $f(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$ is the Gaussian density. The previous integral can be easily computed through some quadrature. We can also remark than since default times are monotonic transforms of Gaussian variables, the copula of default times is indeed a Gaussian copula.

2.2 One factor mean variance Gaussian mixtures

The previously described one factor Gaussian copula can be extended to a variety of factor models with easy implementation. In a mean variance or location scale mixture model, we write $X_i = m_i(V) + \sigma_i(V)\varepsilon_i$, where V is a one dimensional mixing latent variable with density function f and the ε_i are independent standard Gaussian random variables[7]. Conditionally on V, the X_i's are independent Gaussian random variables with mean $m_i(V)$ and standard deviation $\sigma_i(V)$.

The calibration to the marginal distributions of default times consists in determining increasing real functions g_i, $i = 1, \dots, n$ such that $\tau_i = g_i(X_i)$ and τ_i has distribution function F_i. We have $F_i(g_i(t)) = Q(\tau_i <$

[6] Such a correlation structure is appropriate for computations: it involves only n parameters and provides tractable expressions for survival functions. Here, we depart from the BIS notations, where $X_i = \sqrt{\rho_i}V + \sqrt{1-\rho_i}\bar{V}_i$. In the BIS settings, if $\rho_i = \rho$ is independent of i, then $\mathrm{cov}(X_i, X_j) = \rho$. Thus ρ can readily be seen as a correlation parameter.

[7] The ε_i do not need to be Gaussian in order to obtain this dimensionality reduction.

$g_i(t)) = Q(X_i < t) = \int \Phi\left(\frac{t-m_i(v)}{\sigma_i(v)}\right) f(v)dv$. Thus, we have the calibrating equation:

$$g_i(t) = F_i^{-1}\left(\int \Phi\left(\frac{t-m_i(v)}{\sigma_i(v)}\right) f(v)dv\right), \tag{2.1}$$

which completes the description of the model. Here, the conditional default probabilities are given by:

$$Q(\tau_i \le t \mid V) = p_t^{i|V} = \Phi\left(\frac{g_i^{-1}(t) - m_i(V)}{\sigma_i(V)}\right)$$

2.3 One factor structure and Archimedean copulas

Another interesting examples of our framework corresponds to the frailty model commonly used in survival analysis. Let f be the density of a positive mixing variable V and $\psi(s) = \int_0^\infty e^{-sv} f(v)dv$, the Laplace transform of f. We define G_i as $\forall t \ge 0$, $G_i(t) = \exp\left(-\psi^{-1}(F_i(t))\right)$ where F_i is the cdf of default time τ_i. G_i defines a distribution function and $F_i(t) = \psi\left(-\ln G_i(t)\right) = \int_0^\infty G_i^v(t) f(v)dv$. Let us remark that G_i^v is a proper distribution function and that conditionally on $V = v$, the distribution of τ_i is G_i^v. We define the joint distribution of default times by: $F(t_1, \dots, t_n) = \int_0^\infty \prod_{i=1}^n G_i^v(t_i) f(v)dv$. $\forall t_1, \dots, t_n$, $Q\left(\tau_1 < t_1, \dots, \tau_n < t_n \mid V\right) = \prod_{i=1}^n G_i^V(t_i)$; thus, conditionally on V the default times are independent. We then have:

$$p_t^{i|V} = G_i^V(t) = \exp\left(-V\psi^{-1}(F_i(t))\right).$$

Since:

$$F(t_1, \dots, t_n) = \int_0^\infty \prod_{i=1}^n G_i^v(t_i) f(v)dv = \psi\left(-\sum_{i=1}^n \ln G_i(t_i)\right) = \psi\left(\sum_{i=1}^n \psi^{-1}\left(F_i(t_i)\right)\right),$$

we conclude that the joint distribution function can be computed directly as:

$$F(t_1, \dots, t_n) = \psi\left(\sum_{i=1}^n \psi^{-1}\left(F_i(t_i)\right)\right),$$

and the copula of default times is given by:

$$C(u_1, \dots, u_n) = \psi\left(\psi^{-1}(u_1) + \dots + \psi^{-1}(u_n)\right).$$

Thus C is an Archimedean copula with generator $\phi = \psi^{-1}$. A typical example is the Clayton copula, where the mixing variable V has a Gamma distribution with parameter $1/\theta$, where $\theta > 0$. More precisely, we have $f(x) = \frac{1}{\Gamma(1/\theta)} e^{-x} x^{(1-\theta)/\theta}$, $\psi^{-1}(s) = s^{-\theta} - 1$, $\psi(s) = (1+s)^{-1/\theta}$. This leads to $C(u_1, \dots, u_n) = \left(u_1^{-\theta} + \dots + u_n^{-\theta} - n + 1\right)^{-1/\theta}$ and $G_i(t) = \exp\left(1 - F_i(t)^{-\theta}\right)$.

3 Survival functions and stochastic intensities

3.1 Survival function of first to default time

We consider here the computation of the survival function of the first to default time that will be further involved in computing basket default swap premiums under various dependence assumptions. The distribution of first to default time can be computed directly, while for more general k-th to default time, we use a moment generating function approach as discussed below. We denote by $\tau^1 = \min(\tau_1, \dots, \tau_n)$ the first to default time and by S^1 the associated survival function. For $t \geq 0$, $S^1(t) = Q(\tau^1 > t) = Q(\tau_1 > t, \dots, \tau_n > t) = S(t, \dots, t)$. Using the previous joint survival expression, we readily get: $S^1(t) = \int \prod_{i=1}^n q_t^{i|v} f(v)dv$ which specializes to the one factor Gaussian, mean variance Gaussian mixtures and Archimedean cases.

3.2 Stochastic intensities up to first to default time

We now derive stochastic intensities of default times under filtration $\mathbb{H}$ on the set $\tau^1 > t$ that we denote by $\lambda_i(t)$, $i = 1, \dots, n$. We recall that the $\mathbb{H}_i$ stochastic intensity of τ_i is given by $h_i(t)1_{\tau_i \leq t}$.

Property 3.1 stochastic intensities (before first to default time)
On $\{\tau^1 > t\}$, the $\mathbb{H}$ stochastic intensities $\lambda_i(t)$, $i = 1, \dots, n$ are given by:

$$\lambda_i(t) = -\frac{1}{S^1(t)} \frac{\partial S(t, \dots, t)}{\partial t_i} = -\frac{\int \frac{dq_t^{i|v}}{dt} \left(\prod_{j \neq i} q_t^{j|v} \right) f(v)dv}{\int \prod_{i=1}^n q_t^{i|v} f(v)dv}. \tag{3.2}$$

Let us firstly remark that while $S^1(t)$ has already been computed in the case of factor models, we can write a tractable expression of $\frac{\partial S(t,\dots,t)}{\partial t_i}$. Since $S(t_1, \dots, t_n) = \int \prod_{i=1}^n q_{t_i}^{i|v} f(v)dv$, we get $\frac{\partial S(t,\dots,t)}{\partial t_i} = \int \frac{dq_t^{i|v}}{dt} \left(\prod_{j \neq i} q_t^{j|v} \right) f(v)dv$. To prove the property, let us remark that $Q(\tau_i \in]t, t+dt] \mid \mathcal{H}_t)$, as a $\mathcal{H}_t$ random variable, is constant on $\{\tau^1 > t\}$. Then it can be easily checked that on $\{\tau^1 > t\}$, $Q(\tau_i \in]t, t+dt] \mid \mathcal{H}_t) = Q(\tau_i \in]t, t+dt] \mid \tau^1 > t)$[8]. Since $\lambda_i(t) = \lim_{dt \to 0^+} \dfrac{Q(\tau_i \in]t, t+dt] \mid \mathcal{H}_t)}{dt}$, we get $1_{\{\tau^1 > t\}} \lambda_i(t) = 1_{\{\tau^1 > t\}} \lim_{dt \to 0^+} \dfrac{Q(\tau_i \in]t, t+dt] \mid \tau^1 > t)}{dt}$.

$\lambda_i(t)$ can be seen as the probability of name i defaulting in the next small time interval $]t, t+dt]$ provided that none of the reference credits have defaulted prior to t[9].

3.3 Number of defaults distribution

When considering m out of n default swaps, with $m > 1$, it is important to compute the probability of k names being in default at time t where $k = 0, \dots, n$. We thereafter denote by $N(t) = \sum_{i=1}^n 1_{\{\tau_i \leq t\}}$, $N(t)$

[8] We write:

$$Q(\tau_i \in]t, t+dt] \mid \mathcal{H}_t) = A 1_{\{\tau^1 > t\}} + Z 1_{\{\tau^1 \leq t\}},$$

where A is a constant. Thus $E\left[Q(\tau_i \in]t, t+dt] \mid \mathcal{H}_t) 1_{\{\tau^1 > t\}}\right] = AE\left[1_{\{\tau^1 > t\}}\right]$. On the other hand, we have $E\left[Q(\tau_i \in]t, t+dt] \mid \mathcal{H}_t) 1_{\{\tau^1 > t\}}\right] = E\left[1_{\{\tau_i \in]t, t+tdt]\}} 1_{\{\tau^1 > t\}}\right]$ from the definition of conditional expectation. This provides the stated result for A.

[9] Let us remark that while for notational convenience $\lambda_i(t)$ is indexed by i only, it depends on the chosen set of names $1, \dots, n$ since every set of names defines a filtration. Fortunately for pricing purpose, the computed premiums do not depend of such an arbitrary choice. In other words, enlarging the filtration $\mathbb{G}$ by observing default arrivals of other names than those involved in the basket structure does not change the basket default swap premiums (though the default intensities do change).

being the counting process associated with the number of defaults and by $N_i(t) = 1_{\{\tau_i \le t\}}$ the counting process associated with default of name i. Let us now compute the probability generating function of $N(t)$:

$$\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = \sum_{k=0}^{n} Q(N(t) = k)u^k.$$

Let us remark that $N_i(t)$ is a Bernoulli random variable and $E\left[u^{N_i(t)} \mid V\right] = q_t^{i|V} + p_t^{i|V} \times u$, where $q_t^{i|V} = Q(\tau_i > t \mid V)$ and $p_t^{i|V} = Q(\tau_i \le t \mid V)$ are the conditional survival and default probabilities. Using iterated expectations theorem and the conditional independence of the $N_i(t)$, we obtain:

$$\psi_{N(t)}(u) = E\left[\prod_{i=1}^{n}\left(q_t^{i|V} + p_t^{i|V} \times u\right)\right] = \int \prod_{i=1}^{n}\left(q_t^{i|V} + p_t^{i|V} \times u\right) f(v)dv,$$

where f stands for the factor density. Since $\psi_{N(t)}(u)$ can be written as $E\left[u^n \phi_n(V) + \ldots + \phi_0(V)\right]$, where $\phi_k(V)$ is given by a formal expansion of $\prod_{i=1}^{n}\left(q_t^{i|V} + p_t^{i|V}\right)$, we can obtain the probability of k names being in default at time t as : $Q(N(t) = k) = E\left[\phi_k(V)\right] = \int \phi_k(v) f(v)dv$.

For the different models (one factor Gaussian model, one factor mean-variance mixture model, frailty model), we have just to input the corresponding conditional default probabilities $p_t^{i|V}$. Let us remark that for practical purpose, the formal expansion approach to the computation of the probabilities $P(N(t) = k)$, $k = 0, \ldots, n$ is well suited for small dimensional problems. More generally one can use FFT approaches to obtain the distribution function from its pgf.

3.4 Survival distribution of k-th to default time

We denote by τ^k the time of the k-th default and by $S^k(t) = Q(\tau^k > t)$, $F^k(t) = 1 - S^k(t)$ the corresponding survival and distribution functions. We can write:

$$S^k(t) = Q(\tau^k > t) = Q(N(t) < k) = \sum_{l=0}^{k-1} Q(N(t) = l),$$

which involves only the known $Q(N(t) = k)$, $k = 0, 1, \ldots, n$.

4 Pricing of basket default swaps

We consider thereafter the pricing of various basket default swaps. In a first to default swap, there is a default payment at the first to default time. The payment corresponds to the non recovered part of bond in default[10]. In an m out of n basket default swap ($m \le n$, where n is the number of names), there is a default payment a the m-th default time. We can also consider some basket default swaps that provide protection for defaults ranking between d_m and d_M, with $1 \le d_m \le d_M \le n$. The default leg here is simply the sum of default legs of m out of n default swaps, with $d_m \le m < d_M$. We detail below the premium payments

[10]More precisely, the recovery is based on the nominal plus the accrued coupon. In the following, we will make the simplifying assumption of a recovery based on the nominal only. However, this assumption can easily be relaxed by considering a time dependent nominal in the pricing formulas.

of such a basket credit derivative. In a homogeneous basket, the nominals and the recovery rates of the reference credits are assumed to be equal. However, the marginal default probabilities may differ.

We compute separately the price of the premium leg and of the default payment leg at time zero. The basket premium is such that the prices of the two legs are equal. For simplicity, we assume independence between default dates and interest rates, since the important issue for basket type credit derivatives is the modelling of dependence between default dates[11]. Given the products studied, we will then only need discount bond prices and we may well assume deterministic interest rates. $B(t)$ will thereafter denote the discount price (at time 0) for maturity t. Similarly, we assume that the recovery rates on the underlying bonds are independent from default times and interest rates. Since the payoffs of basket default swaps are linear in these recovery rates, only the expected recovery rates are involved. For notational simplicity, we will thus confuse recovery rates and their expectation[12].

4.1 m out of n basket default swaps: premium leg

We consider here a basket default swap on a set of n reference credits, with protection payment arising between defaults of rank d_m (included) and d_M (excluded). We denote by t_i, $i = 1, \dots, I$ the premium payments dates (with $t_I = T$ where T is the maturity date of the basket default swap) and by X the periodic premium. $\Delta_{i-1,i}$ represents the length of period $[t_{i-1}, t_i]$ and $B(t_i)$ is the discount factor for maturity t_i. For simplicity, we do not take into account accrued premium payments due to defaults between premium payments dates[13]. Let us firstly detail the premium payments and consider some payment date t_i. If $N(t_i) \geq d_M$, the basket payments are exhausted[14]. If $N(t_i) < d_m$, the premium is due on a full nominal of $d_M - d_m$. In between, if $d_m \leq N(t_i) < d_M$, the premium is paid on the outstanding protected nominal, i.e. $d_M - N(t_i)$.

From the distribution function of $N(t)$ we can compute the premium payment leg for m out of n basket default swaps. The discounted expectation of premium payment of date t_i can then be written as:

$$\Delta_{i-1,i} X B(t_i) \times \left((d_M - d_m) Q(N(t_i) < d_m) + \sum_{k=d_m}^{d_M} (d_M - k) Q(N(t_i) = k) \right),$$

where the probabilities of k names being in default at time t, $Q(N(t) = k)$ have already been computed. We can eventually write the price of the premium payment leg by summing over possible premium payment dates:

$$\sum_{i=1}^{I} \Delta_{i-1,i} X B(t_i) \times \left((d_M - d_m) \sum_{k=0}^{d_m - 1} Q(N(t_i) = k) + \sum_{k=d_m}^{d_M} (d_M - k) Q(N(t_i) = k) \right).$$

This price only involves the distribution of the number of defaults through $Q(N(t_i) = k)$.

[11] However, for such products as quanto default swaps, defaultable interest rate swaps, credit spread options, the dependence between defaults and interest rates is an important issue.

[12] Let us remark that CDO tranches do not fulfil that linearity in the recovery rates. The distribution of recovery rates can have some effect on the price of such tranches (see below).

[13] The accrued premium adjustments can be dealt with analytically. See the CDO pricing section for an example.

[14] We recall that $N(t)$ is the number of names in default at time t.

4.2 m out of n homogeneous default swaps: default leg

Let us now consider the default payment leg of a homogeneous basket default swap: we denote by 1, the nominal of a given reference credit and by δ the unique recovery rate. As will be seen, the homogeneity assumption allows the computation of the price of the default payment leg knowing only the distribution of k-th to default times. We will consider the default payments at dates $\tau^{d_m+1}, \ldots, \tau^k, \ldots, \tau^{d_M}$ provided that these dates are before the maturity of the basket default swap $T = t_I$. Straightforward algebra shows that the payoff of the default leg is equal to the sum of the payoffs of default legs paying $1-\delta$ at the k-th default, $d_m \leq k < d_M$, provided that the k-th default is before $T = t_I$. Then, we simply have to compute the current price of a k-th to default payment and sum over possible k. We remark that the corresponding discounted payoff can be written as $(1-\delta)1_{[0,T]}(\tau^k)B(\tau^k)$ where $B(t)$ denotes the discount factor for maturity t. Under the previous independence assumptions on interest rates and recovery rates, as a direct consequence of transfer theorem, we can then write the price of the k-th to default payment leg as:

$$E\left[(1-\delta)1_{[0,T]}(\tau^k)B(\tau^k)\right] = -(1-\delta)\int_0^T B(t)dS^k(t), \tag{4.1}$$

Integrating by parts, we can write the price of the payment leg of the k-th to default swap by using:

$$(1-\delta) \times \left(1 - S^k(T)B(T) + \int_0^T S^k(t)dB(t)\right). \tag{4.2}$$

Under the usual smoothness assumptions we have that $fw(t)B(t) = -\frac{dB(t)}{dt}$ where $fw(t)$ is the spot forward rate. Thus, the payment leg can be written as $(1-\delta) \times \left(1 - S^k(T)B(T) - \int_0^T fw(t)B(t)S^k(t)dt\right)$.

As an example the price of the default payment leg of a First to Default swap (we drop the $1-\delta$ term) is given by:

$$1 - S^1(T)B(T) - \int_0^T fw(t)B(t)S^1(t)dt,$$

where $S^1(t) = \int \prod_{i=1}^n q_t^{i|v} f(v)dv$ is the survival function of first to default time, $q_t^{i|v}$ are the conditional survival probabilities and f the factor density. This will specialize to one factor Gaussian, one factor mean-variance Gaussian mixture and Archimedean copulas cases.

To illustrate the pricing approach for homogeneous first to default swaps, we study how the first to default annual premium varies with the number of names, up to $n = 50$ names, all with equal unit nominal. The underlying credit spreads are assumed to be constant and equal to 80 bp, the recovery rate is equal to $\delta = 40\%$ and the maturity of the first to default swap is five years. For the Gaussian copula model, we have set the correlation parameter to $\rho = 30\%$ and for the Clayton copula model, the dependence parameter is $\theta = 0.1728$. This parameter is set in order to have the same first to default swap premium when $n = 25$.

Table 1 reports the results: the premium increases with the number of names, but less quickly than in the independence case. Let us remark that for $n = 1$, the first to default swap collapses into a plain default swap and the corresponding premium is 80 bp in the two cases.We can see that prices computed with the Clayton copula are larger for small baskets but smaller for large baskets. However the overall changes in premiums with respect to the number of names are rather similar and the discrepancies between the two pricing models are small.

number of names	Clayton	Gaussian
1	80	80
5	335	331
10	571	564
15	759	752
20	917	913
25	1055	1055
30	1177	1183
35	1288	1301
40	1390	1411
45	1485	1514
50	1573	1611

Table 1: first to default swaps annual premium (bp)

It is also possible to study how the basket default premiums vary with the rank of the default protection. We consider ten names. Credit spreads are distributed uniformly between 60 bp and 150 bp. The recovery rates are assumed to be equal to $\delta = 40\%$. The maturity of the basket default swaps is equal to five years. Table 2 displays basket default swaps annual premiums, for a first to default, second to default up to last to default. We compare the premiums computed under the one factor Gaussian model, with a correlation of $\rho = 30\%$, and under the Clayton copula model with a dependence parameter of $\theta = 0.193$. The parameter in the Clayton copula model is such that the premiums of the first to default swap are equal which eases comparisons between models.

Rank	Clayton	Gaussian
1	723	723
2	277	274
3	122	123
4	55	56
5	24	25
6	10	11
7	3.6	4.3
8	1.2	1.5
9	0.28	0.39
10	0.04	0.06

Table 2: basket default swaps annual premium (bp)

The striking result is that basket premiums almost do not depend on the choice of copula. Let us recall that all marginal distributions of default dates are equal and that the dependence parameters in the two copulas are set in such a way that first to default premiums are equal. Then, whatever the default rank, all basket premiums are very similar. Though the copulas involve rather different dependence structure, this is not of first importance in this basket default swap pricing example. If we look in greater detail at the premiums, we find that second to default swaps have slightly higher premiums when priced with the Clayton copula. For higher rank default swaps, we find slightly higher premiums with the Gaussian copula.

4.3 Default leg of first to default swap: non homogeneous case

In the general case of the default leg of a non homogeneous m out of n basket default swap, the computations are a bit more involved. We consider a series of n names with nominal N_i and recovery rates δ_i, $i = 1, \dots, n$. We denote by $M_i = N_i \times (1 - \delta_i)$ the payment in case of default[15]. We consider a first to default swap with maturity T. As before, let us denote by $\tau^1 = \min(\tau_1, \dots, \tau_n)$ the first to default time. If $\tau^1 \leq T$, there is a default payment at that time that depends on the name in default: if name i is in default, the payment is equal to M_i. In the models under study, the default times admit a joint density. As a consequence, the probabilities of simultaneous defaults $Q(\tau_j = \tau_k)$ for $j \neq k$ are equal to zero. Thus, there is no ambiguity about the first to default name and the default payment can be decomposed as the sum of n default payments, each of them corresponding to a specific name being the first to default. Let us denote by $\tau^{(-i)} = \min_{j \neq i} \tau_j$ the first to default time for the set of reference credits, i excluded. i is the first to default name if and only if $\tau^{(-i)} > \tau_i$ or equivalently if and only if $\tau^1 \geq \tau_i$. The discounted default payment can then be expressed as:

$$\sum_{i=1}^{n} M_i B(\tau_i) 1_{\tau^1 \geq \tau_i} 1_{\tau_i \leq T}.$$

Let us consider the price of the default payment leg as the limit of the price in a discrete model[16]. We denote by π_k, $k \in \mathbb{N}$, a sequence of partitions of $[0, T]$ with mesh converging to zero. The time zero price of the default leg is given by:

$$\lim_{k \to \infty} \sum_{i=1}^{n} \sum_{t_l \in \pi_k} M_i B(t_l) Q\left(\tau_i \in]t_l, t_{l+1}], \tau_j > t_l, j \neq i\right) = -\sum_{i=1}^{n} M_i \int_0^T \partial_i S(t, \dots, t) B(t) dt, \tag{4.3}$$

where $\partial_i S(t, \dots, t) = \frac{\partial S(t, \dots, t)}{\partial t_i}$ denotes the partial derivative of the joint survival function with respect to the ith component at point $(t, \dots, t)$. Indeed $-\partial_i S(t, \dots, t) dt \approx Q(\tau_j > t, j \neq i, \tau_i \in]t, t + dt])$[17]. From the definition of the conditional hazard rates, we have $\lambda_i(t) S^1(t) = -\partial_i S(t, \dots, t)$, where $S^1(t)$ denotes the survival function of first to default time. Thus, the price of the default payment leg can also be written as:

$$\sum_{i=1}^{n} M_i \int_0^T \lambda_i(t) S^1(t) B(t) dt.$$

We can use the expressions of $\lambda_i S^1(t)$ for the one factor Gaussian, mean-variance Gaussian mixture and Archimedean copulas derived from equation (3.2), to readily obtain the price of the default payment leg of a first to default swap in the non homogeneous case.

[15] The nominals will normally be equal but the estimated recoveries may well differ.

[16] See below, default leg of a m out of n basket default swap, for a more detailed discussion.

[17] We can provide a more rigorous, while more abstract, derivation of this statement. We need to compute for every $i = 1, \dots, n$, $E\left[B(\tau_i) 1_{\tau^1 \geq \tau_i} 1_{\tau_i \leq T}\right]$. From iterated expectations theorem, this can be written as $E\left[E\left[B(\tau_i) 1_{\tau^1 \geq \tau_i} 1_{\tau_i \leq T} \mid \tau_i\right]\right] = E\left[B(\tau_i) Q(\tau^1 \geq \tau_i \mid \tau_i) 1_{\tau_i \leq T}\right]$ or from transfer theorem as $\int_0^T B(t) Q(\tau^1 \geq t \mid \tau_i = t) f_i(t) dt$, where f_i stands for the marginal density of τ_i. By the same conditioning technique and using the continuity of the distribution, we can write:

$$S(t_1, \dots, t_n) = Q(\tau_1 \geq t_1, \dots, \tau_n \geq t_n) = \int_{t_i}^{\infty} Q(\tau_j \geq t_j, \forall j \neq i \mid \tau_i = u) f_i(u) du.$$

Differentiating with respect to t_i provides $Q(\tau^{(-i)} \geq t \mid \tau_i = t) f_i(t) = -\partial_i S(t, \dots, t)$, which allows to conclude since $Q(\tau^{(-i)} \geq t \mid \tau_i = t) = Q(\tau^1 \geq t \mid \tau_i = t)$.

4.4 default leg of m out of n default swaps: non homogeneous case

We consider a series of n names with nominal N_i and recovery rates δ_i, $i = 1, \ldots, n$ and the default leg of a m out of n basket default swap ($1 \le m \le n$) with maturity T. We consider here a single default payment ; more general cases can be treated straightforwardly by summing up (see subsection on homogeneous baskets). We denote by $M_i = N_i \times (1 - \delta_i)$ the payment in case of default. τ^m denotes the m-th default time. If $\tau^m \le T$, there is a default payment at that time that depends on the name in default: if name i is in default, the payment is equal to M_i. We recall that $N_i(t) = 1_{\{\tau_i \le t\}})$ and $N(t) = \sum_{j=1}^n N_j(t)$. We define $N^{(-i)}(t) = \sum_{j \neq i} N_j(t)$ and $N^m(t) = 1_{\{\tau^m \le t\}}$. The discounted payoff can be written as:

$$\sum_{i=1}^n M_i B(\tau_i) 1_{\tau_i \le T} 1_{N^{(-i)}(\tau_i) = m-1} = \sum_{i=1}^n \int_0^T M_i B(t) 1_{N^{(-i)}(t) = m-1} dN_i(t) \tag{4.4}$$

where $B(t)$ is the maturity t discount factor. We can see $N^{(-i)}(\tau_i) = m - 1$ if and only if the m-th default is associated with name i[18]. We have thus decomposed the default payments into n payoffs, each of them being similar to a plain CDS default payment activated upon some event being satisfied.

Let us now compute the probability of name i being the m-th to default time and that default time being in the interval $]t, t']$, $t' > t$. Let us remark that:

$$\{\tau^m = \tau_i, \tau^m \in]t, t']\} = \{N(t) = m - 1, \tau_i \in]t, t']\},$$

The latter set corresponding to $m - 1$ names being in default at time t and default date of name i being in the interval $]t, t']$. Since $\{\tau_i \in]t, t']\} = \{N_i(t') - N_i(t) = 1\}$, we can write:

$$\{N(t) = m - 1, \tau_i \in]t, t']\} = \{N(t) = m - 1, N_i(t') - N_i(t) = 1\}.$$

Lastly, for events such that τ_i is after t, $N(t) = N^{(-i)}(t)$. Thus, we need to compute:

$$Q\left(N^{(-i)}(t) = m - 1, N_i(t') - N_i(t) = 1\right).$$

We consider the slightly more general issue of computing $Q\left(N^{(-i)}(t^*) = m - 1, N_i(t') - N_i(t) = 1\right)$, where $t^* \le t \le t'$. This can be done by using the joint pgf of $\left(N^{(-i)}(t^*), N_i(t') - N_i(t)\right)$ defined by $\psi(u, v) = E\left[u^{N_i(t') - N_i(t)} v^{N^{(-i)}(t^*)}\right]$. We compute ψ by conditioning on the latent variable V. Conditionally on V, $N_i(t') - N_i(t)$ is a Bernoulli random variable with $Q\left(N_i(t') - N_i(t) = 1 \mid V\right) = Q(\tau_i \le t' \mid V) - Q(\tau_i \le t \mid V) = p_{t'}^{i|V} - p_t^{i|V}$. We can write $\psi(u, v)$ as:

$$\psi(u, v) = \sum_{k=1}^n Q\left(N_i(t') - N_i(t) = 0, N^{(-i)}(t^*) = k\right) v^k + \sum_{k=1}^{n-1} Q\left(N_i(t') - N_i(t) = 1, N^{(-i)}(t^*) = k\right) u v^k.$$

On the other hand,

$$\psi(u, v) = E\left[E\left[u^{N_i(t') - N_i(t)} v^{N^{(-i)}(t^*)} \mid V\right]\right].$$

By conditional independence:

$$\psi(u, v) = E\left[E\left[u^{N_i(t') - N_i(t)} \mid V\right] \times E\left[v^{N^{(-i)}(t^*)} \mid V\right]\right],$$

[18] This triggering effect is a consequence of the absence of simultaneous defaults. As for the first to default swap, we would need further payment definitions in case of simultaneous defaults.

which leads to:

$$\psi(u,v) = E\left[\left(1 - p_{t'}^{i|V} + p_t^{i|V} + \left(p_{t'}^{i|V} - p_t^{i|V}\right)u\right) \times \prod_{j \neq i}\left(1 - p_{t^*}^{j|V} + p_{t^*}^{j|V} v\right)\right].$$

As a consequence, we obtain:

$$\sum_{k=1}^{n-1} Q\left(N_i(t') - N_i(t) = 1, N^{(-i)}(t^*) = k\right) v^k = E\left[\left(p_{t'}^{i|V} - p_t^{i|V}\right) \times \prod_{j \neq i}\left(1 - p_{t^*}^{j|V} + p_{t^*}^{j|V} v\right)\right],$$

where the term within the expectation can be computed by formal expansion.

We now turn back to the pricing of the default payment leg and compute:

$$\sum_{i=1}^{n} E\left[\int_0^T M_i B(t) 1_{N^{(-i)}(t)=m-1} dN_i(t)\right], \tag{4.5}$$

by looking after the terms: $E\left[\int_0^T B(t) 1_{N^{(-i)}(t)=m-1} dN_i(t)\right]$, $i = 1, \dots, n$. $\int_0^T B(t) 1_{N^{(-i)}(t)=m-1} dN_i(t)$ is a plain stochastic integral with respect to the pure jump process $N_i(t)$. Let us consider a given sequence of partitions of $[0,T]$, π_k with mesh converging to zero. We define the processes:

$$V_{i,k}(t) = \sum_{t_l \in \pi_k} 1_{N^{(-i)}(t_l)=m-1} 1_{]t_l, t_{l+1}]}(t).$$

$V_{i,k}$ is an adapted process (with respect to the filtration generated by the set of default times) with càglàd paths. The sequence of processes $V_{i,k}$ converges uniformly on compacts in probability[19] towards $1_{N^{(-i)}(.)=m-1}$. By continuity properties of stochastic integrals,

$$\int_0^T B(t) 1_{N^{(-i)}(t)=m-1} dN_i(t) = \lim_{k\to\infty} \sum_{t_l \in \pi_k} B(t_l) 1_{N^{(-i)}(t_{l-1})=m-1} \left(N_i(t_{l+1}) - N_i(t_l)\right),$$

where the limit is taken in probability[20]. The random variables:

$$\int_0^T B(t) 1_{N^{(-i)}(t)=m-1} dN_i(t), \quad \sum_{t_l \in \pi_k} B(t_l) 1_{N^{(-i)}(t_{l-1})=m-1} \left(N_i(t_{l+1}) - N_i(t_l)\right),$$

are uniformly integrable[21]. Therefore, we conclude:

$$E\left[\int_0^T B(t) 1_{N^{(-i)}(t)=m-1} dN_i(t)\right] = \lim_{k\to\infty} E\left[\sum_{t_l \in \pi_k} B(t_l) 1_{N^{(-i)}(t_{l-1})=m-1} \left(N_i(t_{l+1}) - N_i(t_l)\right)\right],$$

or equivalently as $\lim_{k\to\infty} \sum_{t_l \in \pi_k} B(t_l) Q\left(N^{(-i)}(t_{l-1}) = m-1, N_i(t_{l+1}) - N_i(t_l) = 1\right)$. Let us for instance consider the partitions of $[0,T]$ given by $\pi_k = \{0, \frac{T}{k}, \dots, \frac{lT}{k}, \dots, T\}$. We can write:

$$\sum_{k=1}^{n-1} \left[Q\left(N_i(t_{l+1}) - N_i(t_l) = 1, N^{(-i)}(t_l) = k\right) - Q\left(N_i(t_{l+1}) - N_i(t_l) = 1, N^{(-i)}(t_{l-1}) = k\right)\right] v^k,$$

[19] Under the standing assumption of a smooth joint survival function. This implies that there exists some hazard rate for the different rank statistics.

[20] We do not need ucp here.

[21] They take value in [0,1].

as:
$$\sum_{k=1}^{n-1} E\left[\left(p_{t_{l+1}}^{i|V} - p_{t_l}^{i|V}\right) \times \left(\prod_{j\neq i}\left(q_{t_l}^{j|V} + p_{t_l}^{j|V}v\right) - \prod_{j\neq i}\left(q_{t_{l-1}}^{j|V} + p_{t_{l-1}}^{j|V}v\right)\right)\right] v^k = o\left(\frac{1}{k}\right),$$
for smooth conditional default probabilities $p_t^{i|V}$. As a consequence, we can consider the limit:
$$\lim_{k\to\infty}\sum_{t_l\in\pi_k} B(t_l)Q\left(N^{(-i)}(t_l) = m-1, N_i(t_{l+1}) - N_i(t_l) = 1\right).$$
For smooth conditional default probabilities $p_t^{i|V}$, we define:
$$Z_k^i(t) = \lim_{t'\to t}\frac{1}{t'-t}Q\left(N_i(t') - N_i(t) = 1, N^{(-i)}(t) = k\right), \quad k = 1,\dots,n-1,\ i = 1,\dots,n.$$
The $Z_k^i(t)$ are given by: $\sum_{k=1}^{n-1} Z_k^i(t)v^k = E\left[\frac{dp_t^{i|V}}{dt} \times \prod_{j\neq i}\left(q_t^{j|V} + p_t^{j|V}v\right)\right]$. The price of the default payment leg is then given by:
$$\sum_{i=1}^{n}\int_0^T B(t)M_iZ_{m-1}^i(t)dt. \tag{4.6}$$
We can provide some simple alternative expressions. Let us denote by:
$$Z_k^{i|V}(t) = \lim_{t'\to t}\frac{1}{t'-t}Q\left(N_i(t') - N_i(t) = 1, N^{(-i)}(t) = k \mid V\right).$$
Then,
$$\sum_{k=1}^{n-1} Z_k^{i|V}(t)v^k = \frac{dp_t^{i|V}}{dt} \times \prod_{j\neq i}\left(q_t^{j|V} + p_t^{j|V}v\right),$$
and $E\left[Z_k^{i|V}(t)\right] = Z_k^i(t)$, where the expectation is taken upon V. Moreover,
$$Z_k^{i|V}(t) = \frac{dp_t^{i|V}}{dt} \times Q\left(N^{(-i)}(t) = k \mid V\right).$$
As a consequence the price of the default payment leg can be written as:
$$\sum_{i=1}^{n} E\left[\int_0^T B(t)M_iQ\left(N^{(-i)}(t) = m-1 \mid V\right)dp_t^{i|V}\right], \tag{4.7}$$
where the expectation is taken over V and $Q\left(N^{(-i)}(t) = m-1 \mid V\right)$ is obtained from the formal expansion of the polynomial $\prod_{j\neq i}\left(q_t^{j|V} + p_t^{j|V}v\right)$. Let us mention another derivation of the previous result. We need to compute $E\left[B(\tau_i)1_{\tau_i\leq T}1_{N^{(-i)}(\tau_i)=m-1}\right]$ for $i = 1,\dots,n$. We write $E\left[B(\tau_i)1_{\tau_i\leq T}1_{N^{(-i)}(\tau_i)=m-1} \mid V, \tau_i = t\right]$ as $1_{t\leq T}B(t)E\left[1_{N^{(-i)}(t)=m-1} \mid V, \tau_i = t\right]$. Let us remark that $N^{(-i)}(t)$ only involves τ_k for $k \neq i$. Thanks to the conditional on V independence of default times, we can simplify the previous conditional expectation as $B(t)Q\left(N^{(-i)}(t) = m-1 \mid V\right)$. By integrating over the conditional (on V) distribution of τ_i, we obtain:
$$E\left[B(\tau_i)1_{\tau_i\leq T}1_{N^{(-i)}(\tau_i)=m-1} \mid V\right] = \int_0^T B(t)Q\left(N^{(-i)}(t) = m-1 \mid V\right)dp_t^{i|V},$$
which readily leads back to the price of the default payment leg.

As can be seen from the previous equations, one can readily compute the price of the default payment leg of a general m out of n default swap, once the conditional (on the latent variable) probabilities of default are given. Putting in the relevant probabilities provides the price of the default payment leg for the one factor Gaussian model, the mean variance mixture model and the Archimedean copula model.

5 Pricing of CDO's

5.1 Portfolio Loss Distributions

We will thereafter consider a synthetic CDO with some given maturity T. This is based upon n reference credits, $j = 1, \dots, n$[22], with nominals N_i, $i = 1, \dots, n$ and maturity also equal to T. δ_j denotes the recovery rate for credit j and $M_j = (1 - \delta_j)N_j$ the corresponding Loss Given Default. As above, we denote by τ_j the default time of name j and by $N_j(t) = 1_{\tau_j \le t}$ the corresponding counting process. $L(t)$ will denote the cumulative loss on the credit portfolio at time t:

$$L(t) = \sum_{j=1}^{n} M_j N_j(t), \tag{5.1}$$

which is also a pure jump process. As a starting point, we compute the characteristic function of the cumulative losses for different time horizons, $\varphi_{L(t)}(u) = E[\exp(iuL(t))]$. We assume conditional independence of default times upon factor V. We moreover assume that the σ-algebras $\sigma(\delta_1), \dots, \sigma(\delta_n)$, $\sigma(V, \tau_1, \dots, \tau_n)$ are independent. We can write from iterated expectations theorem: $\varphi_{L(t)}(u) = E[E[\exp(iuL(t)) \mid V]]$. From the independence of $N_i(t)$ conditionally on V and the independence assumption over the recovery rates, we get $E[\exp(iuL(t)) \mid V] = \Pi_{j=1}^{n} E[\exp(iuM_jN_j(t)) \mid V]$. This gives : $E[\exp(iuL(t)) \mid V] = \prod_{j=1}^{n} \left(q_t^{j|V} + p_t^{j|V} \varphi_{1-\delta_j}(uN_j) \right)$, where $p_t^{j|V}$ and $q_t^{j|V}$ are the conditional default and survival probabilites and $\varphi_{1-\delta_j}$ denotes the characteristic function of $1 - \delta_j$. For instance, in the case where δ_j follows a Beta distribution, we have $\varphi_{1-\delta_j}(uN_j) = M(a_j, a_j + b_j, iuN_j)$ where M is a Kummer function, a_j, b_j some parameters. This leads to:

$$\varphi_{L(t)}(u) = \int \prod_{j=1}^{n} \left(q_t^{j|V} + p_t^{j|V} \varphi_{1-\delta_j}(uN_j) \right) f(v) dv, \tag{5.2}$$

which requires a numerical integration over the factor distribution. We can then get the distribution of $L(t)$ by some Fourier inversion technique.

5.2 Payoff description

In a CDO, default losses on the credit portfolio are split along some thresholds (attachment points) and allocated to the various tranches. As an example, we consider a three tranches CDO, denoted as equity, mezzanine and senior. We denote by A and B the thresholds, $0 \le A < B \le \sum N_j$. The cumulative default payments on the mezzanine tranche are denoted by $M(t)$[23]. $M(t)$ is equal to zero if $L(t) \le A$, to $L(t) - A$ if $A \le L(t) \le B$ and to $B - A$ if $L(t) \ge B$. This can be summarized as : $M(t) = (L(t) - A)1_{[A,B]}(L(t)) + (B - A)1_{]B, \sum N_i]}(L(t))$. Equity and senior tranches are treated similarly. We can notice that $M(t)$ is a pure jump process derived from $L(t)$. We will thereafter denote $M(t) = \omega(L(t))$ where ω is non decreasing function.

The holder of a synthetic CDO tranche receives at time T a unique principal payment of $M(\infty) - M(T)$, where $M(\infty)$ denotes the initial nominal of the tranche and $M(\infty) - M(t)$ will thereafter denote the outstanding

[22] Or on plain Credit Default Swaps.

[23] In order to simplify notations, we use a unique terminology $M(t)$ to denote the cumulative losses on the three tranches.

nominal of the tranche. The interest payments are usually equal to a floating rate plus a fixed margin, that is specific to each tranche, and based on the outstanding nominal on the tranche. As for the basket case, let us set some payment dates, t_i, $i = 1, \dots, I$ with $t_I = T$. The interest payment at t_i is equal to $\Delta_{i-1,i}(M(\infty) - M(t_i))\left(Libor_{t_{i-1}} + X\right)$ where X is the CDO margin, $\Delta_{i-1,i}$ represent the length of period $[t_{i-1}, t_i]$ and $Libor_{t_{i-1}}$ is the t_{i-1} Libor rate for this period. Since the interest is based on outstanding nominal at the end of the period, there are some accrued interest payments. In case name j (say) defaults between t_{i-1} and t_i, the associated accrued interest payment is equal to $(\tau_j - t_{i-1})(Libor_{t_{i-1}} + X)\left(M(\tau_j) - M(\tau_j^-)\right)$[24].

As a consequence, a CDO tranche can be decomposed into a non defaultable amortizing floating rate note[25] plus a default swap transaction where the CDO margin is exchanged against the default payments on the tranche. More precisely, the default payments are simply the increments of $M(t)$, ie there is a payment of $M(t) - M(t^-)$ at every jump time of $M(t)$. The CDO margin payments are equal to $X\Delta_{i-1,i}(M(\infty) - M(t_i))$ at regular payment dates t_i, $i = 1, \dots, I$ plus some accrued payments such as $X(\tau_j - t_{i-1})\left(M(\tau_j) - M(\tau_j^-)\right)$ on default dates τ_j. As above for basket credit derivatives, we will value separately the default and margin leg. The CDO margin is such that the two legs have same value.

5.3 Pricing the default payment leg of a CDO tranche

We can notice that the discounted payoff corresponding to default payments can written as:

$$\int_0^T B(t)dM(t) = \sum_{j=1}^{n} B(\tau_j)N_j(T)\left(M(\tau_j) - M(\tau_j^-)\right),$$

where $B(t)$ stands for the discount factor for maturity t[26] and T is the maturity of the CDO. Since $M(t)$ is an increasing process, we can define Stieltjes integrals with respect to $M(t)$. The price of the default payment leg of the given tranche as:

$$E\left[\int_0^T B(t)dM(t)\right],$$

Using the Stieltjes integration by parts formula, we get $\int_0^T B(t)dM(t) = B(T)M(T) + \int_0^T fw(t)B(t)M(t)dt$, where $fw(t)$ denotes the spot forward rate. Using Fubini theorem, we then have:

$$E\left[\int_0^T B(t)dM(t)\right] = B(T)E[M(T] + \int_0^T fw(t)B(t)E[M(t)]dt.$$

Let us remark that we only need the first moment of the cumulative loss on the tranche which can be obtained from the loss distribution on the portfolio of reference credits:

$$E[M(t)] = (B - A)Q_{L(t)}(]B, \infty[) + \int_A^B (x - a)Q_{L(t)}(dx),$$

[24] Here, the Libor rate for the accrued interest payment is the Libor rate in between dates t_{i-1} and τ_j. For simplicity, we use the same notation as for the Libor rate involved in regular interest payments. The accrued interest has to be paid at τ_j or equivalently it can be compounded up to t_i.

[25] The amortizing dates correspond to default dates of the names in the CDO which are obviously not known in advance. Thus, the self replicating valuation approach to floating rate notes rigorously applies only when interest rates are deterministic. In this case the floating rate note is at par.

[26] As mentioned above, we assume deterministic interest rates.

where $Q_{L(t)}$ is the distribution of $L(t)$.

We also propose a second pricing approach which emphasizes the contribution of different names to the default leg. For simplicity, we assume here that the recovery rates are deterministic. Since the discounted payoff is equal to $\sum_{j=1}^{n} B(\tau_j)N_j(T)\left(M(\tau_j)-M(\tau_j^-)\right)$, we need to compute $E\left[B(\tau_j)N_j(T)\left(M(\tau_j)-M(\tau_j^-)\right)\right]$. We denote by $L^{(-j)}(t)=\sum_{k\neq j} M_k N_k(t)$. We can notice that $L(\tau_j^-)=L^{(-j)}(\tau_j)$ and $L(\tau_j)=L^{(-j)}(\tau_j)+M_j$. We can write

$$E\left[B(\tau_j)N_j(T)\left(\omega\left(L^{(-j)}(\tau_j)+M_j\right)-\omega\left(L^{(-j)}(\tau_j)\right)\right)\mid V,\tau_j=t\right],$$

as $1_{t\leq T}B(t)E\left[\omega\left(L^{(-j)}(t)+M_j\right)-\omega\left(L^{(-j)}(t)\right)\mid V,\tau_j=t\right]$. Since $L^{(-j)}(t)$ only involves τ_k for $k\neq j$ and thanks to the conditional on V independence of default times, we can simplify the previous expression as: $1_{t\leq T}B(t)E\left[\omega\left(L^{(-j)}(t)+M_j\right)-\omega\left(L^{(-j)}(t)\right)\mid V\right]$. By integrating over the conditional on V distribution of τ_j, we obtain:

$$E\left[B(\tau_j)N_j(T)\left(M(\tau_j)-M(\tau_j^-)\right)\mid V\right]=\int_0^T B(t)E\left[\omega\left(L^{(-j)}(t)+M_j\right)-\omega\left(L^{(-j)}(t)\right)\mid V\right]dp_t^{j|V}.$$

Eventually, the price of the default leg of the CDO is provided by some integration over the distribution of V:

$$\sum_{j=1}^{n} E\left[\int_0^T B(t)E\left[\omega\left(L^{(-j)}(t)+M_j\right)-\omega\left(L^{(-j)}(t)\right)\mid V\right]dp_t^{j|V}\right]. \tag{5.3}$$

The conditional on V distributions of $L^{(-j)}(t)+M_j$ and $L^{(-j)}(t)$ can be obtained from their conditional characteristic functions whose expressions are respectively $\varphi_{1-\delta_j}(uN_j)\prod_{k\neq j}\left(q_t^{k|V}+p_t^{k|V}\varphi_{1-\delta_k}(uN_k)\right)$ and $\prod_{k\neq j}\left(q_t^{k|V}+p_t^{k|V}\varphi_{1-\delta_k}(uN_k)\right)$. One can then compute how much different names contribute to the default payment leg of the CDO which is quite useful for risk management purposes.

5.4 Pricing the margin leg of a CDO

Let us firstly consider the valuation of the accrued margins. Let us denote by $t_{k(j)}$ the payment date immediately before τ_j, i.e. $t_{k(j)-1}\leq\tau_j\leq t_{k(j)}$. At default time of name j, there is an accrued margin payment $X(\tau_j-t_{k(j)-1})\left(M(\tau_j)-M(\tau_j^-)\right)$. The discounted payoff corresponding to accrued margin payments can be written:

$$X\sum_{j=1}^{n} B(\tau_j)N_j(T)\left(\tau_j-t_{k(j)-1}\right)\times\left(M(\tau_j)-M(\tau_j^-)\right). \tag{5.4}$$

Let us denote by $t_0=0,t_1,\ldots,t_i,\ldots,t_I=T$ the premium payment dates. We can then write the discounted payoff as:

$$X\sum_{j=1}^{n} B(\tau_j)N_j(T)\left(\tau_j-t_{k(j)-1}\right)\times\left(\omega\left(L^{(-j)}(\tau_j)+M_j\right)-\omega\left(L^{(-j)}(\tau_j)\right)\right). \tag{5.5}$$

Using the same technique as above, we write the price of accrued margin payments as:

$$\sum_{j=1}^{n} E\left[\sum_{i=1}^{I}\int_{t_{i-1}}^{t_i} B(t)(t-t_{i-1})E\left[\omega\left(L^{(-j)}(t)+M_j\right)-\omega\left(L^{(-j)}(t)\right)\mid V\right]dp_t^{j|V}\right]. \tag{5.6}$$

Eventually, we need to compute the price of margin payments made at regular payment dates. The discounted payoff is equal to $X\sum_{i=1}^{I} B(t_i) \times (\omega(\infty) - \omega(L(t_i)))$. Using the independence of default times conditionally on V, we can write the corresponding price as:

$$X\sum_{i=1}^{I} B(t_i) E\left[\omega(\infty) - \omega(L(t_i))\right], \tag{5.7}$$

which is readily obtained from the distribution of $L(t_i)$.

We now provide some practical examples. We have considered 100 names, all with a recovery rate of 40 % and equal unit nominal. The credit spreads are uniformly distributed between 60 bp and 150 bp. The CDO maturity is equal to five years. We have considered CDO margins for equity, mezzanine and senior tranches both under a one factor Gaussian copula assumption and for the Clayton copula. The thresholds for the computation of the tranches are $A = 3\%$ and $B = 10\%$. We firstly consider the Gaussian model and compute the margins with respect to the correlation parameter ρ:

ρ	equity	mezzanine	senior
0 %	6176	694	0.05
10 %	4046	758	5.8
30 %	2303	698	23
50 %	1489	583	40
70 %	933	470	56

Table 3: CDO margins (bp), Gaussian copula

In order to make comparisons with the Clayton copula based pricing model, we proceed the following way. For a given level of correlation in the Gaussian model, we look for the θ parameter in the Clayton copula such that the margins on the equity tranche are the same for the two models[27]. We then compute the margins of the mezzanine and the senior tranche. By construction, the equity margins are the same in the two tables. We can see that the mezzanine and senior tranche margins remain almost unchanged. This is a rather striking result: by matching the left hand side of the loss distribution, we also match well the right tail that is associated with the senior tranche.

ρ	θ	equity	mezzanine	senior
0 %	0	6176	694	0.05
10 %	0.054	4046	759	5.1
30 %	0.1964	2303	694	22
50 %	0.399	1489	588	38
70 %	0.758	933	472	55

Table 4: CDO margins (bp), Clayton copula

[27] $\rho = 0$ corresponds to the independence case. Then, the two models coincide.

6 Conclusion

Under the assumption that default times are independent conditionally on a low dimensional factor, we can derive semi-analytical expressions of basket default swaps and synthetic CDO premiums. This eases comparisons between models as we did when studying prices computed under the Gaussian and Clayton copulas.

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