New results for the pricing and hedging of CDOs

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Presentation related to papers
A note on the risk management of CDOs (2007)
Hedging default risks of CDOs in Markovian contagion models (2007)
Comparison results for credit risk portfolios (2007)
Available on www.defaultrisk.com
New results for the pricing and hedging of CDOs

• Hedging issues
  – Hedging of default risk in contagion models
    ➢ Markov chain approach to contagion models
    ➢ Comparison of models deltas with “market deltas”
  – Hedging of credit spread risk in intensity models

• Pricing issues with factor models
  – Comparison of CDO pricing models through stochastic orders
  – Comprehensive approach to copula, structural and multivariate Poisson models
Hedging Default and Credit Spread Risks within CDOs

• Purpose of the presentation
  - Not trying to embrace all risk management issues
  - Focus on very specific aspects of default and credit spread risk

• Overlook of the presentation
  - Economic background
  - Tree approach to hedging defaults
  - Hedging credit spread risks for large portfolios
I - Economic Background

• Hedging CDOs context

• About 1 000 papers on defaultrisk.com

• About 10 papers dedicated to hedging issues
  – In interest rate or equity markets, pricing is related to the cost of the hedge
  – In credit markets, pricing is disconnect from hedging

• Need to relate pricing and hedging

• What is the business model for CDOs?

• Risk management paradigms
  – Static hedging, risk-return arbitrage, complete markets
I - Economic Background

• Static hedging

• Buy a portfolio of credits, split it into tranches and sell the tranches to investors
  ➢ No correlation or model risk for market makers
  ➢ No need to dynamically hedge with CDS

• Only « budget constraint »:
  ➢ Sum of the tranche prices greater than portfolio of credits price
  ➢ Similar to stripping ideas for Treasury bonds

• No clear idea of relative value of tranches
  ➢ Depends on demand from investors
  ➢ Markets for tranches might be segmented
I - Economic Background

- Risk – return arbitrage
- Historical returns are related to ratings, factor exposure
  - CAPM, equilibrium models
  - In search of high alphas
  - Relative value deals, cross-selling along the capital structure
- Depends on the presence of « arbitrageurs »
  - Investors with small risk aversion
    - Trading floors, hedge funds
  - Investors without too much accounting, regulatory, rating constraints
I - Economic Background

- The ultimate step: complete markets
  - As many risks as hedging instruments
  - News products are only designed to save transactions costs and are used for risk management purposes
  - Assumes a high liquidity of the market

- Perfect replication of payoffs by dynamically trading a small number of «underlying assets»
  - Black-Scholes type framework
  - Possibly some model risk

- This is further investigated in the presentation
  - Dynamic trading of CDS to replicate CDO tranche payoffs
I - Economic Background

• Default risk
  – Default bond price jumps to recovery value at default time.
  – Drives the CDO cash-flows

• Credit spread risk
  – Changes in defaultable bond prices prior to default
    ➢ Due to shifts in credit quality or in risk premiums
  – Changes in the marked to market of tranches

• Interactions between credit spread and default risks
  – Increase of credit spreads increase the probability of future defaults
  – Arrival of defaults may lead to jump in credit spreads
    ➢ Contagion effects (Jarrow & Yu)
I - Economic Background

• Credit deltas in copula models
• CDS hedge ratios are computed by bumping the marginal credit curves
  – Local sensitivity analysis
  – Focus on credit spread risk
  – Deltas are copula dependent
  – Hedge over short term horizons
    ➢ Poor understanding of gamma, theta, vega effects
    ➢ Does not lead to a replication of CDO tranche payoffs
• Last but not least: not a hedge against defaults…
• Credit deltas in copula models
  – Stochastic correlation model (Burstchell, Gregory & Laurent, 2007)
Main assumptions and results

- Credit spreads are driven by defaults
  - Contagion model
  - Credit spreads are deterministic between two defaults
- Homogeneous portfolio
  - Only need of the CDS index
  - No individual name effect
- Markovian dynamics
  - Pricing and hedging CDOs within a binomial tree
  - Easy computation of dynamic hedging strategies
  - Perfect replication of CDO tranches
We will start with two names only

Firstly in a static framework
- Look for a First to Default Swap
- Discuss historical and risk-neutral probabilities

Further extending the model to a dynamic framework
- Computation of prices and hedging strategies along the tree
- Pricing and hedging of tranchelets

Multiname case: homogeneous Markovian model
- Computation of risk-neutral tree for the loss
- Computation of dynamic deltas

Technical details can be found in the paper:
- “hedging default risks of CDOs in Markovian contagion models”
Some notations:

- $\tau_1, \tau_2$ default times of counterparties 1 and 2,
- $\mathcal{H}_t$ available information at time $t$,
- $P$ historical probability,
- $\alpha_1^P, \alpha_2^P$ : (historical) default intensities:
  \[ P[\tau_i \in [t, t+dt|H_t]] = \alpha_i^P dt, \ i = 1, 2 \]

Assumption of « local » independence between default events

- Probability of 1 and 2 defaulting altogether:
  \[ P[\tau_1 \in [t, t+dt], \tau_2 \in [t, t+dt|H_t]] = \alpha_1^P dt \times \alpha_2^P dt \text{ in } (dt)^2 \]
- Local independence: simultaneous joint defaults can be neglected
Building up a tree:

- Four possible states: $(D,D)$, $(D,ND)$, $(ND,D)$, $(ND,ND)$
- Under no simultaneous defaults assumption $p_{(D,D)} = 0$
- Only three possible states: $(D,ND)$, $(ND,D)$, $(ND,ND)$
- Identifying (historical) tree probabilities:

\[
\begin{align*}
\alpha_1^P dt & \quad (D, ND) \\
\alpha_2^P dt & \quad (ND, D) \\
1 - (\alpha_1^P + \alpha_2^P) dt & \quad (ND, ND)
\end{align*}
\]

\[
\begin{align*}
p_{(D,D)} = 0 & \Rightarrow p_{(D,ND)} = p_{(D,D)} + p_{(D,ND)} = p_{(D,D)} = \alpha_1^P dt \\
p_{(D,D)} = 0 & \Rightarrow p_{(ND,D)} = p_{(D,D)} + p_{(ND,D)} = p_{(ND,D)} = \alpha_2^P dt \\
p_{(ND,ND)} = 1 - p_{(D,D)} - p_{(ND,D)} &
\end{align*}
\]
II - Tree approach to hedging defaults

- Stylized cash flows of short term digital CDS on counterparty 1:
  - $\alpha_1^O dt$ CDS 1 premium

  $0 \quad \frac{\alpha_1^P dt}{\alpha_2^P dt} \quad 1 - \alpha_1^O dt \quad (D, ND)$
  $\quad \frac{\alpha_2^P dt}{-\alpha_1^O dt} \quad (ND, D)$
  $\frac{1 - (\alpha_1^P + \alpha_2^P) dt}{-\alpha_1^O dt} \quad (ND, ND)$

- Stylized cash flows of short term digital CDS on counterparty 2:

  $0 \quad \frac{\alpha_1^P dt}{-\alpha_2^O dt} \quad (D, ND)$
  $\quad \frac{\alpha_2^P dt}{1 - \alpha_2^O dt} \quad (ND, D)$
  $\frac{1 - (\alpha_1^P + \alpha_2^P) dt}{-\alpha_2^O dt} \quad (ND, ND)$
II - Tree approach to hedging defaults

- Cash flows of short term digital first to default swap with premium $\alpha_F^0 dt$:
  
  $\alpha_1^p dt \quad 1 - \alpha_F^0 dt \quad (D, ND)$
  
  $\alpha_2^p dt \quad 1 - \alpha_F^0 dt \quad (ND, D)$

  $1 - (\alpha_1^p + \alpha_2^p) dt \quad -\alpha_F^0 dt \quad (ND, ND)$

- Cash flows of holding CDS 1 + CDS 2:

  $\alpha_1^p dt \quad 1 - (\alpha_1^O + \alpha_2^O) dt \quad (D, ND)$

  $\alpha_2^p dt \quad 1 - (\alpha_1^O + \alpha_2^O) dt \quad (ND, D)$

  $1 - (\alpha_1^p + \alpha_2^p) dt \quad -(\alpha_1^O + \alpha_2^O) dt \quad (ND, ND)$

- Perfect hedge of first to default swap by holding 1 CDS 1 + 1 CDS 2
  
  - Delta with respect to CDS 1 = 1, delta with respect to CDS 2 = 1
Absence of arbitrage opportunities imply:

\[ \alpha_F^O = \alpha_1^O + \alpha_2^O \]

Arbitrage free first to default swap premium

- Does not depend on historical probabilities \( \alpha_1^P, \alpha_2^P \)

Three possible states: \((D, ND), (ND, D), (ND, ND)\)

Three tradable assets: CDS1, CDS2, risk-free asset

For simplicity, let us assume \( r = 0 \)
II - Tree approach to hedging defaults

- Three state contingent claims
  - Example: claim contingent on state $(D, ND)$
  - Can be replicated by holding
  - $1 \text{ CDS} + \alpha_1^O \, dt$ risk-free asset

- Replication price $= \alpha_1^O \, dt$
Similarly, the replication prices of the \((ND, D)\) and \((ND, ND)\) claims

\[
\begin{align*}
\alpha_1^p dt & : 0 \quad (D, ND) \\
\alpha_2^p dt & : 1 \quad (ND, D) \\
0 - (\alpha_1^p + \alpha_2^p) dt & : 0 \quad (ND, ND)
\end{align*}
\]

Replication price of:

\[
\begin{align*}
\alpha_1^p dt & : 0 \quad (D, ND) \\
\alpha_2^p dt & : 0 \quad (ND, D) \\
0 - (\alpha_1^p + \alpha_2^p) dt & : 1 \quad (ND, ND)
\end{align*}
\]

Replication price = \(\alpha_1^o dt \times a + \alpha_2^o dt \times b + \left(1 - (\alpha_1^o + \alpha_2^o) dt\right) c\)
II - Tree approach to hedging defaults

- Replication price obtained by computing the expected payoff
  - Along a risk-neutral tree

\[ \alpha_1^0 dt \times a + \alpha_2^0 dt \times b + \left( 1 - (\alpha_1^0 + \alpha_2^0) dt \right) c \]

- Risk-neutral probabilities
  - Used for computing replication prices
  - Uniquely determined from short term CDS premiums
  - No need of historical default probabilities
Computation of deltas

- Delta with respect to CDS 1: $\delta_1$
- Delta with respect to CDS 2: $\delta_2$
- Delta with respect to risk-free asset: $p$

$p$ also equal to up-front premium

$$\begin{align*}
a &= p + \delta_1 \times (1 - \alpha_1^o dt) + \delta_2 \times (-\alpha_2^o dt) \\
b &= p + \delta_1 \times (-\alpha_1^o dt) + \delta_2 \times (1 - \alpha_2^o dt) \\
c &= p + \delta_1 \times (-\alpha_1^o dt) + \delta_2 \times (-\alpha_2^o dt)
\end{align*}$$

- As for the replication price, deltas only depend upon CDS premiums
\section*{II - Tree approach to hedging defaults}

\begin{itemize}
  \item **Dynamic case:**

  \begin{itemize}
    \item \(\lambda_2^0 \, dt\) CDS 2 premium after default of name 1
    \item \(\kappa_1^0 \, dt\) CDS 1 premium after default of name 2
    \item \(\pi_1^0 \, dt\) CDS 1 premium if no name defaults at period 1
    \item \(\pi_2^0 \, dt\) CDS 2 premium if no name defaults at period 1
  \end{itemize}

  \begin{itemize}
    \item Change in CDS premiums due to contagion effects
      \begin{itemize}
        \item Usually, \(\pi_1^0 < \alpha_1^0 < \lambda_1^0\) and \(\pi_2^0 < \alpha_2^0 < \lambda_2^0\)
      \end{itemize}
  \end{itemize}
\end{itemize}
II - Tree approach to hedging defaults

• Computation of prices and hedging strategies by backward induction
  – use of the dynamic risk-neutral tree
  – Start from period 2, compute price at period 1 for the three possible nodes
  – + hedge ratios in short term CDS 1,2 at period 1
  – Compute price and hedge ratio in short term CDS 1,2 at time 0

• Example to be detailed:
  – computation of CDS 1 premium, maturity = 2
  – $p_1dt$ will denote the periodic premium
  – Cash-flow along the nodes of the tree
II - Tree approach to hedging defaults

- Computations CDS on name 1, maturity = 2

\[
0 \quad (D,D)
\]
\[
\lambda_2^O \ dt \quad 0 \quad (D,D)
\]
\[
1-p_1 dt \quad (D,ND)
\]
\[
1-\lambda_2^O \ dt \quad 0 \quad (D,ND)
\]
\[
\alpha_1^O \ dt \quad 1-p_1 dt \quad (D,ND)
\]
\[
\alpha_2^O \ dt \quad -p_1 dt \quad (ND,D)
\]
\[
1-(\alpha_1^O + \alpha_2^O) dt \quad -p_1 dt \quad (ND,ND)
\]
\[
-kappa_1^O \ dt \quad -p_1 dt \quad (ND,D)
\]
\[
-kappa_2^O \ dt \quad -p_1 dt \quad (ND,ND)
\]
\[
1-(\pi_1^O + \pi_2^O) dt \quad -p_1 dt \quad (ND,ND)
\]

- Premium of CDS on name 1, maturity = 2, time = 0, \( p_1 dt \) solves for:

\[
0 = (1-p_1)\alpha_1^O + \left(-p_1 + (1-p_1)\kappa_1^O - p_1(1-\kappa_1^O)\right)\alpha_2^O
\]
\[
+ \left(-p_1 + (1-p_1)\pi_1^O - p_1\pi_2^O - p_1(1-\pi_1^O - \pi_2^O)\right)(1-\alpha_1^O - \alpha_2^O)
\]
Example: stylized zero coupon CDO tranchelets

- Zero-recovery, maturity 2
- Aggregate loss at time 2 can be equal to 0,1,2
  
  - Equity type tranche contingent on no defaults
  - Mezzanine type tranche: one default
  - Senior type tranche: two defaults

\[
\begin{align*}
\alpha_1^O \, dt \times \kappa_2^O \, dt + \alpha_2^O \, dt \times \kappa_1^O \, dt \\
\text{up-front premium default leg}
\end{align*}
\]

\[
1 - \left( \alpha_1^O + \alpha_2^O \right) \, dt
\]

II - Tree approach to hedging defaults
II - Tree approach to hedging defaults

- mezzanine tranche
  - Time pattern of default payments
    \[
    \alpha_1^O dt + \alpha_2^O dt + \left(1 - \left(\alpha_1^O + \alpha_2^O\right) dt\right)\left(\pi_1^O + \pi_2^O\right) dt
    \]

- Possibility of taking into account discounting effects
- The timing of premium payments
- Computation of dynamic deltas with respect to short or actual CDS on names 1,2
II - Tree approach to hedging defaults

• In theory, one could also derive dynamic hedging strategies for index CDO tranches
  – Numerical issues: large dimensional, non recombining trees
  – Homogeneous Markovian assumption is very convenient
    ➢ CDS premiums at a given time $t$ only depend upon the current number of defaults $N(t)$
    – CDS premium at time 0 (no defaults) $\alpha_1^O dt = \alpha_2^O dt = \alpha^O (t = 0, N(0) = 0)$
    – CDS premium at time 1 (one default) $\lambda_2^O dt = \kappa_1^O dt = \alpha^O (t = 1, N(t) = 1)$
    – CDS premium at time 1 (no defaults) $\pi_1^O dt = \pi_2^O dt = \alpha^O (t = 1, N(t) = 0)$
II - Tree approach to hedging defaults

- Homogeneous Markovian tree

\[
\begin{align*}
\alpha_0 (0,0) & \quad (D, ND) \\
\alpha_0 (0,0) & \quad (ND, D) \\
1 - 2\alpha_1 (0,0) & \quad (ND, ND)
\end{align*}
\]

- If we have \( N(1) = 1 \), one default at \( t=1 \)
- The probability to have \( N(2) = 1 \), one default at \( t=2 \)
- Is \( 1 - \alpha_0 (1,1) \) and does not depend on the defaulted name at \( t=1 \)
- \( N(t) \) is a Markov process
- Dynamics of the number of defaults can be expressed through a binomial tree
II - Tree approach to hedging defaults

- From name per name to number of defaults tree

\[
\begin{align*}
\alpha^Q_0 (0,0) & \quad (D, ND) \\
\alpha^Q_1 (0,0) & \quad (ND, D) \\
1 - 2\alpha^Q_1 (0,0) & \quad (ND, ND)
\end{align*}
\]

\[
\begin{align*}
\alpha^Q_0 (1,0) & \quad (D, ND) \\
1 - \alpha^Q (1,1) & \quad (ND, D) \\
1 - 2\alpha^Q_1 (1,0) & \quad (ND, ND)
\end{align*}
\]

\[
\begin{align*}
\alpha^Q_0 (1,1) & \quad (D, D) \\
1 - \alpha^Q_0 (1,1) & \quad (D, ND) \\
1 - 2\alpha^Q_0 (1,0) & \quad (ND, D)
\end{align*}
\]

\[
\begin{align*}
(\alpha^Q_0 (0,0) & \quad (D, ND) \\
(ND, D) & \quad (ND, ND) \\
(ND, ND) & \quad (ND, ND)
\end{align*}
\]

Number of defaults tree

\[
\begin{align*}
N(0) = 0 & \quad N(1) = 1 & \quad N(2) = 0 \\
N(1) = 1 & \quad N(2) = 1 \\
N(0) = 0 & \quad N(1) = 0 & \quad N(2) = 2
\end{align*}
\]
II - Tree approach to hedging defaults

- Easy extension to $n$ names
  - Predefault name intensity at time $t$ for $N(t)$ defaults: $\alpha^O_i(t, N(t))$
  - Number of defaults intensity: sum of surviving name intensities:
    \[ \lambda(t, N(t)) = (n - N(t)) \alpha^O_i(t, N(t)) \]
    \[ (n - 2) \alpha^O_i(2,2) \]
    \[ N(3) = 3 \]
    \[ 1 - (n - 1) \alpha^O_i(2,1) \]
    \[ (n - 1) \alpha^O_i(1,1) \]
    \[ N(2) = 2 \]
    \[ N(3) = 2 \]
    \[ 1 - (n - 1) \alpha^O_i(1,1) \]
    \[ N(1) = 1 \]
    \[ N(0) = 0 \]
    \[ 1 - n\alpha^O_i(0,0) \]
    \[ n\alpha^O_i(0,0) \]
    \[ 1 - n\alpha^O_i(1,0) \]
    \[ n\alpha^O_i(1,0) \]
    \[ 1 - n\alpha^O_i(2,0) \]
    \[ 1 - n\alpha^O_i(2,0) \]
    \[ 1 - n\alpha^O_i(2,0) \]

- $\alpha^O_i(0,0), \alpha^O_i(1,0), \alpha^O_i(1,1), \alpha^O_i(2,0), \alpha^O_i(2,1), \ldots$ can be easily calibrated
- on marginal distributions of $N(t)$ by forward induction.
Previous recombining binomial risk-neutral tree provides a framework for the valuation of payoffs depending upon the number of defaults

- CDO tranches
- Credit default swap index

What about the credit deltas?

- In a homogeneous framework, deltas with respect to CDS are all the same
- Perfect dynamic replication of a CDO tranche with a credit default swap index and the default-free asset
- Credit delta with respect to the credit default swap index
- \( \Delta \text{change in PV of the tranche} / \Delta \text{change in PV of the CDS index} \)
II - Tree approach to hedging defaults

- Example: number of defaults distribution at 5Y generated from a Gaussian copula
  - Correlation parameter: 30%
  - Number of names: 125
  - Default-free rate: 3%
  - 5Y credit spreads: 20 bps
  - Recovery rate: 40%

- Figure shows the probabilities of $k$ defaults for a 5Y horizon
II - Tree approach to hedging defaults

• Calibration of loss intensities
  – For simplicity, assumption of time homogeneous intensities
  – Figure below represents loss intensities, with respect to the number of defaults
  – Increase in intensities: contagion effects
### II - Tree approach to hedging defaults

- Dynamics of the 5Y CDS index spread
  - In bp pa

<table>
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<th>28</th>
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Dynamics of credit deltas:
- [0,3\%] equity tranche, buy protection
- With respect to the 5Y CDS index
- For selected time steps

Hedging strategy leads to a perfect replication of equity tranche payoff
- Prior to first defaults, deltas are above 1!
- When the number of defaults is > 6, the tranche is exhausted

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<td>2.52%</td>
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II - Tree approach to hedging defaults

- Credit deltas of the tranche
  - Sum of credit deltas of premium and default legs

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- premium leg
- default leg
### Credit deltas of the premium leg of the equity tranche

- **Premiums based on outstanding nominal**
- **Arrival of defaults reduces the commitment to pay**
  - Smaller outstanding nominal
  - Increase in credit spreads (contagion) involve a decrease in expected outstanding nominal
- **Negative deltas**
  - This is only significant for the equity tranche
    - Associated with much larger spreads

<table>
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<tr>
<th>Nb Defaults</th>
<th>Outstanding Nominal</th>
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<th></th>
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</table>

**II - Tree approach to hedging defaults**
Credit deltas for the default leg of the equity tranche

- Are actually between 0 and 1
- Gradually decrease with the number of defaults
  
  ➢ Concave payoff, negative gammas
- Credit deltas increase with time
  
  ➢ Consistent with a decrease in time value
  ➢ At maturity date, when number of defaults < 6, delta=1

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>OutStanding Nominal</th>
<th>Weeks</th>
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<tr>
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Dynamics of credit deltas
- Junior mezzanine tranche [3,6%]
- Deltas lie in between 0 and 1
- When the number of defaults is above 12, the tranche is exhausted

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<tr>
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<tr>
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</tr>
<tr>
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• Dynamics of credit deltas (junior mezzanine tranche)
  – Gradually increase and then decrease with the number of defaults
  – Call spread payoff (convex, then concave)
  – Initial delta = 16% (out of the money option)

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<th>28</th>
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II - Tree approach to hedging defaults

- **Comparison analysis**
  - After six defaults, the [3,6%] should be like a [0,3%] equity tranche
  - However, credit delta is much lower
    - 12% instead of 84%
  - But credit spreads after six defaults are much larger
    - 127 bps instead of 19 bps
  - Expected loss of the tranche is much larger
  - Which is associated with smaller deltas
### Dynamics of credit deltas ([6,9%] tranche)
- Initial credit deltas are smaller (deeper out of the money call spread)

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### II - Tree approach to hedging defaults

- Small dependence of credit deltas with respect to recovery rate
  - Equity tranche, $R=30\%$

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<td>4</td>
<td>0.76%</td>
<td>0.000</td>
<td>0.060</td>
<td>0.066</td>
<td>0.074</td>
<td>0.084</td>
<td>0.095</td>
<td>0.109</td>
</tr>
<tr>
<td>5</td>
<td>0.20%</td>
<td>0.000</td>
<td>0.011</td>
<td>0.011</td>
<td>0.013</td>
<td>0.014</td>
<td>0.015</td>
<td>0.017</td>
</tr>
<tr>
<td>6</td>
<td>0.00%</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

- Equity tranche, $R=40\%$

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>OutStanding Nominal</th>
<th>0</th>
<th>14</th>
<th>28</th>
<th>42</th>
<th>56</th>
<th>70</th>
<th>84</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.00%</td>
<td>0.967</td>
<td>0.993</td>
<td>1.016</td>
<td>1.035</td>
<td>1.052</td>
<td>1.065</td>
<td>1.075</td>
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<tr>
<td>1</td>
<td>2.52%</td>
<td>0.742</td>
<td>0.786</td>
<td>0.828</td>
<td>0.869</td>
<td>0.908</td>
<td>0.943</td>
<td>0.976</td>
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<tr>
<td>2</td>
<td>2.04%</td>
<td>0.439</td>
<td>0.484</td>
<td>0.532</td>
<td>0.583</td>
<td>0.637</td>
<td>0.691</td>
<td>0.743</td>
</tr>
<tr>
<td>3</td>
<td>1.56%</td>
<td>0.206</td>
<td>0.233</td>
<td>0.265</td>
<td>0.301</td>
<td>0.343</td>
<td>0.391</td>
<td>0.440</td>
</tr>
<tr>
<td>4</td>
<td>1.08%</td>
<td>0.082</td>
<td>0.093</td>
<td>0.106</td>
<td>0.121</td>
<td>0.141</td>
<td>0.164</td>
<td>0.188</td>
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<tr>
<td>5</td>
<td>0.60%</td>
<td>0.029</td>
<td>0.032</td>
<td>0.035</td>
<td>0.039</td>
<td>0.045</td>
<td>0.051</td>
<td>0.058</td>
</tr>
<tr>
<td>6</td>
<td>0.12%</td>
<td>0.004</td>
<td>0.005</td>
<td>0.005</td>
<td>0.006</td>
<td>0.006</td>
<td>0.007</td>
<td>0.008</td>
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<tr>
<td>7</td>
<td>0.00%</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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</table>
Small dependence of credit deltas with respect to recovery rate

- Initial delta with respect to the credit default swap index

<table>
<thead>
<tr>
<th>Tranches</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
<th>50%</th>
<th>60%</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0-3%]</td>
<td>0.9960</td>
<td>0.9824</td>
<td>0.9746</td>
<td>0.9670</td>
<td>0.9527</td>
<td>0.9456</td>
</tr>
<tr>
<td>[3-6%]</td>
<td>0.1541</td>
<td>0.1602</td>
<td>0.1604</td>
<td>0.1616</td>
<td>0.1659</td>
<td>0.1604</td>
</tr>
<tr>
<td>[6-9%]</td>
<td>0.0164</td>
<td>0.0165</td>
<td>0.0168</td>
<td>0.0168</td>
<td>0.0168</td>
<td>0.0169</td>
</tr>
</tbody>
</table>

- Only a small dependence of credit deltas with respect to recovery rates

Which is rather fortunate
II - Tree approach to hedging defaults

- Dependence of credit deltas with respect to correlation
  - Default leg, equity tranche

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>OutStanding Nominal</th>
<th>Weeks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.00%</td>
<td>0.968 0.974 0.978 0.982 0.985 0.987 0.990</td>
</tr>
<tr>
<td>1</td>
<td>2.52%</td>
<td>0.933 0.944 0.953 0.962 0.969 0.976</td>
</tr>
<tr>
<td>2</td>
<td>2.04%</td>
<td>0.835 0.856 0.876 0.895 0.912 0.928</td>
</tr>
<tr>
<td>3</td>
<td>1.56%</td>
<td>0.653 0.683 0.714 0.744 0.774 0.804</td>
</tr>
<tr>
<td>4</td>
<td>1.08%</td>
<td>0.405 0.433 0.464 0.496 0.531 0.568</td>
</tr>
<tr>
<td>5</td>
<td>0.60%</td>
<td>0.170 0.185 0.202 0.221 0.243 0.268</td>
</tr>
<tr>
<td>6</td>
<td>0.12%</td>
<td>0.027 0.030 0.033 0.037 0.041 0.046</td>
</tr>
<tr>
<td>7</td>
<td>0.00%</td>
<td>0.000 0.000 0.000 0.000 0.000 0.000</td>
</tr>
</tbody>
</table>

$\rho = 10\%$

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>OutStanding Nominal</th>
<th>Weeks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.00%</td>
<td>0.814 0.843 0.869 0.893 0.915 0.933 0.949</td>
</tr>
<tr>
<td>1</td>
<td>2.52%</td>
<td>0.614 0.658 0.702 0.746 0.787 0.827</td>
</tr>
<tr>
<td>2</td>
<td>2.04%</td>
<td>0.341 0.384 0.431 0.482 0.535 0.591</td>
</tr>
<tr>
<td>3</td>
<td>1.56%</td>
<td>0.140 0.165 0.194 0.229 0.269 0.315</td>
</tr>
<tr>
<td>4</td>
<td>1.08%</td>
<td>0.045 0.054 0.064 0.078 0.095 0.117</td>
</tr>
<tr>
<td>5</td>
<td>0.60%</td>
<td>0.013 0.015 0.017 0.020 0.024 0.030</td>
</tr>
<tr>
<td>6</td>
<td>0.12%</td>
<td>0.002 0.002 0.002 0.003 0.003 0.003</td>
</tr>
<tr>
<td>7</td>
<td>0.00%</td>
<td>0.000 0.000 0.000 0.000 0.000 0.000</td>
</tr>
</tbody>
</table>

$\rho = 30\%$
Equity deltas decrease as correlation increases

Value of equity default leg under different correlation assumptions

Number of defaults on the x-axis
II - Tree approach to hedging defaults

- Smaller correlation
  - Prior to first default, higher expected losses on the tranche
    ➢ Should lead to smaller deltas
  - But smaller contagion effects
    ➢ When shifting from zero to one default
    ➢ The expected loss on the index jumps due to...
      - Default arrival and jumps in credit spreads
      - Smaller jumps in credit spreads for smaller correlation
    ➢ Smaller correlation is associated with smaller jumps in the expected loss of the index
    ➢ Leads to higher deltas
      - Since we have negative gamma
II - Tree approach to hedging defaults

• Computing deltas with market inputs
  – Base correlations (5Y), as for iTraxx, June 2007

<table>
<thead>
<tr>
<th></th>
<th>3%</th>
<th>6%</th>
<th>9%</th>
<th>12%</th>
<th>22%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>16%</td>
<td>24%</td>
<td>30%</td>
<td>35%</td>
<td>50%</td>
</tr>
</tbody>
</table>

– Probabilities of $k$ defaults
II - Tree approach to hedging defaults

- Loss intensities for the Gaussian copula and market case examples

- Number of defaults on the x-axis
Credit spread dynamics

- Base correlation inputs

- Similar to Gaussian copula at the first default
- Dramatic increases in credit spreads after a few defaults
• Comparison of Gaussian copula and market inputs

- Expected losses on the credit portfolio after 14 weeks
- With respect to the number of observed defaults

• Much bigger contagion effects with steep base correlation
Comparison of credit deltas
- Gaussian copula and market case examples
- Smaller credit deltas for the equity tranche

- Dynamic correlation effects
- After the first default, due to magnified contagion,
- New defaults are associated with big shifts in correlation

II - Tree approach to hedging defaults

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>OutStanding Nominal</th>
<th>0</th>
<th>14</th>
<th>28</th>
<th>42</th>
<th>56</th>
<th>70</th>
<th>84</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.00%</td>
<td>0.645</td>
<td>0.731</td>
<td>0.814</td>
<td>0.890</td>
<td>0.953</td>
<td>1.003</td>
<td>1.038</td>
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<tr>
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<td>0.000</td>
<td>0.329</td>
<td>0.402</td>
<td>0.488</td>
<td>0.584</td>
<td>0.684</td>
<td>0.777</td>
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<tr>
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<td>0.000</td>
<td>0.091</td>
<td>0.115</td>
<td>0.149</td>
<td>0.197</td>
<td>0.264</td>
<td>0.351</td>
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<td>0.023</td>
<td>0.028</td>
<td>0.035</td>
<td>0.045</td>
<td>0.062</td>
<td>0.090</td>
</tr>
<tr>
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<td>1.08%</td>
<td>0.000</td>
<td>0.008</td>
<td>0.008</td>
<td>0.009</td>
<td>0.011</td>
<td>0.013</td>
<td>0.018</td>
</tr>
<tr>
<td>5</td>
<td>0.60%</td>
<td>0.000</td>
<td>0.004</td>
<td>0.004</td>
<td>0.003</td>
<td>0.003</td>
<td>0.003</td>
<td>0.004</td>
</tr>
<tr>
<td>6</td>
<td>0.12%</td>
<td>0.000</td>
<td>0.001</td>
<td>0.001</td>
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<td>0.001</td>
</tr>
<tr>
<td>7</td>
<td>0.00%</td>
<td>0.000</td>
<td>0.000</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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</tbody>
</table>
Comparison of credit deltas

- Market and model deltas at inception
- Equity tranche

<table>
<thead>
<tr>
<th></th>
<th>[0-3%]</th>
<th>[3-6%]</th>
<th>[6-9%]</th>
<th>[9-12%]</th>
<th>[12-22%]</th>
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<tbody>
<tr>
<td>market deltas</td>
<td>27</td>
<td>4.5</td>
<td>1.25</td>
<td>0.6</td>
<td>0.25</td>
</tr>
<tr>
<td>model deltas</td>
<td>21.5</td>
<td>4.63</td>
<td>1.63</td>
<td>0.9</td>
<td>NA</td>
</tr>
</tbody>
</table>

- Figures are roughly the same
  - Though the base copula market and the contagion model are quite different models
  - Smaller equity tranche deltas for contagion model
  - Base correlation sticky deltas underestimate the increase in contagion after the first defaults
- Recent market shifts go in favour of the contagion model
II - Tree approach to hedging defaults

- Comparison of credit deltas
  - Arnsdorf & Halperin (2007)
  - Credit spread deltas in a 2D Markov chain

<table>
<thead>
<tr>
<th></th>
<th>[0-3%]</th>
<th>[3-6%]</th>
<th>[6-9%]</th>
<th>[9-12%]</th>
<th>[12-22%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>market deltas</td>
<td>26.5</td>
<td>4.5</td>
<td>1.25</td>
<td>0.65</td>
<td>0.25</td>
</tr>
<tr>
<td>model deltas</td>
<td>21.9</td>
<td>4.81</td>
<td>1.64</td>
<td>0.79</td>
<td>0.38</td>
</tr>
</tbody>
</table>

- Confirms previous results
- Model deltas in A&H are smaller than market deltas for the equity tranche
- Credit spreads deltas in A&H are quite similar to credit deltas in the 1D Markov chain
What do we learn from this hedging approach?

- Thanks to stringent assumptions:
  - credit spreads driven by defaults
  - homogeneity
  - Markov property
- It is possible to compute a dynamic hedging strategy
  - Based on the CDS index
- That fully replicates the CDO tranche payoffs
  - Model matches market quotes of liquid tranches
  - Very simple implementation
  - Credit deltas are easy to understand
- Improve the computation of default hedges
  - Since it takes into account credit contagion
- Credit spread dynamics needs to be improved
When dealing with the risk management of CDOs, traders
− concentrate upon credit spread and correlation risk
− Neglect default risk

What about default risk?
− For large indices, default of one name has only a small direct
effect on the aggregate loss

Is it possible to build a framework where hedging default
risk can be neglected?

And where one could only consider the hedging of credit
spread risk?
− See paper “A Note on the risk management of CDOs”
Main and critical assumption
- Default times follow a multivariate Cox process
  - For instance, affine intensities
  - Duffie & Garleanu, Mortensen, Feldhütter, Merrill Lynch

\[ \tau_i = \inf \left\{ t \in \mathbb{R}^+ : U_i \geq \exp \left( - \int_0^t \lambda_{i,u} \, du \right) \right\}, \quad i = 1, \ldots, n \]  

where \( \lambda_1, \ldots, \lambda_n \) are strictly positive, \( \mathcal{F} \) - progressively measurable processes, \( U_1, \ldots, U_n \) are independent random variables uniformly distributed on \([0,1]\) under \( Q \) and \( \mathcal{F} \) and \( \sigma(U_1, \ldots, U_n) \) are independent under \( Q \).

No contagion effects
III - Hedging credit spread risks for large portfolios

- No contagion effects
  - Credit spreads drive defaults but defaults do not drive credit spreads
  - For a large portfolio, default risk is perfectly diversified
  - Only remains credit spread risks: parallel & idiosyncratic

- Main result
  - With respect to dynamic hedging, default risk can be neglected
  - Only need to focus on dynamic hedging of credit spread risks
    - With CDS
  - Similar to interest rate derivatives markets
III - Hedging credit spread risks for large portfolios

- Formal setup
  - $\tau_1, \ldots, \tau_n$ default times
  - $N_i(t) = 1_{\{\tau_i \leq t\}}, i = 1, \ldots, n$ default indicators
  - $H_t = \bigvee_{i=1,\ldots,n} \sigma(N_i(s), s \leq t)$ natural filtration of default times
  - $F_t$ background (credit spread filtration)
  - $G_t = H_t \vee F_t$ enlarged filtration, $P$ historical measure
  - $l_i(t, T), i = 1, \ldots, n$ time $t$ price of an asset paying $N_i(T)$ at time $T$
Sketch of the proof

Step 1: consider some smooth shadow risky bonds
  – Only subject to credit spread risk
  – Do not jump at default times

Projection of the risky bond prices on the credit spread filtration

**Definition 3.2** The default free $T$ forward loss process associated with name $i \in \{0, \ldots, n\}$, denoted by $p^i(\cdot, T)$ is such that for $0 \leq t \leq T$:

$$p^i(t, T) \overset{\Delta}{=} \mathbb{E}^Q \left[ p^i(T) \mid \mathcal{F}_t \right] = \mathbb{E}^Q \left[ N_i(T) \mid \mathcal{F}_t \right] = Q(\tau_i \leq T \mid \mathcal{F}_t). \quad (3.2)$$

**Lemma 3.1** $p^i(t, T), i = 1, \ldots, n$ are projections of the forward price processes $l^i(t, T)$ on $\mathcal{F}_t$:

$$p^i(t, T) = \mathbb{E}^Q \left[ l^i(t, T) \mid \mathcal{F}_t \right], \quad (3.3)$$

for $i = 1, \ldots, n$ and $0 \leq t \leq T$. 
III - Hedging credit spread risks for large portfolios

• Step 2: Smooth the aggregate loss process
• ... and thus the tranche payoffs
  – Remove default risk and only consider credit spread risk
  – Projection of aggregate loss on credit spread filtration

Definition 3.1 We denote by \( p^i(.) \), the default-free running loss process associated with name \( i \in \{0, \ldots, n\} \), which is such that for \( 0 \leq t \leq T \):

\[
p^i(t) \triangleq E^Q[N_i(t) \mid \mathcal{F}_t] = Q(\tau_i \leq t \mid \mathcal{F}_t) = 1 - \exp(-\Lambda_{i,t}). \tag{3.1}
\]

Definition 3.5 default-free aggregate running loss process The default free aggregate running loss at time \( t \) is such that for \( 0 \leq t \leq T \):

\[
p_n(t) \triangleq \frac{1}{n} \sum_{i=1}^{n} p^i(t). \tag{3.7}
\]
III - Hedging credit spread risks for large portfolios

- Step 3: compute perfect hedge ratios of the smoothed payoff

  ➢ With respect to the smoothed risky bonds
    - Smoothed payoff and risky bonds only depend upon credit spread dynamics
    - Both idiosyncratic and parallel credit spread risks
    - Similar to a multivariate interest rate framework
    - Perfect hedging in the smooth market

Assumption 2 There exists some bounded $\mathcal{F}$ - predictable processes $\theta_1(\cdot), \ldots, \theta_n(\cdot)$ such that:

\[
(p_n(T) - K)^+ = E^Q [(p_n(T) - K)^+] + \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{T} \theta_i(t) dp^i(t, T) + z_n, \tag{4.2}
\]

where $z_n$ is $\mathcal{F}_T$-measurable, of $Q$-mean zero and $Q$-strongly orthogonal to $p^1(\cdot, T), \ldots, p^n(\cdot, T)$. 
• Step 4: apply the hedging strategy to the **true** defaultable bonds

• **Main result**
  
  – Bound on the hedging error following the previous hedging strategy
  – When hedging an actual CDO tranche with actual defaultable bonds
  – Hedging error decreases with the number of names

  ➢ Default risk diversification

**Proposition 1** Under Assumptions (1) and (2), the hedging error $\varepsilon_n$ defined as:

$$
\varepsilon_n = (l_n(T) - K)^+ - E^Q [(l_n(T) - K)^+] - \frac{1}{n} \sum_{i=1}^n \int_0^T \theta_i(t) dl^i(t, T),
$$

is such that $E^P[||\varepsilon_n||]$ is bounded by:

$$
\frac{1}{\sqrt{2n}} \left( 1 + \left( E^Q \left( \left( \frac{dP}{dQ} \right)^2 \right) \right)^{1/2} \right) + \frac{1}{n} \left( E^Q \left[ \left( \frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \left( \sum_{i=1}^n Q(\tau_i \leq T) + E^Q [B_i|T]| \right)^{1/2}
$$

$$
+ E^P[||\varepsilon_n||].
$$
III - Hedging credit spread risks for large portfolios

- Provides a hedging technique for CDO tranches
  - Known theoretical properties
  - Takes into account idiosyncratic and parallel gamma risks
  - Good theoretical properties rely on no simultaneous defaults, no contagion effects assumptions
  - Empirical work remains to be done

- Thought provocative
  - To construct a practical hedging strategy, do not forget default risk
  - Equity tranche [0,3%]
  - iTraxx or CDX first losses cannot be considered as smooth
III - Hedging credit spread risks for large portfolios

• Linking pricing and hedging?
• The black hole in CDO modeling?
• Standard valuation approach in derivatives markets
  ➢ Complete markets
  ➢ Price = cost of the hedging/replicating portfolio
• Mixing of dynamic hedging strategies
  – for credit spread risk
• And diversification/insurance techniques
  – For default risk
Comparing hedging approaches

- Two different models have been investigated
- Contagion homogeneous Markovian models
  - Perfect hedge of default risks
  - Easy implementation
  - Poor dynamics of credit spreads
  - No individual name effects
- Multivariate Cox processes
  - Rich dynamics of credit spreads
  - But no contagion effects
  - Thus, default risk can be diversified at the index level
  - Replication of CDO tranches is feasible by hedging only credit spread risks.
Comparison results for credit risk portfolios

- Pricing issues with factor models
  - Comparison of CDO pricing models through stochastic orders
  - Comprehensive approach to copula, structural and multivariate Poisson models
  - Relevance of the conditional default probabilities
    - Drive the tranche pricing
  - For simplicity, we further restrict to homogeneous portfolios
  - We provide a general comparison of pricing models methodology
  - By looking for the distribution of conditional default probabilities
Contents

1. Comparison of Exchangeable Bernoulli random vectors
   - Exchangeability assumption
   - De Finetti Theorem and Factor representation
   - Stochastic orders

2. Application to Credit Risk Management
   - Multivariate Poisson model
   - Structural model
   - Factor copula models
     - Archimedean copula
     - Additive copula framework
Exchangeability assumption

- $n$ defaultable firms
- $\tau_1, \ldots, \tau_n$ default times
- $(D_1, \ldots, D_n) = (1\{\tau_1 \leq t\}, \ldots, 1\{\tau_n \leq t\})$ default indicators
- Homogeneity assumption: default dates are assumed to be exchangeable

**Definition (Exchangeability)**

A random vector $(\tau_1, \ldots, \tau_n)$ is exchangeable if its distribution function is invariant by permutation: $\forall \sigma \in S_n$

$$(\tau_1, \ldots, \tau_n) \overset{d}{=} (\tau_{\sigma(1)}, \ldots, \tau_{\sigma(n)})$$

- Same marginals
Suppose that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables.

There exists a random factor $\tilde{p}$ such that $D_1, \ldots, D_n$ are independent knowing $\tilde{p}$.

Denote by $F_\tilde{p}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum i d_i} (1 - p)^{n - \sum i d_i} F_\tilde{p}(dp)$$

$\tilde{p}$ is characterized by:

$$\frac{1}{n} \sum_{i=1}^{n} D_i \overset{a.s.}{\rightarrow} \tilde{p} \text{ as } n \rightarrow \infty$$
Stochastic orders

- \( X \leq_{cx} Y \) if \( E[f(X)] \leq E[f(Y)] \) for all convex functions \( f \)
- \( X \leq_{sl} Y \) if \( E[(X - K)^+] \leq E[(Y - K)^+] \) for all \( K \in \mathbb{R} \)
  - \( X \leq_{sl} Y \) and \( E[X] = E[Y] \) \( \iff \) \( X \leq_{cx} Y \)
- \( X \leq_{sm} Y \) if \( E[f(X)] \leq E[f(Y)] \) for all supermodular functions \( f \)

**Definition (Supermodular function)**

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is **supermodular** if for all \( x \in \mathbb{R}^n \), \( 1 \leq i < j \leq n \) and \( \varepsilon, \delta > 0 \) holds

\[
 f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n) \\
\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)
\]

- consequences of new defaults are always worse when other defaults have already occurred
Stochastic orders

- \((D_1, \ldots, D_n)\) and \((D_1^*, \ldots, D_n^*)\) two exchangeable default indicator vectors
- \(M_i\): loss given default
- Aggregate losses:

\[
L_t = \sum_{i=1}^{n} M_i D_i
\]

\[
L_t^* = \sum_{i=1}^{n} M_i D_i^*
\]

Müller (1997)

Stop-loss order for portfolios of dependent risks.

\((D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \Rightarrow L_t \leq_{sl} L_t^*\)
Stochastic orders

Theorem

Let $D = (D_1, \ldots, D_n)$ and $D^* = (D_1^*, \ldots, D_n^*)$ be two exchangeable Bernoulli random vectors with (resp.) $F$ and $F^*$ as mixture distributions. Then:

$$F \leq_{cx} F^* \Rightarrow D \leq_{sm} D^* \quad \text{and}$$

Theorem

Let $D_1, \ldots, D_n, \ldots$ and $D_1^*, \ldots, D_n^*, \ldots$ be two exchangeable sequences of Bernoulli random variables. We denote by $F$ (resp. $F^*$) the distribution function associated with the mixing measure. Then,

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow F \leq_{cx} F^*.$$
Multivariate Poisson model

- $\tilde{N}_t^i$ Poisson with parameter $\tilde{\lambda}$: idiosyncratic risk
- $N_t$ Poisson with parameter $\lambda$: systematic risk
- $(B_j^i)_{i,j}$ Bernoulli random variable with parameter $p$
- All sources of risk are independent
- $N_t^i = \tilde{N}_t^i + \sum_{j=1}^{N_t} B_j^i, \ i = 1 \ldots n$
- $\tau_i = \inf\{t > 0|N_t^i > 0\}, \ i = 1 \ldots n$
Multivariate Poisson model

- $\tau_i \sim \text{Exp} (\bar{\lambda} + p\lambda)$
- $D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n$ are independent knowing $N_t$
- $\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)$
- Conditional default probability:
  \[
  \tilde{p} = 1 - (1 - p)^{N_t} \exp(-\bar{\lambda}t)
  \]
Comparison of two multivariate Poisson models with parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$

Supermodular order comparison requires equality of marginals:
$$\lambda + p\lambda = \bar{\lambda}^* + p^*\lambda^*$$

Comparison directions:
- $p = p^*$: $\bar{\lambda}$ v.s. $\lambda$
- $\lambda = \lambda^*$: $\bar{\lambda}$ v.s. $p$
Theorem \((p = p^*)\)

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*\), then:

\[
\lambda \leq \lambda^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]
**Theorem \((\lambda = \lambda^*)\)**

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda\), then:

\[
p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \bar{p} \leq_{cx} \bar{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]
Hull, Predescu and White (2005)

- Consider $n$ firms
- Let $X^i_t$, $i = 1 \ldots n$ be their asset dynamics
  \[ X^i_t = \rho W_t + \sqrt{1 - \rho^2} W^i_t, \quad i = 1 \ldots n \]
- $W$, $W^i$, $i = 1 \ldots n$ are independent standard Wiener processes
- Default times as first passage times:
  \[ \tau_i = \inf\{t \in \mathbb{R}^+ | X^i_t \leq f(t)\}, \quad i = 1 \ldots n, \quad f : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous} \]
- $D_i = 1\{\tau_i \leq \tau\}, \quad i = 1 \ldots n$ are independent knowing $\sigma(W_t, \ t \in [0, T])$
- $\frac{1}{n} \sum_{i=1}^n D_i \xrightarrow{a.s.} \tilde{p}$
For any fixed time horizon $T$, denote by $D_i = 1\{\tau_i \leq T\}$, $i = 1 \ldots n$ and $D_i^* = 1\{\tau_i^* \leq T\}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \implies (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$
Archimedean copula

<table>
<thead>
<tr>
<th>Copula name</th>
<th>Generator $\varphi$</th>
<th>$V$-distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>$t^{-\theta} - 1$</td>
<td>Gamma$(1/\theta)$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$(-\ln(t))^\theta$</td>
<td>$\alpha$-Stable, $\alpha = 1/\theta$</td>
</tr>
<tr>
<td>Franck</td>
<td>$-\ln \left( \frac{(1 - e^{-\theta t})}{(1 - e^{-\theta})} \right)$</td>
<td>Logarithmic series</td>
</tr>
</tbody>
</table>

**Theorem**

$$\alpha \leq \alpha^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$
Additive copula framework

- \( V_i = \rho V + \sqrt{1 - \rho^2} \bar{V}_i \)
- \( V, V_i \ i = 1 \ldots n \) independent
- Laws of \( V, V_i \ i = 1 \ldots n \) do not depend on the dependence parameter \( \rho \)
- Standard copula models:
  - Gaussian, Student \( t \)
  - Double \( t \): Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2005)
  - Double Variance Gamma: Moosbrucker(2005)

**Theorem**

\[
\rho \leq \rho^* \Rightarrow \bar{p} \leq_{\text{cx}} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{\text{sm}} (D_1^*, \ldots, D_n^*)
\]
Conclusion

- Characterization of supermodular order for exchangeable Bernoulli random vectors
- Comparison of CDO tranche premiums in several pricing models
- Unified way of presenting default risk models