New results for the pricing and hedging of CDOs

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Jean-Paul LAURENT
Professor, ISFA Actuarial School, University of Lyon,
Scientific consultant, BNP PARIBAS
http://laurent.jeanpaul.free.fr

Presentation related to papers
***A note on the risk management of CDOs* (2007)**
***Hedging default risks of CDOs in Markovian contagion models* (2007)**
***Comparison results for credit risk portfolios* (2007)**
Available on www.defaultrisk.com

New results for the pricing and hedging of CDOs

- **Hedging issues**
  - **Hedging of default risk in contagion models**
    - Markov chain approach to contagion models
    - Comparison of models deltas with “market deltas”
  - **Hedging of credit spread risk in intensity models**
- **Pricing issues with factor models**
  - **Comparison of CDO pricing models through stochastic orders**
  - **Comprehensive approach to copula, structural and multivariate Poisson models**

Hedging Default and Credit Spread Risks within CDOs

- Purpose of the presentation
  - Not trying to embrace all risk management issues
  - Focus on very specific aspects of default and credit spread risk
- Overlook of the presentation
  - Economic background
  - Tree approach to hedging defaults
  - Hedging credit spread risks for large portfolios

I - Economic Background

- Hedging CDOs context
- About 1 000 papers on defaultrisk.com
- About 10 papers dedicated to hedging issues
  - **In interest rate or equity markets, pricing is related to the cost of the hedge**
  - **In credit markets, pricing is disconnect from hedging**
- Need to relate pricing and hedging
- What is the business model for CDOs?
- Risk management paradigms
  - Static hedging, risk-return arbitrage, complete markets

I - Economic Background

- Static hedging
- Buy a portfolio of credits, split it into tranches and sell the tranches to investors
  - ➢ No correlation or model risk for market makers
  - ➢ No need to dynamically hedge with CDS
- Only « budget constraint »:
  - ➢ Sum of the tranche prices greater than portfolio of credits price
  - ➢ Similar to stripping ideas for Treasury bonds
- <u>No clear idea of relative value of tranches</u>
  - ➢ Depends of demand from investors
  - ➢ Markets for tranches might be segmented

I - Economic Background

- Risk – return arbitrage
- Historical returns are related to ratings, factor exposure
  - CAPM, equilibrium models
  - **In search of high alphas**
  - **Relative value deals, cross-selling along the capital structure**
- Depends on the presence of « arbitrageurs »
  - Investors with small risk aversion
    - ➢ Trading floors, hedge funds
  - Investors without too much accounting, regulatory, rating constraints

I - Economic Background

- The ultimate step : complete markets
  - As many risks as hedging instruments
  - News products are only designed to save transactions costs and are used for risk management purposes
  - Assumes a high liquidity of the market
- Perfect replication of payoffs by dynamically trading a small number of « underlying assets »
  - Black-Scholes type framework
  - Possibly some model risk
- This is further investigated in the presentation
  - Dynamic trading of CDS to replicate CDO tranche payoffs

I - Economic Background

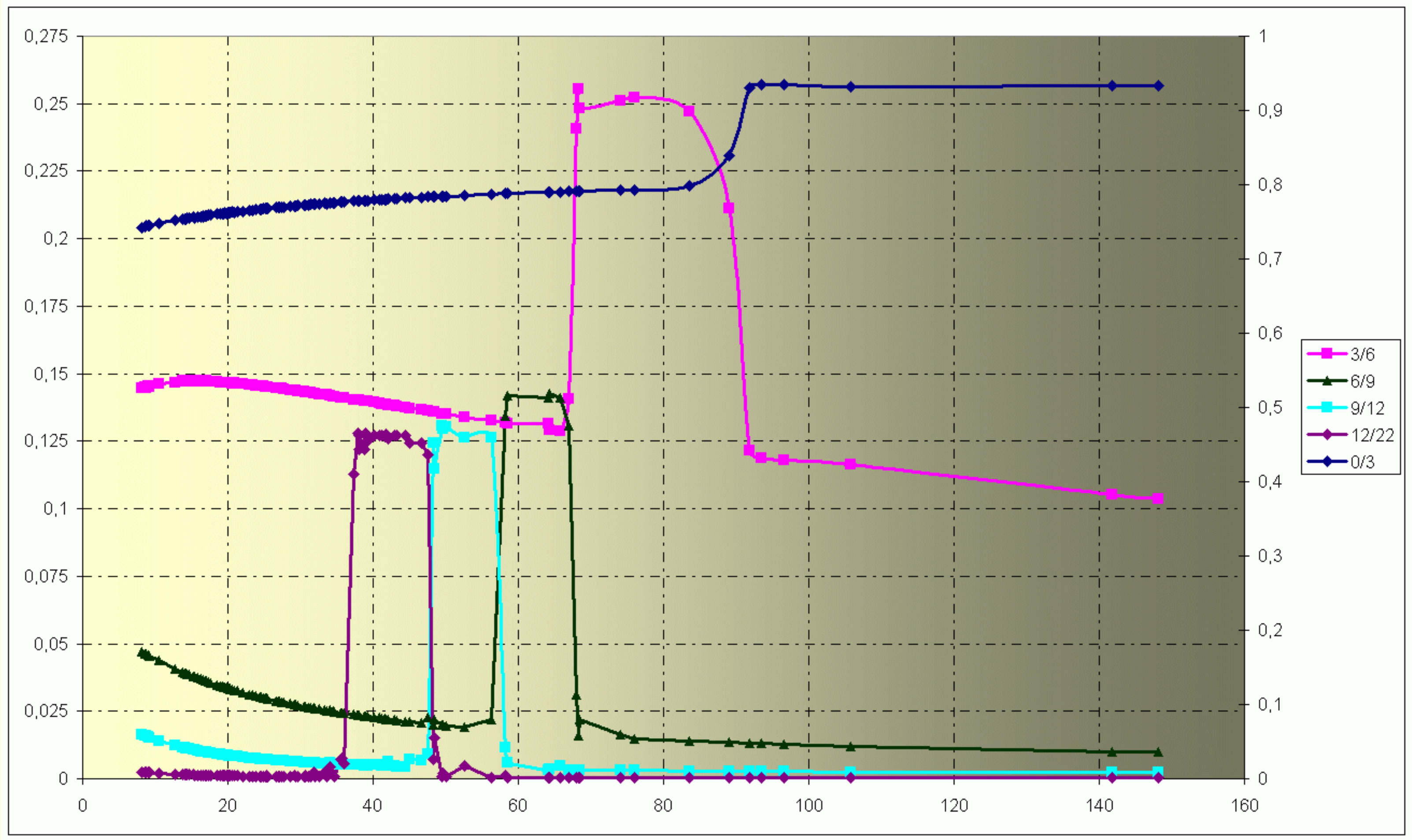
- Default risk
  - Default bond price jumps to recovery value at default time.
  - Drives the CDO cash-flows
- Credit spread risk
  - Changes in defaultable bond prices prior to default
    - Due to shifts in credit quality or in risk premiums
  - Changes in the marked to market of tranches
- Interactions between credit spread and default risks
  - Increase of credit spreads increase the probability of future defaults
  - Arrival of defaults may lead to jump in credit spreads
    - Contagion effects (Jarrow & Yu)

I - Economic Background

- Credit deltas in copula models
- CDS hedge ratios are computed by bumping the marginal credit curves
  - Local sensitivity analysis
  - Focus on credit spread risk
  - Deltas are copula dependent
  - Hedge over short term horizons
    - Poor understanding of gamma, theta, vega effects
    - Does not lead to a replication of CDO tranche payoffs
- Last but not least: not a hedge against defaults…

I - Economic Background

- Credit deltas in copula models
  - Stochastic correlation model (Burstchell, Gregory & Laurent, 2007)



II - Tree approach to hedging defaults

- Main assumptions and results
  - Credit spreads are driven by defaults
    - ➢Contagion model
    - ➢Credit spreads are deterministic between two defaults
  - Homogeneous portfolio
    - ➢Only need of the CDS index
    - ➢No individual name effect
  - Markovian dynamics
    - ➢Pricing and hedging CDOs within a binomial tree
    - ➢Easy computation of dynamic hedging strategies
    - ➢Perfect replication of CDO tranches

II - Tree approach to hedging defaults

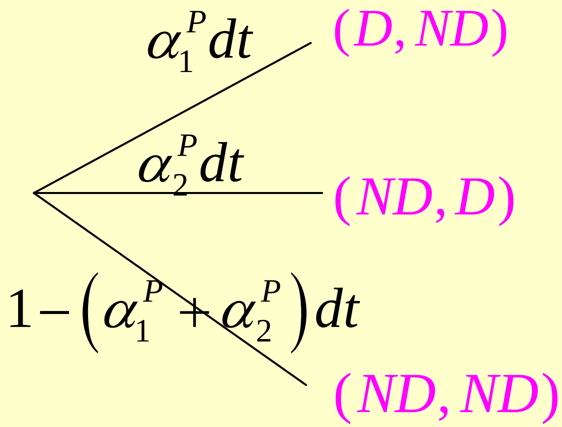
- We will start with two names only
- Firstly in a static framework
  - Look for a First to Default Swap
  - Discuss historical and risk-neutral probabilities
- Further extending the model to a dynamic framework
  - Computation of prices and hedging strategies along the tree
  - Pricing and hedging of tranchelets
- Multiname case: homogeneous Markovian model
  - Computation of risk-neutral tree for the loss
  - Computation of dynamic deltas
- Technical details can be found in the paper:
  - "hedging default risks of CDOs in Markovian contagion models"

II - Tree approach to hedging defaults

- Some notations :
  - τ_1, τ_2 default times of counterparties 1 and 2,
  - $\mathcal{H}_t$ available information at time t,
  - P historical probability,
  - α_1^P, α_2^P : (historical) default intensities:
    - $$P\left[\tau_i \in [t, t+dt[\,\middle|\, H_t\right] = \alpha_i^P dt, \; i = 1,2$$
- Assumption of « local » independence between default events
  - Probability of 1 and 2 defaulting altogether:
    - $$P\left[\tau_1 \in [t, t+dt[, \tau_2 \in [t, t+dt[\,\middle|\, H_t\right] = \alpha_1^P dt \times \alpha_2^P dt \text{ in } (dt)^2$$
  - Local independence: simultaneous joint defaults can be neglected

II - Tree approach to hedging defaults

- Building up a tree:
  - Four possible states: (*D*,*D*), (*D*,*ND*), (*ND*,*D*), (*ND*,*ND*)
  - Under no simultaneous defaults assumption $p_{(D,D)}=0$
  - Only three possible states: (*D*,*ND*), (*ND*,*D*), (*ND*,*ND*)
  - Identifying (historical) tree probabilities:



$$\begin{cases} p_{(D,D)} = 0 \Rightarrow p_{(D,ND)} = p_{(D,D)} + p_{(D,ND)} = p_{(D,.)} = \alpha_1^P dt \\ p_{(D,D)} = 0 \Rightarrow p_{(ND,D)} = p_{(D,D)} + p_{(ND,D)} = p_{(.,D)} = \alpha_2^P dt \\ p_{(ND,ND)} = 1 - p_{(D,.)} - p_{(.,D)} \end{cases}$$

II - Tree approach to hedging defaults

- Stylized cash flows of short term digital CDS on counterparty 1:
  - $\alpha_1^Q dt$ CDS 1 premium

$$0 \;\to\; \begin{cases} \alpha_1^P dt: & 1-\alpha_1^Q dt \quad (D, ND) \\ \alpha_2^P dt: & -\alpha_1^Q dt \quad (ND, D) \\ 1-\left(\alpha_1^P+\alpha_2^P\right)dt: & -\alpha_1^Q dt \quad (ND, ND) \end{cases}$$

- Stylized cash flows of short term digital CDS on counterparty 2:

$$0 \;\to\; \begin{cases} \alpha_1^P dt: & -\alpha_2^Q dt \quad (D, ND) \\ \alpha_2^P dt: & 1-\alpha_2^Q dt \quad (ND, D) \\ 1-\left(\alpha_1^P+\alpha_2^P\right)dt: & -\alpha_2^Q dt \quad (ND, ND) \end{cases}$$

II - Tree approach to hedging defaults

- Cash flows of short term digital first to default swap with premium $\alpha_F^Q dt$:

$\alpha_1^P dt$	$1-\alpha_F^Q dt$	(D, ND)
$\alpha_2^P dt$	$1-\alpha_F^Q dt$	(ND, D)
$1-\left(\alpha_1^P+\alpha_2^P\right)dt$	$-\alpha_F^Q dt$	(ND, ND)

Initial value: 0

- Cash flows of holding CDS 1 + CDS 2:

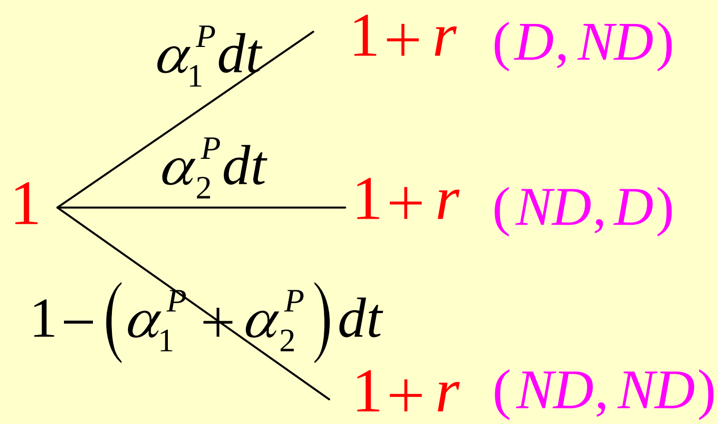
$\alpha_1^P dt$	$1-\left(\alpha_1^Q+\alpha_2^Q\right)dt$	(D, ND)
$\alpha_2^P dt$	$1-\left(\alpha_1^Q+\alpha_2^Q\right)dt$	(ND, D)
$1-\left(\alpha_1^P+\alpha_2^P\right)dt$	$-\left(\alpha_1^Q+\alpha_2^Q\right)dt$	(ND, ND)

Initial value: 0

- Perfect hedge of first to default swap by holding 1 CDS 1 + 1 CDS 2
  - Delta with respect to CDS 1 = 1, delta with respect to CDS 2 = 1

II - Tree approach to hedging defaults

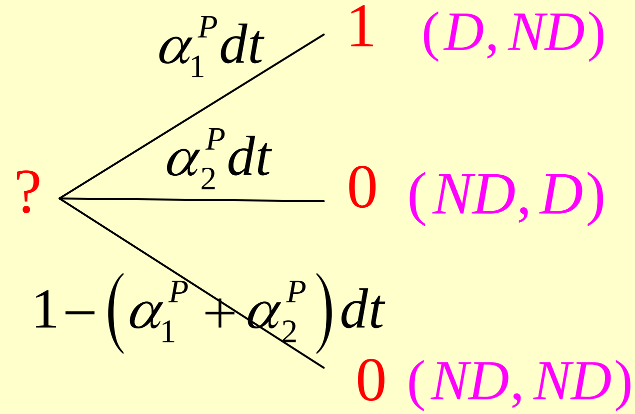
- Absence of arbitrage opportunities imply:
  - $\alpha_F^Q = \alpha_1^Q + \alpha_2^Q$
- Arbitrage free first to default swap premium
  - Does not depend on historical probabilities α_1^P, α_2^P
- Three possible states: (*D*,*ND*), (*ND*,*D*), (*ND*,*ND*)
- Three tradable assets: CDS1, CDS2, risk-free asset

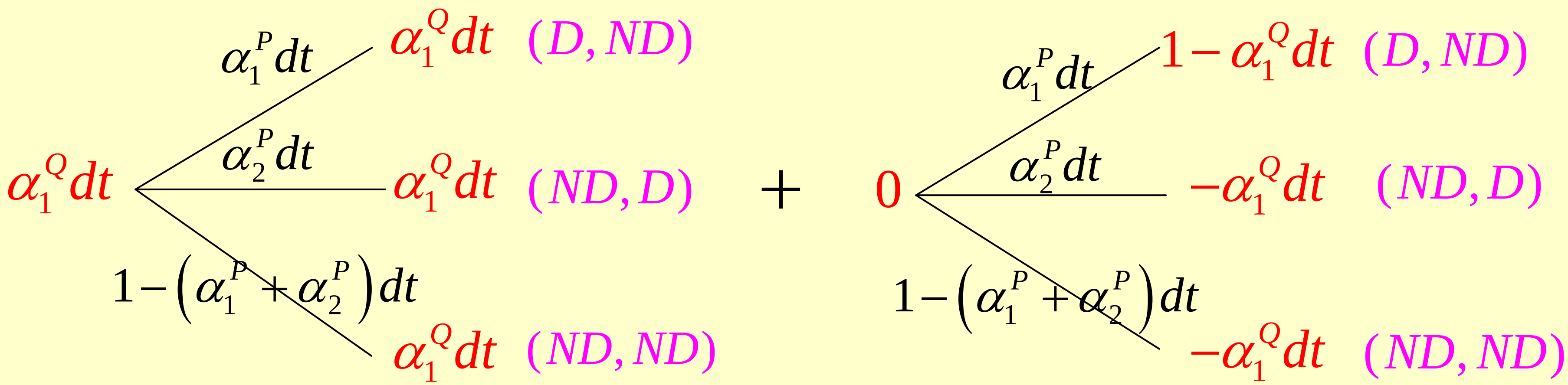


- For simplicity, let us assume $r = 0$

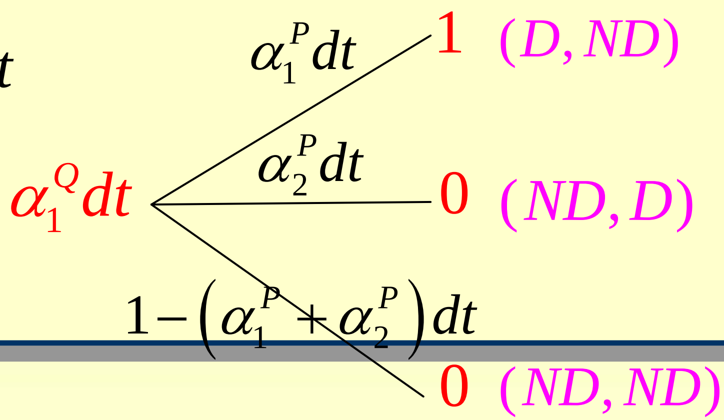
II - Tree approach to hedging defaults

- Three state contingent claims
  - Example: claim contingent on state (D, ND)
  - Can be replicated by holding
  - 1 CDS 1 + $\alpha_1^Q dt$ risk-free asset



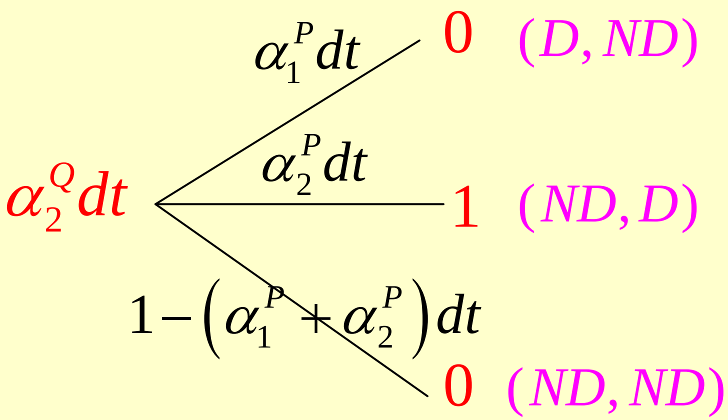


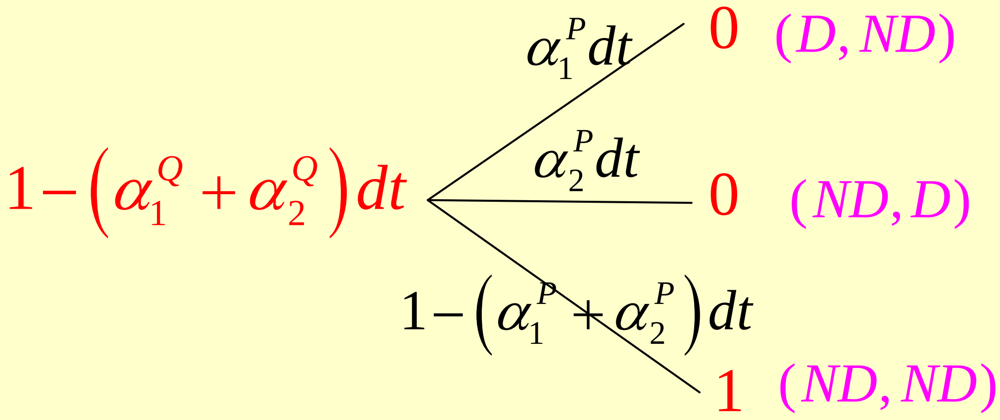
  - Replication price = $\alpha_1^Q dt$



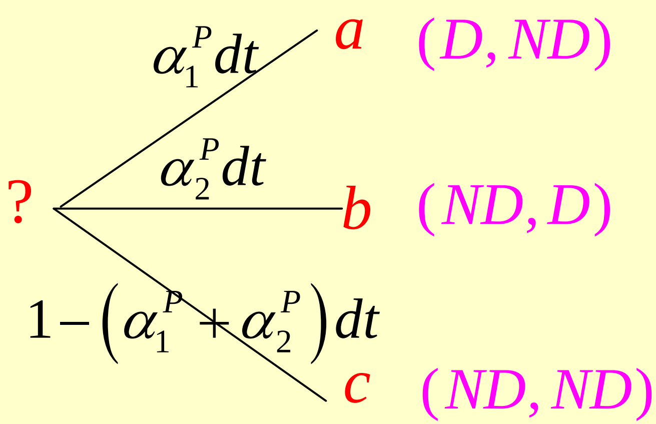
II - Tree approach to hedging defaults

- Similarly, the replication prices of the (ND, D) and (ND, ND) claims





- Replication price of:

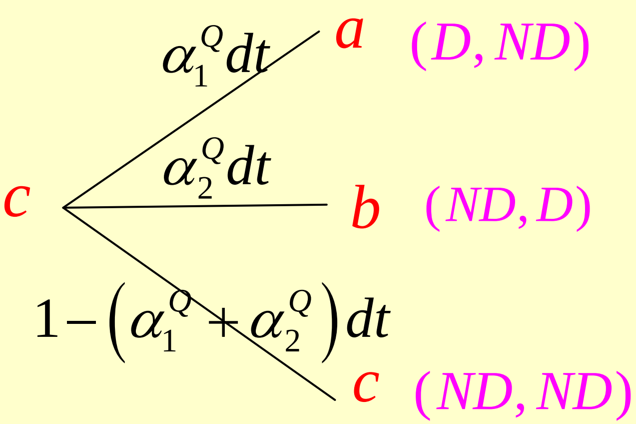


- Replication price = $\alpha_1^Q dt \times a + \alpha_2^Q dt \times b + \left(1-(\alpha_1^Q+\alpha_2^Q)dt\right)c$

II - Tree approach to hedging defaults

- Replication price obtained by computing the expected payoff
  - Along a risk-neutral tree

$$\alpha_1^Q dt \times a + \alpha_2^Q dt \times b + \left(1 - (\alpha_1^Q + \alpha_2^Q) dt\right) c$$



- Risk-neutral probabilities
  - Used for computing replication prices
  - Uniquely determined from short term CDS premiums
  - No need of historical default probabilities

II - Tree approach to hedging defaults

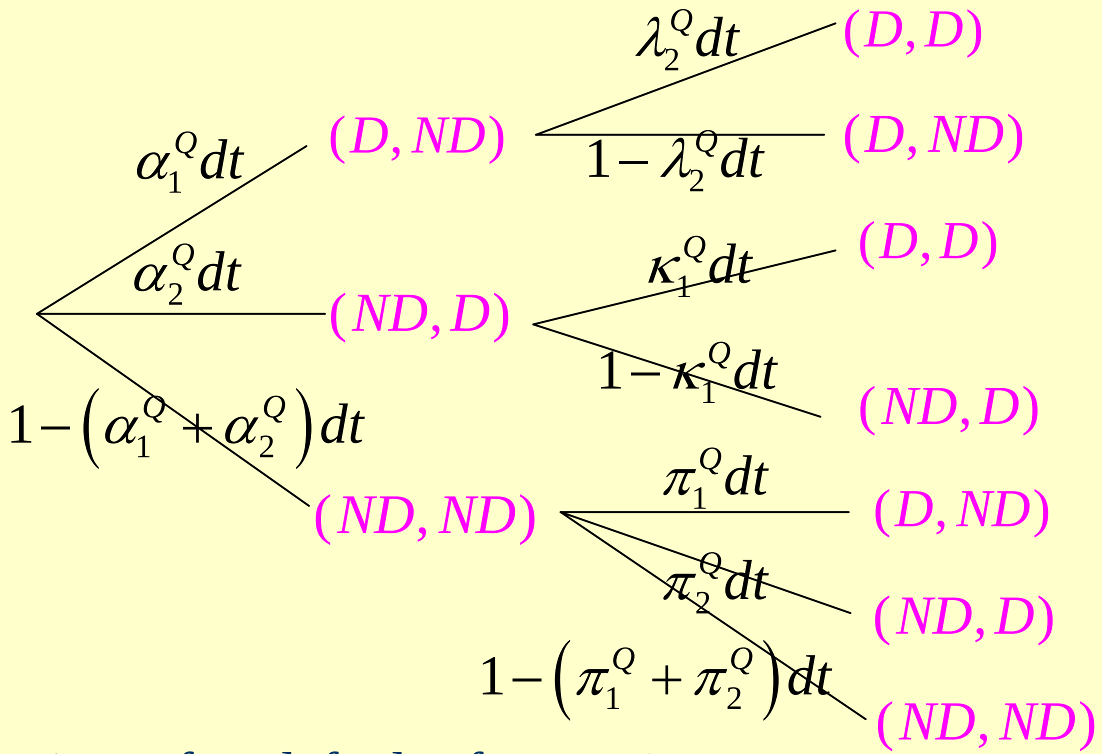
- Computation of deltas
  - Delta with respect to CDS 1: δ_1
  - Delta with respect to CDS 2: δ_2
  - Delta with respect to risk-free asset: p
    - ➢ p also equal to up-front premium

$$
\begin{cases}
a = p + \delta_1 \times \overbrace{\left(1 - \alpha_1^Q dt\right)}^{\text{payoff CDS 1}} + \delta_2 \times \overbrace{\left(-\alpha_2^Q dt\right)}^{\text{payoff CDS 2}} \\
b = p + \delta_1 \times \left(-\alpha_1^Q dt\right) + \delta_2 \times \left(1 - \alpha_2^Q dt\right) \\
c = p + \delta_1 \times \underbrace{\left(-\alpha_1^Q dt\right)}_{\text{payoff CDS 1}} + \delta_2 \times \underbrace{\left(-\alpha_2^Q dt\right)}_{\text{payoff CDS 2}}
\end{cases}
$$

  - As for the replication price, deltas only depend upon CDS premiums

II - Tree approach to hedging defaults

- Dynamic case:



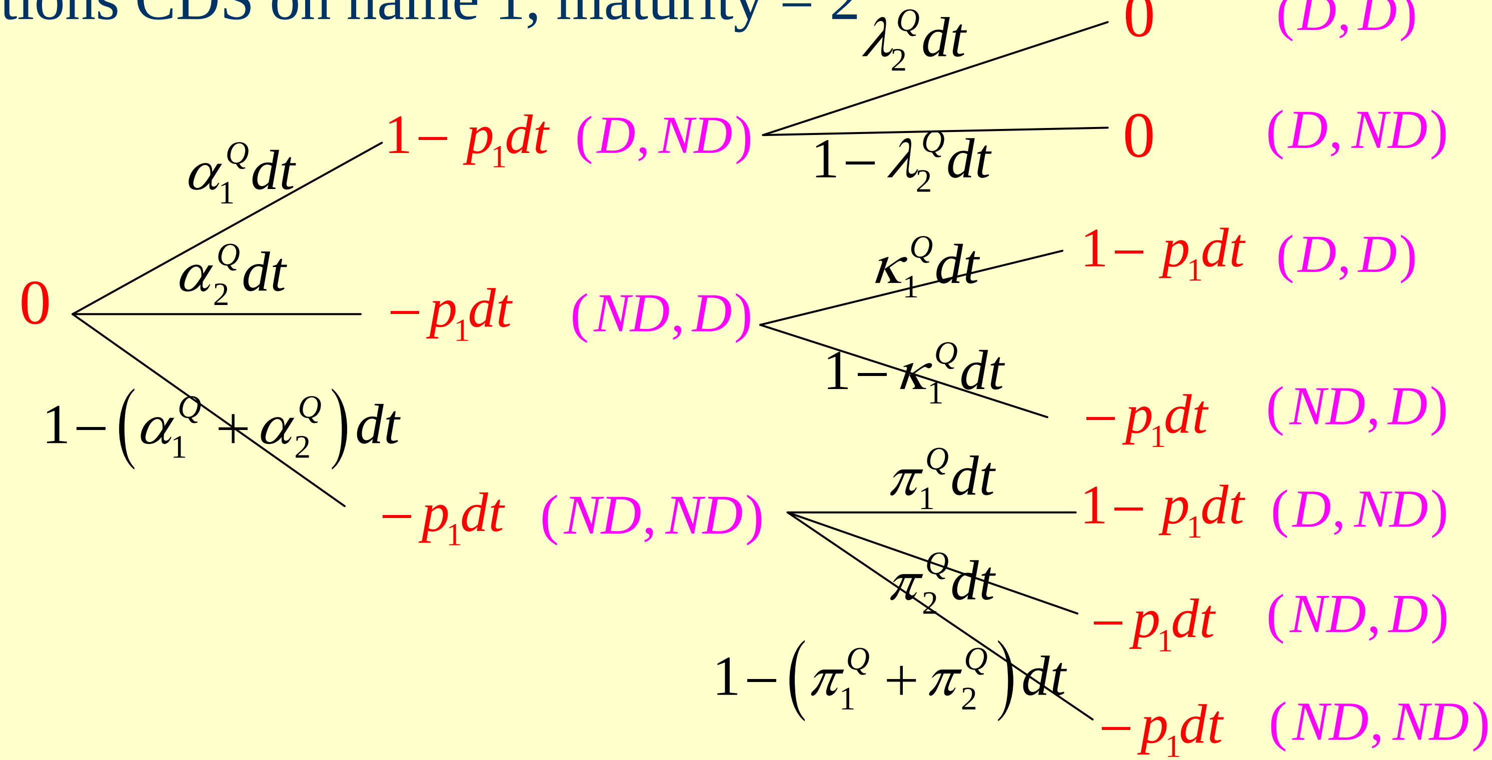
  - $\lambda_2^Q dt$ CDS 2 premium after default of name 1
  - $\kappa_1^Q dt$ CDS 1 premium after default of name 2
  - $\pi_1^Q dt$ CDS 1 premium if no name defaults at period 1
  - $\pi_2^Q dt$ CDS 2 premium if no name defaults at period 1
- Change in CDS premiums due to contagion effects
  - Usually, $\pi_1^Q < \alpha_1^Q < \lambda_1^Q$ and $\pi_2^Q < \alpha_2^Q < \lambda_2^Q$

II - Tree approach to hedging defaults

- Computation of prices and hedging strategies by backward induction
  - use of the dynamic risk-neutral tree
  - Start from period 2, compute price at period 1 for the three possible nodes
  - + hedge ratios in short term CDS 1,2 at period 1
  - Compute price and hedge ratio in short term CDS 1,2 at time 0
- Example to be detailed:
  - computation of CDS 1 premium, maturity = 2
  - $p_1 dt$ will denote the periodic premium
  - Cash-flow along the nodes of the tree

II - Tree approach to hedging defaults

- Computations CDS on name 1, maturity = 2

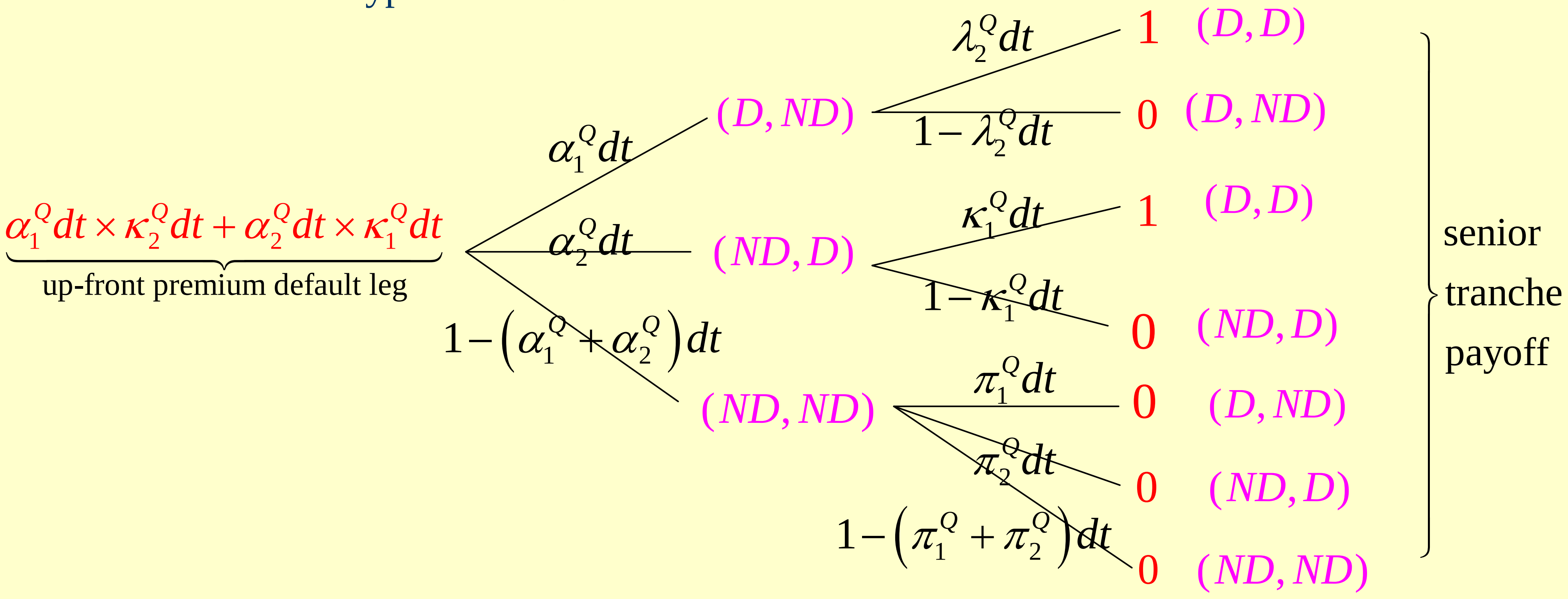


- Premium of CDS on name 1, maturity = 2, time = 0, $p_1 dt$ solves for:

$$0 = (1-p_1)\alpha_1^Q + \left(-p_1 + (1-p_1)\kappa_1^Q - p_1\left(1-\kappa_1^Q\right)\right)\alpha_2^Q$$
$$+\left(-p_1 + (1-p_1)\pi_1^Q - p_1\pi_2^Q - p_1\left(1-\pi_1^Q-\pi_2^Q\right)\right)\left(1-\alpha_1^Q-\alpha_2^Q\right)$$

II - Tree approach to hedging defaults

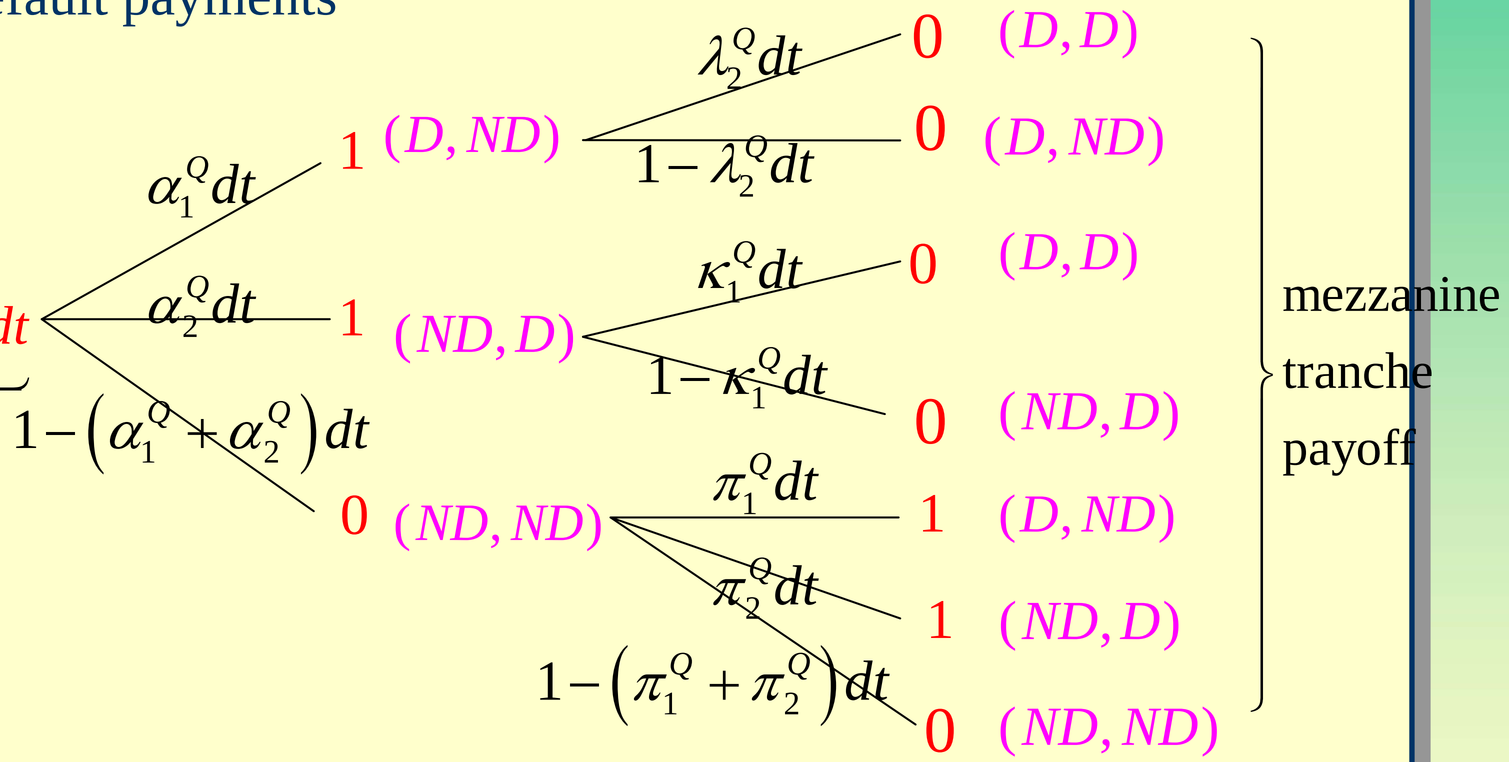
- Example: stylized zero coupon CDO trancheletts
  - Zero-recovery, maturity 2
  - Aggregate loss at time 2 can be equal to 0,1,2
    - Equity type tranche contingent on no defaults
    - Mezzanine type tranche : one default
    - Senior type tranche : two defaults



II - Tree approach to hedging defaults

- mezzanine tranche
  - Time pattern of default payments

$$\underbrace{\alpha_1^Q dt + \alpha_2^Q dt + \left(1-\left(\alpha_1^Q+\alpha_2^Q\right)dt\right)\left(\pi_1^Q+\pi_2^Q\right)dt}_{\text{up-front premium default leg}}$$



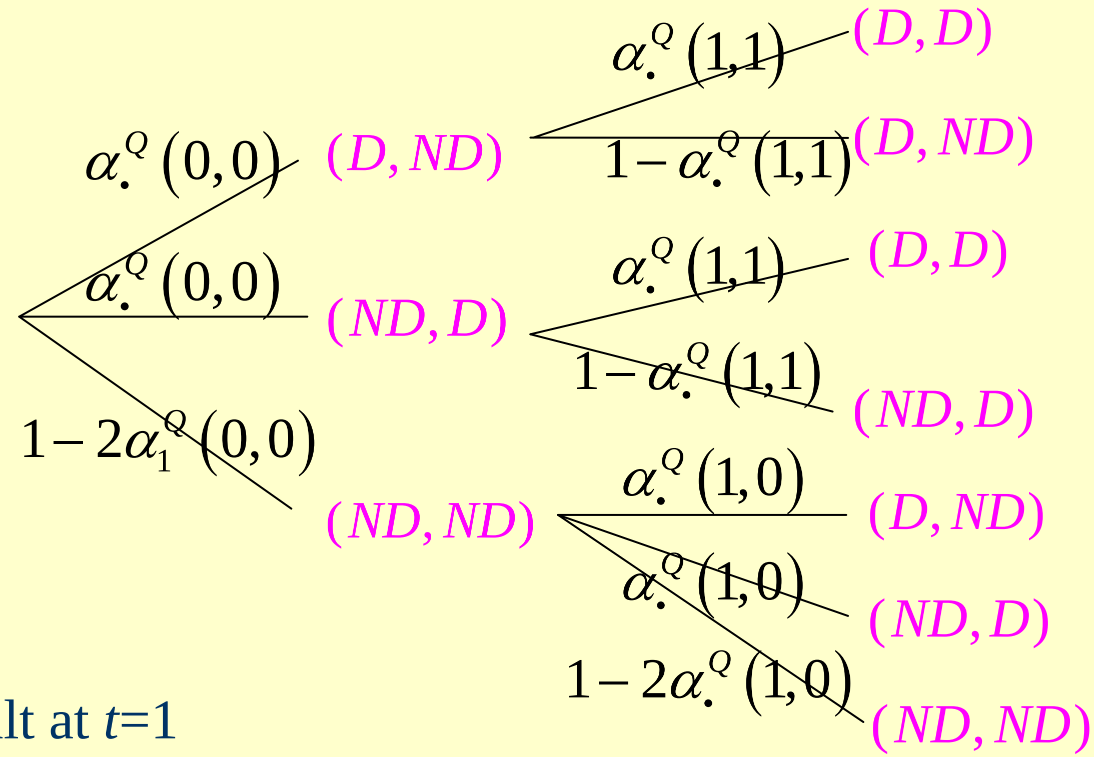
  - Possibility of taking into account discounting effects
  - The timing of premium payments
  - Computation of dynamic deltas with respect to short or actual CDS on names 1,2

II - Tree approach to hedging defaults

- In theory, one could also derive dynamic hedging strategies for index CDO tranches
  - Numerical issues: large dimensional, non recombining trees
  - Homogeneous Markovian assumption is very convenient
    - ➢CDS premiums at a given time *t* only depend upon the current number of defaults $N(t)$
  - CDS premium at time 0 (no defaults) $\alpha_1^Q dt = \alpha_2^Q dt = \alpha_\bullet^Q \left(t = 0, N(0) = 0\right)$
  - CDS premium at time 1 (one default) $\lambda_2^Q dt = \kappa_1^Q dt = \alpha_\bullet^Q \left(t = 1, N(t) = 1\right)$
  - CDS premium at time 1 (no defaults) $\pi_1^Q dt = \pi_2^Q dt = \alpha_\bullet^Q \left(t = 1, N(t) = 0\right)$

II - Tree approach to hedging defaults

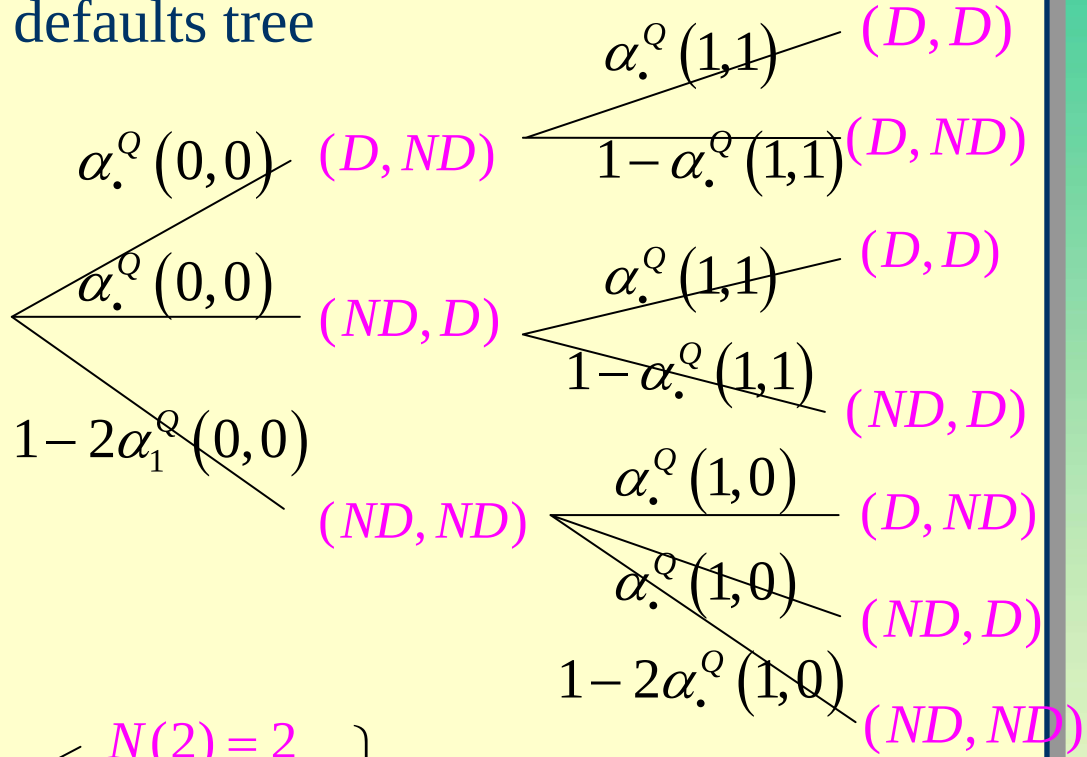
- Homogeneous Markovian tree

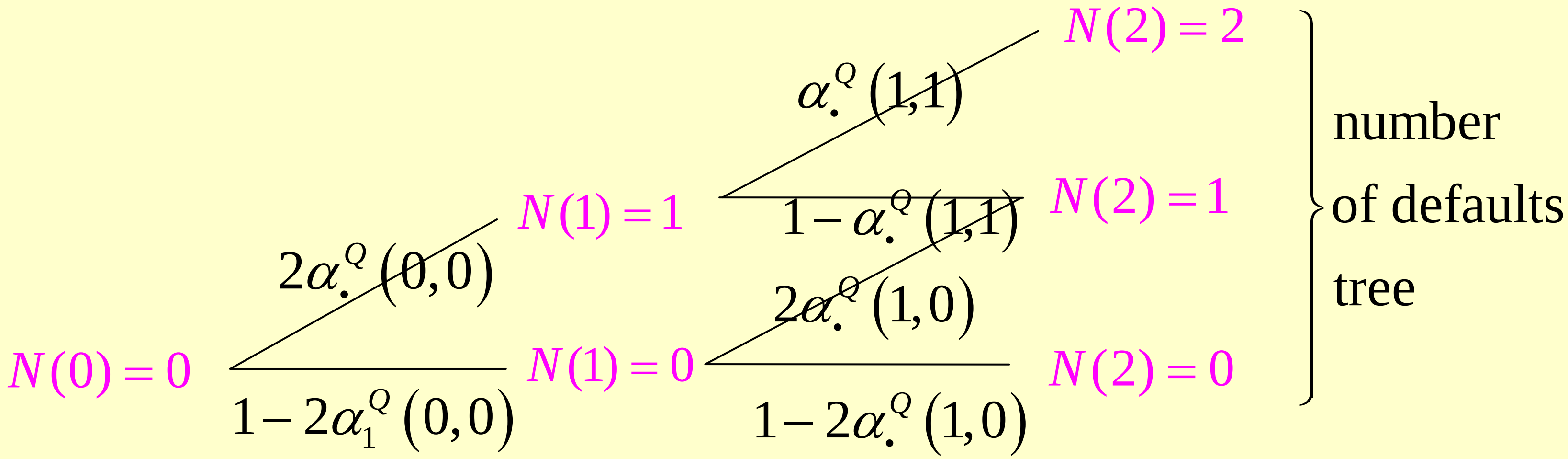


  - If we have $N(1)=1$, one default at t=1
  - The probability to have $N(2)=1$, one default at t=2…
  - Is $1-\alpha_{\bullet}^{Q}(1,1)$ and does not depend on the defaulted name at t=1
  - $N(t)$ is a Markov process
  - Dynamics of the number of defaults can be expressed through a binomial tree

II - Tree approach to hedging defaults

- From name per name to number of defaults tree

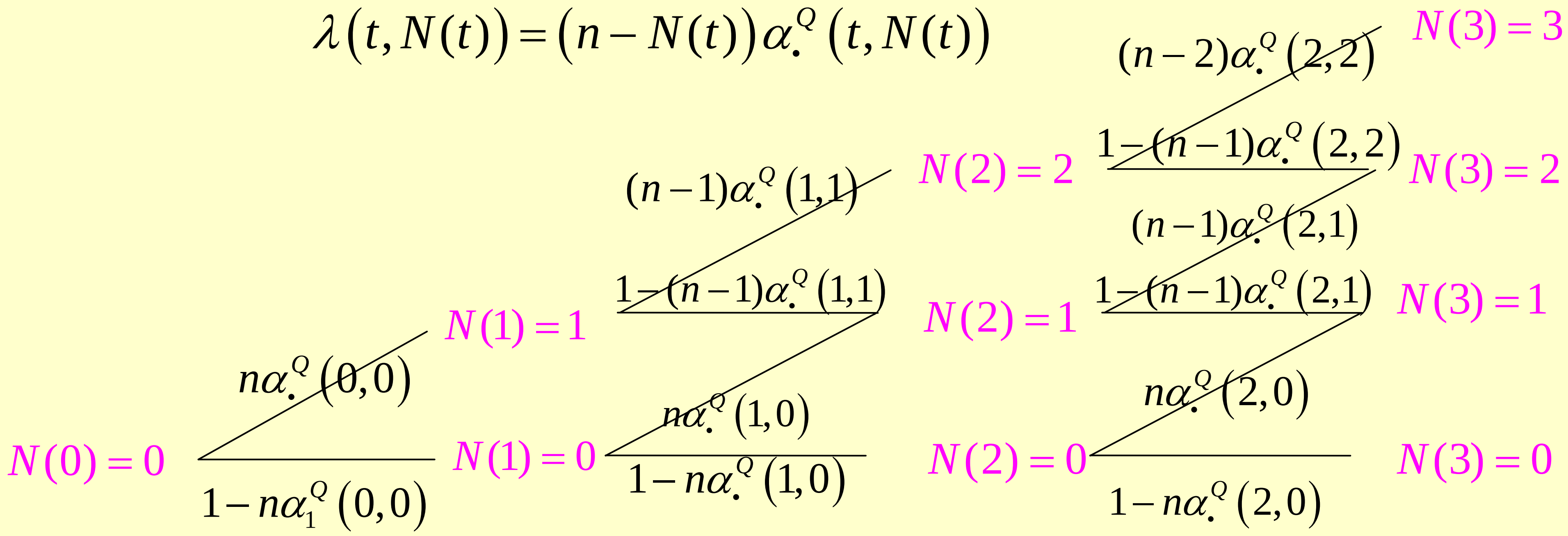




II - Tree approach to hedging defaults

- Easy extension to *n* names
  - Predefault name intensity at time *t* for $N(t)$ defaults: $\alpha_{\bullet}^{Q}\left(t,N(t)\right)$
  - Number of defaults intensity : sum of surviving name intensities:

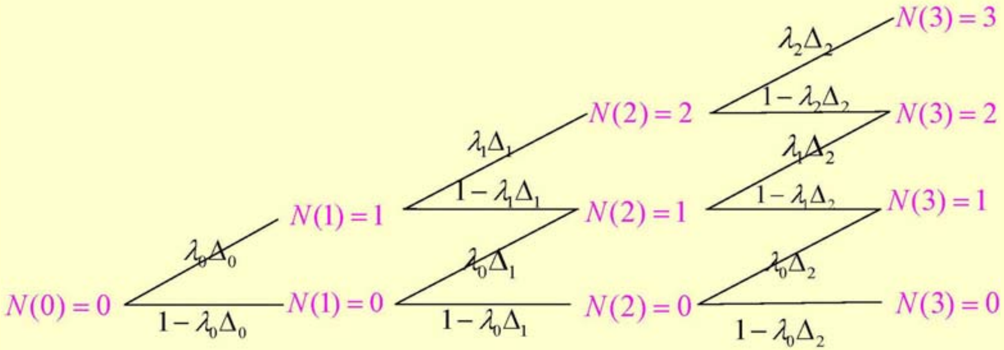
$$\lambda\left(t,N(t)\right)=\left(n-N(t)\right)\alpha_{\bullet}^{Q}\left(t,N(t)\right)$$



  - $\alpha_{\bullet}^{Q}(0,0),\alpha_{\bullet}^{Q}(1,0),\alpha_{\bullet}^{Q}(1,1),\alpha_{\bullet}^{Q}(2,0),\alpha_{\bullet}^{Q}(2,1),\ldots$ can be easily calibrated
  - on marginal distributions of $N(t)$ by forward induction.

II - Tree approach to hedging defaults

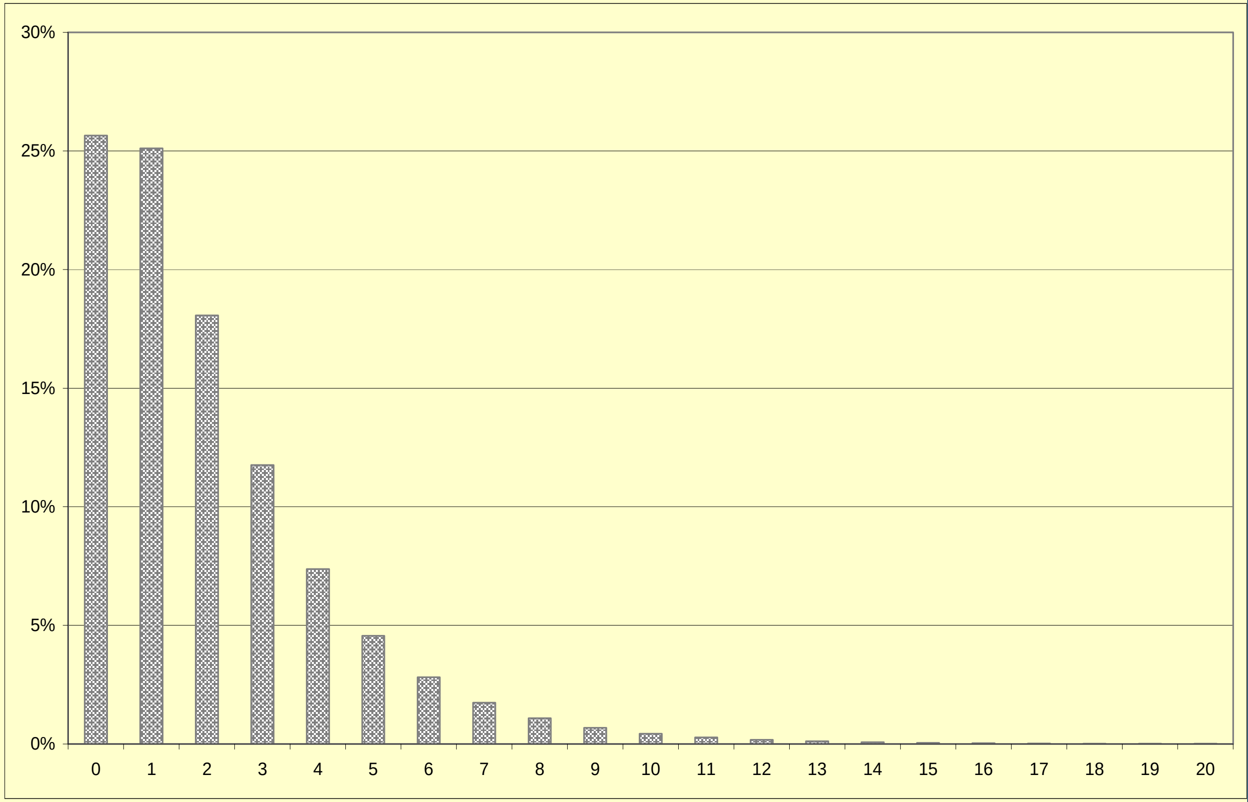
- Previous recombining binomial risk-neutral tree provides a framework for the valuation of payoffs depending upon the number of defaults
  - CDO tranches
  - Credit default swap index



- What about the credit deltas?
  - In a homogeneous framework, deltas with respect to CDS are all the same
  - Perfect dynamic replication of a CDO tranche with a credit default swap index and the default-free asset
  - Credit delta with respect to the credit default swap index
  - = change in PV of the tranche / change in PV of the CDS index

II - Tree approach to hedging defaults

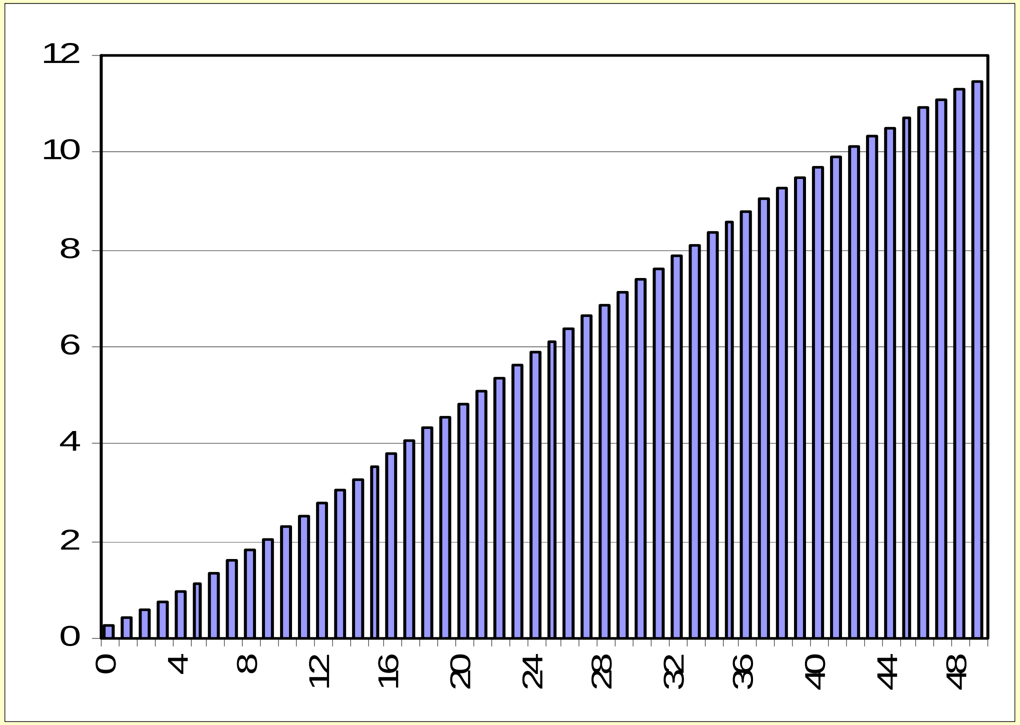
- Example: number of defaults distribution at 5Y generated from a Gaussian copula
  - Correlation parameter: 30%
  - Number of names: 125
  - Default-free rate: 3%
  - 5Y credit spreads: 20 bps
  - Recovery rate: 40%



- Figure shows the probabilities of *k* defaults for a 5Y horizon

II - Tree approach to hedging defaults

- Calibration of loss intensities
  - For simplicity, assumption of time homogeneous intensities
  - Figure below represents loss intensities, with respect to the number of defaults
  - Increase in intensities: contagion effects



II - Tree approach to hedging defaults

- Dynamics of the 5Y CDS index spread
  – In bp pa

		Weeks						
		0	**14**	**28**	**42**	**56**	**70**	**84**
Nb Defaults	**0**	20	19	19	18	18	17	17
	1	0	31	30	29	28	27	26
	2	0	46	44	43	41	40	38
	3	0	63	61	58	56	54	52
	4	0	83	79	76	73	70	67
	5	0	104	99	95	91	87	83
	6	0	127	121	116	111	106	101
	7	0	151	144	138	132	126	120
	8	0	176	169	161	154	146	140
	9	0	203	194	185	176	168	160
	10	0	230	219	209	200	190	181
	11	0	257	246	235	224	213	203
	12	0	284	272	260	248	237	225
	13	0	310	298	286	273	260	248
	14	0	336	324	311	298	284	271
	15	0	0	348	336	323	308	294

II - Tree approach to hedging defaults

- Dynamics of credit deltas:
  - [0,3%] equity tranche, buy protection
  - With respect to the 5Y CDS index
  - For selected time steps

		OutStanding Nominal	Weeks						
			0	**14**	**28**	**42**	**56**	**70**	**84**
Nb Defaults	**0**	3.00%	0.967	0.993	1.016	1.035	1.052	1.065	1.075
	1	2.52%	0	0.742	0.786	0.828	0.869	0.908	0.943
	2	2.04%	0	0.439	0.484	0.532	0.583	0.637	0.691
	3	1.56%	0	0.206	0.233	0.265	0.301	0.343	0.391
	4	1.08%	0	0.082	0.093	0.106	0.121	0.141	0.164
	5	0.60%	0	0.029	0.032	0.035	0.039	0.045	0.051
	6	0.12%	0	0.004	0.005	0.005	0.006	0.006	0.007
	7	0.00%	0	0	0	0	0	0	0

  - Hedging strategy leads to a perfect replication of equity tranche payoff
  - Prior to first defaults, deltas are above 1!
  - When the number of defaults is > 6, the tranche is exhausted

II - Tree approach to hedging defaults

- Credit deltas of the tranche
  - Sum of credit deltas of premium and default legs

premium leg

Nb Defaults		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
	0	3.00%	-0.153	-0.150	-0.146	-0.142	-0.137	-0.132	-0.126
	1	2.52%	0	-0.128	-0.127	-0.126	-0.124	-0.120	-0.116
	2	2.04%	0	-0.098	-0.100	-0.101	-0.102	-0.101	-0.100
	3	1.56%	0	-0.066	-0.068	-0.071	-0.073	-0.074	-0.076
	4	1.08%	0	-0.037	-0.039	-0.041	-0.043	-0.045	-0.047
	5	0.60%	0	-0.016	-0.017	-0.018	-0.019	-0.020	-0.021
	6	0.12%	0	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003
	7	0.00%	0	0	0	0	0	0	0

default leg

Nb Defaults		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
	0	3.00%	0.814	0.843	0.869	0.893	0.915	0.933	0.949
	1	2.52%	0	0.614	0.658	0.702	0.746	0.787	0.827
	2	2.04%	0	0.341	0.384	0.431	0.482	0.535	0.591
	3	1.56%	0	0.140	0.165	0.194	0.229	0.269	0.315
	4	1.08%	0	0.045	0.054	0.064	0.078	0.095	0.117
	5	0.60%	0	0.013	0.015	0.017	0.020	0.024	0.030
	6	0.12%	0	0.002	0.002	0.002	0.003	0.003	0.003
	7	0.00%	0	0	0	0	0	0	0

II - Tree approach to hedging defaults

Nb Defaults	OutStanding Nominal	Weeks 0	14	28	42	56	70	84
0	3.00%	-0.153	-0.150	-0.146	-0.142	-0.137	-0.132	-0.126
1	2.52%	0	-0.128	-0.127	-0.126	-0.124	-0.120	-0.116
2	2.04%	0	-0.098	-0.100	-0.101	-0.102	-0.101	-0.100
3	1.56%	0	-0.066	-0.068	-0.071	-0.073	-0.074	-0.076
4	1.08%	0	-0.037	-0.039	-0.041	-0.043	-0.045	-0.047
5	0.60%	0	-0.016	-0.017	-0.018	-0.019	-0.020	-0.021
6	0.12%	0	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003
7	0.00%	0	0	0	0	0	0	0

- **Credit deltas of the premium leg of the equity tranche**
  - **Premiums based on outstanding nominal**
  - **Arrival of defaults reduces the commitment to pay**
    - ➢ Smaller outstanding nominal
    - ➢ Increase in credit spreads (contagion) involve a decrease in expected outstanding nominal
  - **Negative deltas**
    - ➢ This is only significant for the equity tranche
      - **Associated with much larger spreads**

II - Tree approach to hedging defaults

		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
Nb Defaults	0	3.00%	0.814	0.843	0.869	0.893	0.915	0.933	0.949
	1	2.52%	0	0.614	0.658	0.702	0.746	0.787	0.827
	2	2.04%	0	0.341	0.384	0.431	0.482	0.535	0.591
	3	1.56%	0	0.140	0.165	0.194	0.229	0.269	0.315
	4	1.08%	0	0.045	0.054	0.064	0.078	0.095	0.117
	5	0.60%	0	0.013	0.015	0.017	0.020	0.024	0.030
	6	0.12%	0	0.002	0.002	0.002	0.003	0.003	0.003
	7	0.00%	0	0	0	0	0	0	0

- **Credit deltas for the default leg of the equity tranche**
  - **Are actually between 0 and 1**
  - **Gradually decrease with the number of defaults**
    - Concave payoff, negative gammas
  - **Credit deltas increase with time**
    - Consistent with a decrease in time value
    - At maturity date, when number of defaults < 6, delta=1

II - Tree approach to hedging defaults

- Dynamics of credit deltas
  - Junior mezzanine tranche [3,6%]
  - Deltas lie in between 0 and 1
  - When the number of defaults is above 12, the tranche is exhausted

		OutStanding Nominal	Weeks						
			0	**14**	**28**	**42**	**56**	**70**	**84**
Nb Defaults	**0**	3.00%	0.162	0.139	0.117	0.096	0.077	0.059	0.045
	1	3.00%	0	0.327	0.298	0.266	0.232	0.197	0.162
	2	3.00%	0	0.497	0.489	0.473	0.448	0.415	0.376
	3	3.00%	0	0.521	0.552	0.576	0.591	0.595	0.586
	4	3.00%	0	0.400	0.454	0.508	0.562	0.611	0.652
	5	3.00%	0	0.239	0.288	0.343	0.405	0.473	0.544
	6	3.00%	0	0.123	0.153	0.190	0.236	0.291	0.358
	7	2.64%	0	0.059	0.073	0.090	0.115	0.147	0.189
	8	2.16%	0	0.031	0.036	0.043	0.052	0.066	0.086
	9	1.68%	0	0.019	0.020	0.023	0.026	0.030	0.037
	10	1.20%	0	0.012	0.012	0.013	0.014	0.016	0.018
	11	0.72%	0	0.007	0.007	0.007	0.007	0.008	0.009
	12	0.24%	0	0.002	0.002	0.002	0.002	0.002	0.003
	13	0.00%	0	0	0	0	0	0	0

II - Tree approach to hedging defaults

- **Dynamics of credit deltas (junior mezzanine tranche)**
  - Gradually increase and then decrease with the number of defaults
  - Call spread payoff (convex, then concave)
  - Initial delta = 16% (out of the money option)

		OutStanding Nominal	Weeks						
			0	**14**	**28**	**42**	**56**	**70**	**84**
Nb Defaults	**0**	3.00%	0.162	0.139	0.117	0.096	0.077	0.059	0.045
	1	3.00%	0	0.327	0.298	0.266	0.232	0.197	0.162
	2	3.00%	0	0.497	0.489	0.473	0.448	0.415	0.376
	3	3.00%	0	0.521	0.552	0.576	0.591	0.595	0.586
	4	3.00%	0	0.400	0.454	0.508	0.562	0.611	0.652
	5	3.00%	0	0.239	0.288	0.343	0.405	0.473	0.544
	6	3.00%	0	0.123	0.153	0.190	0.236	0.291	0.358
	7	2.64%	0	0.059	0.073	0.090	0.115	0.147	0.189
	8	2.16%	0	0.031	0.036	0.043	0.052	0.066	0.086
	9	1.68%	0	0.019	0.020	0.023	0.026	0.030	0.037
	10	1.20%	0	0.012	0.012	0.013	0.014	0.016	0.018
	11	0.72%	0	0.007	0.007	0.007	0.007	0.008	0.009
	12	0.24%	0	0.002	0.002	0.002	0.002	0.002	0.003
	13	0.00%	0	0	0	0	0	0	0

II - Tree approach to hedging defaults

- **Comparison analysis**
  - **After six defaults, the [3,6%] should be like a [0,3%] equity tranche**
  - **However, credit delta is much lower**
    - ➢ 12% instead of 84%
  - **But credit spreads after six defaults are much larger**
    - ➢ 127 bps instead of 19 bps
  - **Expected loss of the tranche is much larger**
  - **Which is associated with smaller deltas**

Nb Defaults	OutStanding Nominal	0	14
0	3.00%	0.162	0.139
1	3.00%	0	0.327
2	3.00%	0	0.497
3	3.00%	0	0.521
4	3.00%	0	0.400
5	3.00%	0	0.239
6	3.00%	0	0.123
7	2.64%	0	0.059
8	2.16%	0	0.031
9	1.68%	0	0.019
10	1.20%	0	0.012
11	0.72%	0	0.007
12	0.24%	0	0.002
13	0.00%	0	0

Nb Defaults	OutStanding Nominal	0	14
0	3.00%	0.814	0.843
1	2.52%	0	0.614
2	2.04%	0	0.341
3	1.56%	0	0.140
4	1.08%	0	0.045
5	0.60%	0	0.013
6	0.12%	0	0.002
7	0.00%	0	0

II - Tree approach to hedging defaults

- Dynamics of credit deltas ([6,9%] tranche)
  - Initial credit deltas are smaller (deeper out of the money call spread)

		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
Nb Defaults	0	3.00%	0.017	0.012	0.008	0.005	0.003	0.002	0.001
	1	3.00%	0	0.048	0.036	0.025	0.017	0.011	0.006
	2	3.00%	0	0.133	0.107	0.083	0.061	0.043	0.029
	3	3.00%	0	0.259	0.227	0.193	0.157	0.122	0.090
	4	3.00%	0	0.371	0.356	0.330	0.295	0.253	0.206
	5	3.00%	0	0.405	0.423	0.428	0.420	0.396	0.358
	6	3.00%	0	0.346	0.392	0.433	0.465	0.482	0.481
	7	3.00%	0	0.239	0.292	0.350	0.409	0.465	0.510
	8	3.00%	0	0.139	0.181	0.232	0.293	0.363	0.436
	9	3.00%	0	0.074	0.098	0.132	0.177	0.235	0.307
	10	3.00%	0	0.042	0.053	0.070	0.095	0.132	0.183
	11	3.00%	0	0.029	0.033	0.040	0.051	0.070	0.098
	12	3.00%	0	0.025	0.026	0.028	0.033	0.040	0.053
	13	2.76%	0	0.022	0.022	0.022	0.024	0.026	0.031
	14	2.28%	0	0.020	0.018	0.018	0.018	0.019	0.020
	15	1.80%	0	0	0.015	0.014	0.014	0.014	0.014
	16	1.32%	0	0	0.013	0.011	0.010	0.010	0.010
	17	0.84%	0	0	0.009	0.008	0.007	0.006	0.006
	18	0.36%	0	0	0.005	0.004	0.003	0.003	0.003
	19	0.00%	0	0	0	0	0	0	0

II - Tree approach to hedging defaults

- Small dependence of credit deltas with respect to recovery rate
  - Equity tranche, *R*=30%

Nb Defaults		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
	0	3.00%	0.975	0.997	1.018	1.035	1.050	1.062	1.072
	1	2.44%	0.000	0.735	0.775	0.814	0.852	0.888	0.922
	2	1.88%	0.000	0.417	0.456	0.499	0.544	0.591	0.641
	3	1.32%	0.000	0.178	0.200	0.225	0.253	0.286	0.324
	4	0.76%	0.000	0.060	0.066	0.074	0.084	0.095	0.109
	5	0.20%	0.000	0.011	0.011	0.013	0.014	0.015	0.017
	6	0.00%	0.000	0.000	0.000	0.000	0.000	0.000	0.000

  - Equity tranche, *R*=40%

Nb Defaults		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
	0	3.00%	0.967	0.993	1.016	1.035	1.052	1.065	1.075
	1	2.52%	0	0.742	0.786	0.828	0.869	0.908	0.943
	2	2.04%	0	0.439	0.484	0.532	0.583	0.637	0.691
	3	1.56%	0	0.206	0.233	0.265	0.301	0.343	0.391
	4	1.08%	0	0.082	0.093	0.106	0.121	0.141	0.164
	5	0.60%	0	0.029	0.032	0.035	0.039	0.045	0.051
	6	0.12%	0	0.004	0.005	0.005	0.006	0.006	0.007
	7	0.00%	0	0	0	0	0	0	0

II - Tree approach to hedging defaults

- Small dependence of credit deltas with respect to recovery rate
  - Initial delta with respect to the credit default swap index

	Recovery Rates					
Tranches	**10%**	**20%**	**30%**	**40%**	**50%**	**60%**
[0-3%]	0.9960	0.9824	0.9746	0.9670	0.9527	0.9456
[3-6%]	0.1541	0.1602	0.1604	0.1616	0.1659	0.1604
[6-9%]	0.0164	0.0165	0.0168	0.0168	0.0168	0.0169

  - Only a small dependence of credit deltas with respect to recovery rates

➢ **Which is rather fortunate**

II - Tree approach to hedging defaults

- **Dependence of credit deltas with respect to correlation**
  - **Default leg, equity tranche**

$\rho=10\%$

		OutStanding Nominal	Weeks						
			0	**14**	**28**	**42**	**56**	**70**	**84**
Nb Defaults	**0**	3.00%	0.968	0.974	0.978	0.982	0.985	0.987	0.990
	1	2.52%	0	0.933	0.944	0.953	0.962	0.969	0.976
	2	2.04%	0	0.835	0.856	0.876	0.895	0.912	0.928
	3	1.56%	0	0.653	0.683	0.714	0.744	0.774	0.804
	4	1.08%	0	0.405	0.433	0.464	0.496	0.531	0.568
	5	0.60%	0	0.170	0.185	0.202	0.221	0.243	0.268
	6	0.12%	0	0.027	0.030	0.033	0.037	0.041	0.046
	7	0.00%	0	0	0	0	0	0	0

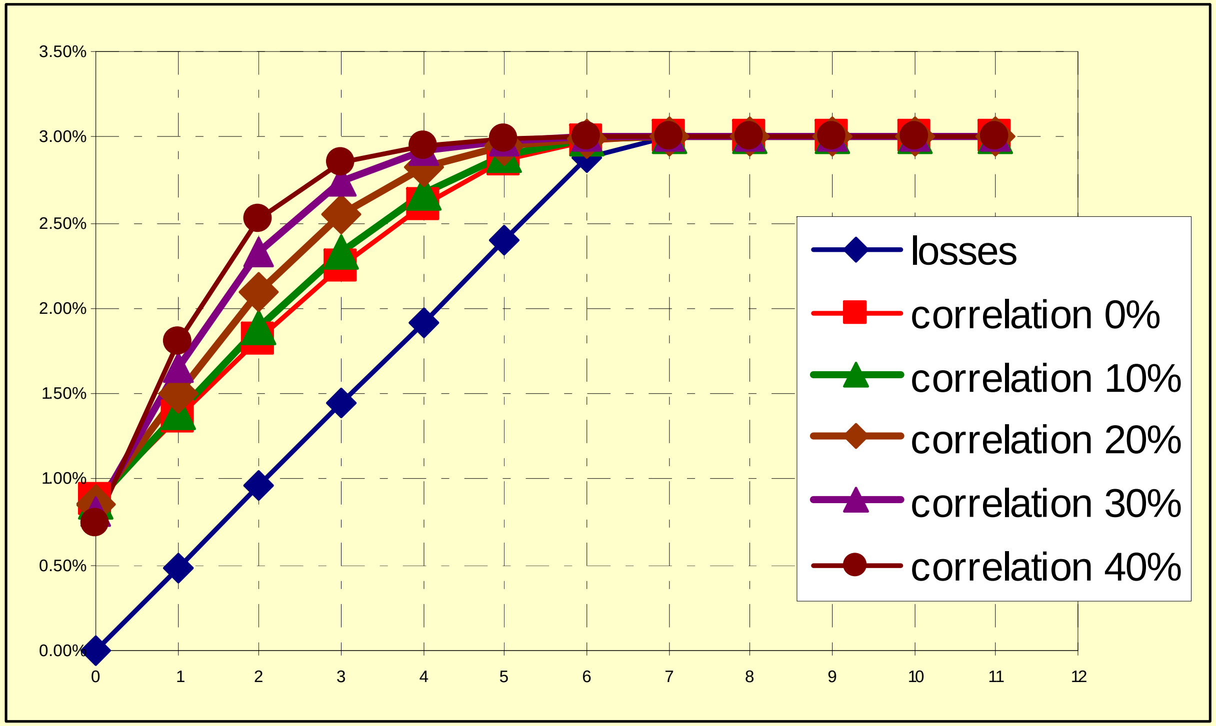
$\rho=30\%$

		OutStanding Nominal	Weeks						
			0	**14**	**28**	**42**	**56**	**70**	**84**
Nb Defaults	**0**	3.00%	0.814	0.843	0.869	0.893	0.915	0.933	0.949
	1	2.52%	0	0.614	0.658	0.702	0.746	0.787	0.827
	2	2.04%	0	0.341	0.384	0.431	0.482	0.535	0.591
	3	1.56%	0	0.140	0.165	0.194	0.229	0.269	0.315
	4	1.08%	0	0.045	0.054	0.064	0.078	0.095	0.117
	5	0.60%	0	0.013	0.015	0.017	0.020	0.024	0.030
	6	0.12%	0	0.002	0.002	0.002	0.003	0.003	0.003
	7	0.00%	0	0	0	0	0	0	0

II - Tree approach to hedging defaults

$$\begin{cases} \rho = 10\%, N(14) = 0, \delta = 97\% \\ \rho = 30\%, N(14) = 0, \delta = 84\% \end{cases}$$

- **Equity deltas decrease as correlation increases**
- **Value of equity default leg under different correlation assumptions**



  – **Number of defaults on the *x* - axis**

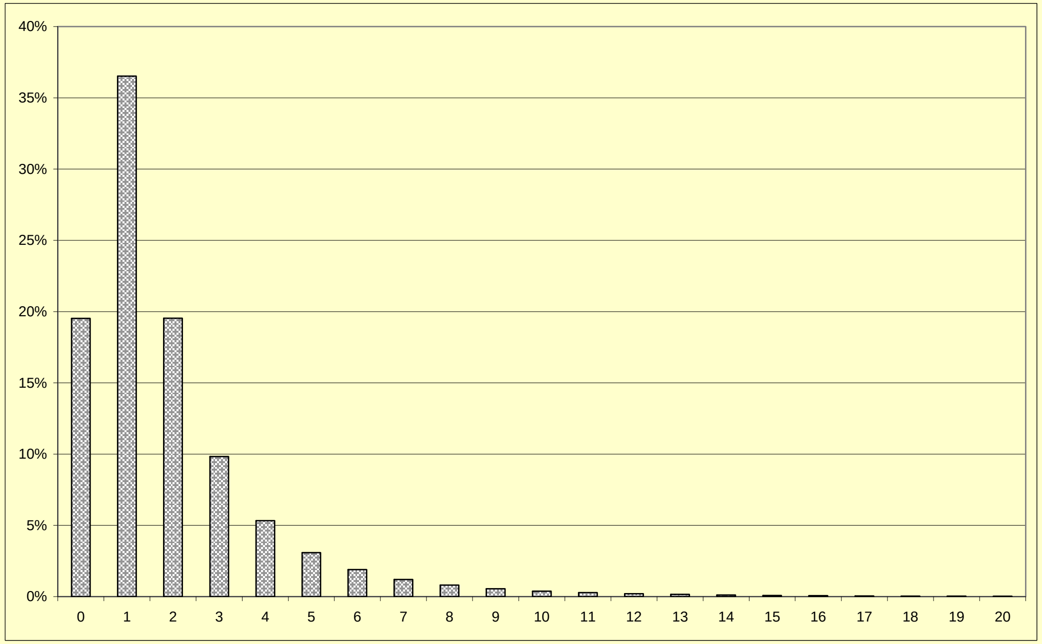
II - Tree approach to hedging defaults

- Smaller correlation
  - Prior to first default, higher expected losses on the tranche
    - ➢ Should lead to smaller deltas
  - But smaller contagion effects
    - ➢ When shifting from zero to one default
    - ➢ The expected loss on the index jumps due to…
      - **Default arrival and jumps in credit spreads**
      - **Smaller jumps in credit spreads for smaller correlation**
    - ➢ Smaller correlation is associated with smaller jumps in the expected loss of the index
    - ➢ Leads to higher deltas
      - **Since we have negative gamma**

II - Tree approach to hedging defaults

- **Computing deltas with market inputs**
  - **Base correlations (5Y), as for iTraxx, June 2007**

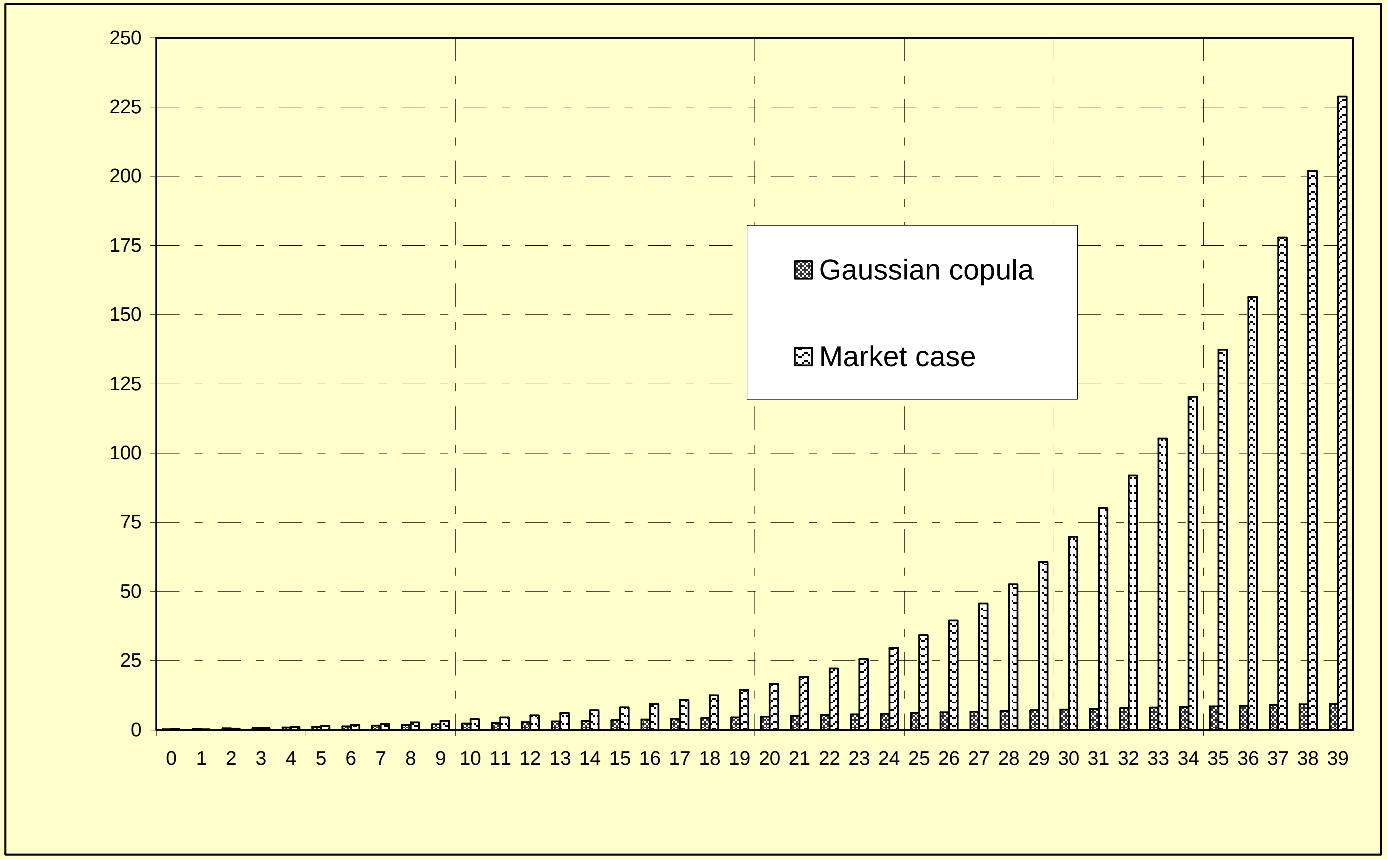
3%	6%	9%	12%	22%
16%	24%	30%	35%	50%



  - **Probabilities of *k* defaults**

II - Tree approach to hedging defaults

- **Loss intensities for the Gaussian copula and market case examples**



  - **Number of defaults on the *x* - axis**

II - Tree approach to hedging defaults

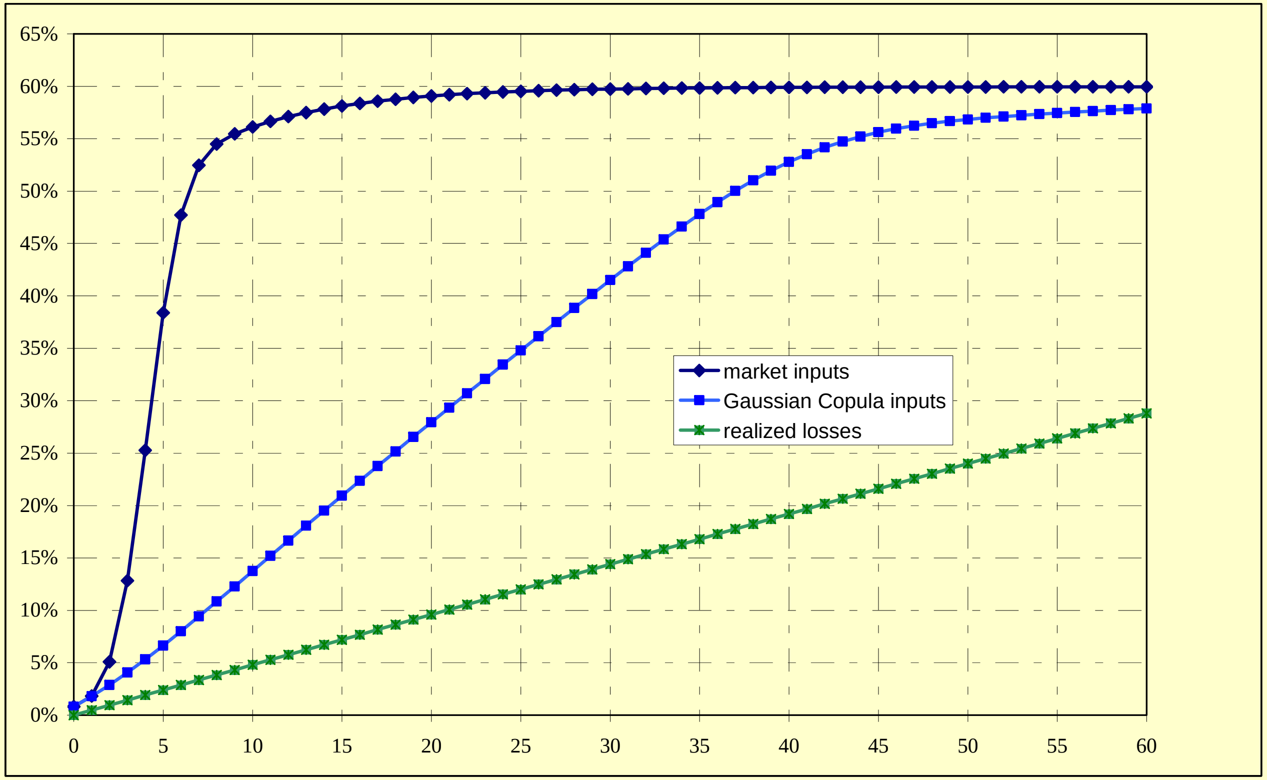
- **Credit spread dynamics**
  - **Base correlation inputs**

		Weeks						
		0	**14**	**28**	**42**	**56**	**70**	**84**
Nb Defaults	**0**	20	19	18	18	17	16	16
	1	0	31	28	25	23	21	20
	2	0	95	80	67	57	49	43
	3	0	269	225	185	150	121	98
	4	0	592	515	437	361	290	228
	5	0	1022	934	834	723	607	490
	6	0	1466	1395	1305	1193	1059	905
	7	0	1870	1825	1764	1680	1567	1420
	8	0	2243	2214	2177	2126	2052	1945
	9	0	2623	2597	2568	2534	2488	2423
	10	0	3035	3003	2971	2939	2903	2859
	11	0	3491	3450	3410	3371	3331	3290
	12	0	4001	3947	3896	3845	3795	3747
	13	0	4570	4501	4434	4369	4306	4245
	14	0	5206	5117	5031	4948	4868	4790
	15	0	5915	5801	5691	5586	5484	5386

  - **Similar to Gaussian copula at the first default**
  - **Dramatic increases in credit spreads after a few defaults**

II - Tree approach to hedging defaults

- **Comparison of Gaussian copula and market inputs**



  - **Expected losses on the credit portfolio after 14 weeks**
  - **With respect to the number of observed defaults**

- **Much bigger contagion effects with steep base correlation**

II - Tree approach to hedging defaults

- **Comparison of credit deltas**
  - Gaussian copula and market case examples
  - Smaller credit deltas for the equity tranche

Nb Defaults		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
	0	3.00%	0.645	0.731	0.814	0.890	0.953	1.003	1.038
	1	2.52%	0.000	0.329	0.402	0.488	0.584	0.684	0.777
	2	2.04%	0.000	0.091	0.115	0.149	0.197	0.264	0.351
	3	1.56%	0.000	0.023	0.028	0.035	0.045	0.062	0.090
	4	1.08%	0.000	0.008	0.008	0.009	0.011	0.013	0.018
	5	0.60%	0.000	0.004	0.004	0.003	0.003	0.003	0.004
	6	0.12%	0.000	0.001	0.001	0.001	0.001	0.001	0.001
	7	0.00%	0.000	0.000	0.000	0.000	0.000	0.000	0.000

  - Dynamic correlation effects
  - After the first default, due to magnified contagion,
  - New defaults are associated with big shifts in correlation

II - Tree approach to hedging defaults

- **Comparison of credit deltas**
  - **Market and model deltas at inception**
  - **Equity tranche**

	[0-3%]	[3-6%]	[6-9%]	[9-12%]	[12-22%]
market deltas	27	4.5	1.25	0.6	0.25
model deltas	21.5	4.63	1.63	0.9	NA

  - **Figures are roughly the same**
    - Though the base copula market and the contagion model are quite different models
  - **Smaller equity tranche deltas for contagion model**
    - Base correlation sticky deltas underestimate the increase in contagion after the first defaults
  - **Recent market shifts go in favour of the contagion model**

II - Tree approach to hedging defaults

- **Comparison of credit deltas**
  - **Arnsdorf & Halperin (2007)**
  - **Credit spread deltas in a 2D Markov chain**

	[0-3%]	[3-6%]	[6-9%]	[9-12%]	[12-22%]
market deltas	26.5	4.5	1.25	0.65	0.25
model deltas	21.9	4.81	1.64	0.79	0.38

  - **Confirms previous results**
  - **Model deltas in A&H are smaller than market deltas for the equity tranche**
  - **Credit spreads deltas in A&H are quite similar to credit deltas in the 1D Markov chain**

II - Tree approach to hedging defaults

- What do we learn from this hedging approach?
  - Thanks to stringent assumptions:
    - **credit spreads driven by defaults**
    - **homogeneity**
    - **Markov property**
  - It is possible to compute a dynamic hedging strategy
    - **Based on the CDS index**
  - That fully replicates the CDO tranche payoffs
    - **Model matches market quotes of liquid tranches**
    - **Very simple implementation**
    - **Credit deltas are easy to understand**
  - Improve the computation of default hedges
    - **Since it takes into account credit contagion**
  - Credit spread dynamics needs to be improved

III - Hedging credit spread risks for large portfolios

- When dealing with the risk management of CDOs, traders
  - concentrate upon credit spread and correlation risk
  - Neglect default risk
- What about default risk ?
  - For large indices, default of one name has only a small direct effect on the aggregate loss
- Is it possible to build a framework where hedging default risk can be neglected?
- And where one could only consider the hedging of credit spread risk?
  - See paper “A Note on the risk management of CDOs”

III - Hedging credit spread risks for large portfolios

- Main and critical assumption
  - Default times follow a multivariate Cox process
    - ➢ For instance, affine intensities
    - ➢ Duffie & Garleanu, Mortensen, Feldhütter, Merrill Lynch

2. *the default times follow a multivariate Cox process:*

$$\tau_i = \inf\left\{t \in \mathbb{R}^+, U_i \geq \exp\left(-\int_0^t \lambda_{i,u} du\right)\right\}, \quad i = 1, \ldots, n \tag{2.2}$$

where $\lambda_1, \ldots, \lambda_n$ *are strictly positive,* $\mathcal{F}$ *- progressively measurable processes,* $U_1, \ldots, U_n$ *are independent random variables uniformly distributed on* $[0,1]$ *under* Q *and* $\mathcal{F}$ *and* $\sigma(U_1, \ldots, U_n)$ *are independent under* Q.

- No contagion effects

III - Hedging credit spread risks for large portfolios

- No contagion effects
  - credit spreads drive defaults but defaults do not drive credit spreads
  - For a large portfolio, default risk is perfectly diversified
  - Only remains credit spread risks: parallel & idiosyncratic
- Main result
  - With respect to dynamic hedging, default risk can be neglected
  - Only need to focus on dynamic hedging of credit spread risks
    - With CDS
  - Similar to interest rate derivatives markets

III - Hedging credit spread risks for large portfolios

- Formal setup
  - $\tau_1, \ldots, \tau_n$ default times
  - $N_i(t) = 1_{\{\tau_i \le t\}}, i = 1, \ldots, n$ default indicators
  - $H_t = \bigvee_{i=1,\ldots,n} \sigma\left(N_i(s), s \le t\right)$ natural filtration of default times
  - F_t background (credit spread filtration)
  - $G_t = H_t \vee F_t$ enlarged filtration, P historical measure
  - $l_i(t,T), i = 1, \ldots, n$ time t price of an asset paying $N_i(T)$ at time T

III - Hedging credit spread risks for large portfolios

- Sketch of the proof
- Step 1: consider some smooth shadow risky bonds
  - Only subject to credit spread risk
  - Do not jump at default times
- Projection of the risky bond prices on the credit spread filtration

Definition 3.2 *The* **default free** T **forward loss process** *associated with name* $i \in \{0, \dots, n\}$, *denoted by* $p^i(., T)$ *is such that for* $0 \leq t \leq T$:

$$p^i(t,T) \triangleq E^Q\left[p^i(T) \mid \mathcal{F}_t\right] = E^Q\left[N_i(T) \mid \mathcal{F}_t\right] = Q(\tau_i \leq T \mid \mathcal{F}_t). \tag{3.2}$$

Lemma 3.1 $p^i(t,T)$, $i = 1, \dots, n$ *are projections of the forward price processes* $l^i(t,T)$ *on* $\mathcal{F}_t$:

$$p^i(t,T) = E^Q\left[l^i(t,T) \mid \mathcal{F}_t\right], \tag{3.3}$$

for $i = 1, \dots, n$ *and* $0 \leq t \leq T$.

III - Hedging credit spread risks for large portfolios

- Step 2: Smooth the aggregate loss process
- … and thus the tranche payoffs
  - Remove default risk and only consider credit spread risk
  - Projection of aggregate loss on credit spread filtration

Definition 3.1 *We denote by $p^i(.)$, the* **default-free running loss process** *associated with name $i \in \{0, \dots, n\}$, which is such that for $0 \le t \le T$:*

$$p^i(t) \triangleq E^Q[N_i(t) \mid \mathcal{F}_t] = Q(\tau_i \le t \mid \mathcal{F}_t) = 1 - \exp\left(-\Lambda_{i,t}\right). \tag{3.1}$$

Definition 3.5 default-free aggregate running loss process *The default free aggregate running loss at time t is such that for $0 \le t \le T$:*

$$p_n(t) \triangleq \frac{1}{n} \sum_{i=1}^{n} p^i(t). \tag{3.7}$$

III - Hedging credit spread risks for large portfolios

- Step 3: compute perfect hedge ratios of the smoothed payoff
  - ➢With respect to the smoothed risky bonds
  - – Smoothed payoff and risky bonds only depend upon credit spread dynamics
  - – Both idiosyncratic and parallel credit spread risks
  - – Similar to a multivariate interest rate framework
  - – Perfect hedging in the smooth market

Assumption 2 *There exists some bounded $\mathcal{F}$ - predictable processes $\theta_1(.), \ldots, \theta_n(.)$ such that:*

$$(p_n(T) - K)^+ = E^Q\left[(p_n(T) - K)^+\right] + \frac{1}{n}\sum_{i=1}^{n}\int_0^T \theta_i(t)dp^i(t,T) + z_n, \tag{4.2}$$

where z_n is $\mathcal{F}_T$-measurable, of Q-mean zero and Q-strongly orthogonal to $p^1(.,T), \ldots, p^n(.,T)$.

III - Hedging credit spread risks for large portfolios

- Step 4: apply the hedging strategy to the <u>true</u> defaultable bonds
- **Main result**
  - Bound on the hedging error following the previous hedging strategy
  - When hedging an actual CDO tranche with actual defaultable bonds
  - Hedging error decreases with the number of names
    - ➢ Default risk diversification

Proposition 1 *Under Assumptions (1) and (2), the hedging error* ε_n *defined as:*

$$\varepsilon_n = (l_n(T) - K)^+ - E^Q\left[(l_n(T) - K)^+\right] - \frac{1}{n}\sum_{i=1}^{n}\int_0^T \theta_i(t)dl^i(t,T), \tag{4.4}$$

is such that $E^P[\| \varepsilon_n \|]$ *is bounded by:*

$$\frac{1}{\sqrt{2n}}\left(1 + \left(E^Q\left[\left(\frac{dP}{dQ}\right)^2\right]\right)^{1/2}\right) + \frac{1}{n}\left(E^Q\left[\left(\frac{dP}{dQ}\right)^2\right]\right)^{1/2}\left(\sum_{i=1}^n\left(Q(\tau_i \le T) + E^Q\left[B_i\right]_T\right)\right)^{1/2}$$
$$+E^P[\| z_n \|]. \tag{4.5}$$

III - Hedging credit spread risks for large portfolios

- Provides a hedging technique for CDO tranches
  - Known theoretical properties
  - Takes into account idiosyncratic and parallel gamma risks
  - Good theoretical properties rely on no simultaneous defaults, no contagion effects assumptions
  - Empirical work remains to be done
- Thought provocative
  - To construct a practical hedging strategy, do not forget default risk
  - Equity tranche [0,3%]
  - iTraxx or CDX first losses cannot be considered as smooth

III - Hedging credit spread risks for large portfolios

- Linking pricing and hedging ?
- The black hole in CDO modeling ?
- Standard valuation approach in derivatives markets
    - ➢Complete markets
    - ➢Price = cost of the hedging/replicating portfolio
- Mixing of dynamic hedging strategies
    - for credit spread risk
- And diversification/insurance techniques
    - For default risk

Comparing hedging approaches

- Two different models have been investigated
- Contagion homogeneous Markovian models
  - Perfect hedge of default risks
  - Easy implementation
  - Poor dynamics of credit spreads
  - No individual name effects
- Multivariate Cox processes
  - Rich dynamics of credit spreads
  - But no contagion effects
  - Thus, default risk can be diversified at the index level
  - Replication of CDO tranches is feasible by hedging only credit spread risks.

Comparison results for credit risk portfolios

- **Pricing issues with factor models**
  - **Comparison of CDO pricing models through stochastic orders**
  - **Comprehensive approach to copula, structural and multivariate Poisson models**
  - **Relevance of the conditional default probabilities**
    - Drive the tranche pricing
  - **For simplicity, we further restrict to homogeneous portfolios**
  - **We provide a general comparison of pricing models methodology**
  - **By looking for the distribution of conditional default probabilities**

Contents

Exchangeability assumption

- n defaultable firms
- $\tau_1, \ldots, \tau_n$ default times
- $(D_1, \ldots, D_n) = (1_{\{\tau_1 \le t\}}, \ldots, 1_{\{\tau_n \le t\}})$ default indicators
- Homogeneity assumption: default dates are assumed to be exchangeable

Definition (Exchangeability)

A random vector $(\tau_1, \ldots, \tau_n)$ is exchangeable if its distribution function is invariant by permutation: $\forall \sigma \in S_n$

$$(\tau_1, \ldots, \tau_n) \stackrel{d}{=} (\tau_{\sigma(1)}, \ldots, \tau_{\sigma(n)})$$

- Same marginals

De Finetti Theorem and Factor representation

- Suppose that $D_1, \dots, D_n, \dots$ is an exchangeable sequence of Bernoulli random variables
- There exists a random factor $\tilde{p}$ such that
- $D_1, \dots, D_n$ are independent knowing $\tilde{p}$
- Denote by $F_{\tilde{p}}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \dots, D_n = d_n) = \int_0^1 p^{\sum_i d_i}(1-p)^{n-\sum_i d_i} F_{\tilde{p}}(dp)$$

- $\tilde{p}$ is characterized by:

$$\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p} \quad \text{as} \quad n \to \infty$$

Stochastic orders

- $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions f
- $X \leq_{sl} Y$ if $E[(X-K)^+] \leq E[(Y-K)^+]$ for all $K \in \boldsymbol{R}$
  - $X \leq_{sl} Y$ and $E[X] = E[Y] \Leftrightarrow X \leq_{cx} Y$
- $X \leq_{sm} Y$ if $E[f(X)] \leq E[f(Y)]$ for all supermodular functions f

Definition (Supermodular function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **supermodular** if for all $x \in \boldsymbol{R}^n$, $1 \leq i < j \leq n$ and $\varepsilon, \delta > 0$ holds

$$f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n)$$

$$\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$$

- consequences of new defaults are always worse when other defaults have already occurred

Stochastic orders

- $(D_1, \dots, D_n)$ and $(D_1^* \dots, D_n^*)$ two exchangeable default indicator vectors
- M_i loss given default
- Aggregate losses:

$$L_t = \sum_{i=1}^{n} M_i D_i$$

$$L_t^* = \sum_{i=1}^{n} M_i D_i^*$$

Müller(1997)
Stop-loss order for portfolios of dependent risks.

$$(D_1, \dots, D_n) \leq_{sm} (D_1^* \dots, D_n^*) \Rightarrow L_t \leq_{sl} L_t^*$$

Stochastic orders

Theorem

Let $\mathbf{D} = (D_1, \ldots, D_n)$ and $\mathbf{D}^* = (D_1^*, \ldots, D_n^*)$ be two exchangeable Bernoulli random vectors with (resp.) F and F^* as mixture distributions. Then:

$$F \leq_{cx} F^* \quad \Rightarrow \quad \mathbf{D} \leq_{sm} \mathbf{D}^* \quad \text{and}$$

Theorem

Let $D_1, \ldots, D_n, \ldots$ and $D_1^*, \ldots, D_n^*, \ldots$ be two exchangeable sequences of Bernoulli random variables. We denote by F (resp. F^*) the distribution function associated with the mixing measure. Then,

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow F \leq_{cx} F^*.$$

Multivariate Poisson model

Duffie(1998), Lindskog and McNeil(2003), Elouerkhaoui(2006)

- $\bar{N}_t^i$ Poisson with parameter $\bar{\lambda}$: idiosyncratic risk
- N_t Poisson with parameter λ: systematic risk
- $(B_j^i)_{i,j}$ Bernoulli random variable with parameter p
- All sources of risk are independent
- $N_t^i = \bar{N}_t^i + \sum_{j=1}^{N_t} B_j^i, \; i = 1 \ldots n$
- $\tau_i = \inf\{t > 0 | N_t^i > 0\}, \; i = 1 \ldots n$

Multivariate Poisson model

- $\tau_i \sim Exp(\bar{\lambda} + p\lambda)$
- $D_i = 1_{\{\tau_i \leq t\}}, \; i = 1 \dots n$ are independent knowing N_t
- $\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)$
- Conditional default probability:

$$\tilde{p} = 1 - (1 - p)^{N_t} \exp(-\bar{\lambda} t)$$

Multivariate Poisson model

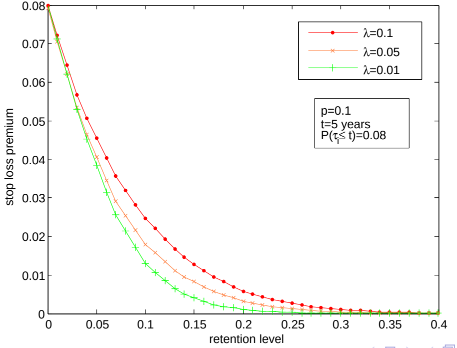
- Comparison of two multivariate Poisson models with parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$
- Supermodular order comparison requires equality of marginals: $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda^*$
- Comparison directions:
  - $p = p^*$: $\bar{\lambda}$ v.s λ
  - $\lambda = \lambda^*$: $\bar{\lambda}$ v.s p

Multivariate Poisson model

Theorem ($p = p^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*$, then:

$$\lambda \leq \lambda^*,\ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

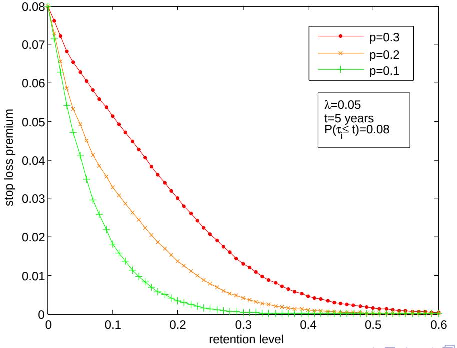


Multivariate Poisson model

Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, then:

$$p \leq p^*, \ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$



Structural Model

Hull, Predescu and White(2005)

- Consider n firms
- Let $X_t^i,\ i = 1 \ldots n$ be their asset dynamics

$$X_t^i = \rho W_t + \sqrt{1-\rho^2} W_t^i, \quad i = 1 \ldots n$$

- $W,\ W^i,\ i = 1 \ldots n$ are independent standard Wiener processes
- Default times as first passage times:

$$\tau_i = \inf\{t \in \boldsymbol{R}^+ | X_t^i \leq f(t)\}, \quad i = 1 \ldots n,\ f : \boldsymbol{R} \to \boldsymbol{R} \text{ continuous}$$

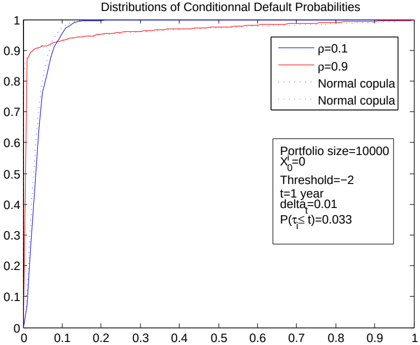
- $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \ldots n$ are independent knowing $\sigma(W_t,\ t \in [0, T])$
- $\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p}$

Structural Model

Theorem

For any fixed time horizon T, denote by $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \dots n$ and $D_i^* = 1_{\{\tau_i^* \leq T\}}$, $i = 1 \dots n$ the default indicators corresponding to (resp.) ρ and ρ^*, then:

$$\rho \leq \rho^* \Rightarrow (D_1, \dots, D_n) \leq_{sm} (D_1^*, \dots, D_n^*)$$



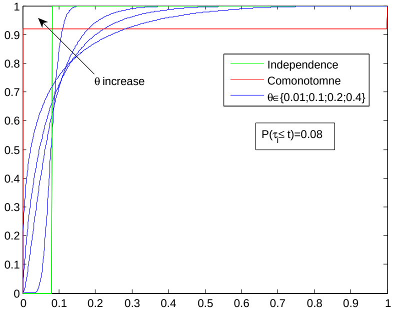
- $\tilde{p}(\rho) \leq_{cx} \tilde{p}(\rho^*)$

Archimedean copula

Copula name	Generator φ	V-distribution
Clayton	$t^{-\theta}-1$	Gamma$(1/\theta)$
Gumbel	$(-\ln(t))^{\theta}$	α-Stable, $\alpha=1/\theta$
Franck	$-\ln\left[(1-e^{-\theta t})/(1-e^{-\theta})\right]$	Logarithmic series

Theorem

$\alpha \leq \alpha^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1,\ldots,D_n) \leq_{sm} (D_1^*,\ldots,D_n^*)$



- $\tilde{p}(\theta) \leq_{cx} \tilde{p}(\theta^*)$

Additive copula framework

- $V_i = \rho V + \sqrt{1-\rho^2}\,\bar{V}_i$
- $V, V_i\ i = 1 \dots n$ independent
- Laws of $V, V_i\ i = 1 \dots n$ do not depend on the dependence parameter ρ
- Standard copula models:
  - Gaussian, Student t
  - Double t: Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2005)
  - Double Variance Gamma: Moosbrucker(2005)

Theorem

$$\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \dots, D_n) \leq_{sm} (D_1^*, \dots, D_n^*)$$

Conclusion

- Characterization of supermodular order for exchangeable Bernoulli random vectors
- Comparison of CDO tranche premiums in several pricing models
- Unified way of presenting default risk models