An overview of the valuation of collateralized derivative contracts

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Abstract

We consider the valuation of collateralized derivative contracts such as interest rate swaps, forward FX contracts or term repos. First, we provide a precise framework regarding collateralization, under which computations are made easy. We allow for posting securities or cash in different currencies. In the latter case, we focus on using overnight index rates on the interbank market, in line with LCH.Clearnet framework. We provide an intuitive way to derive the basic discounting results, keeping in line with the most standard theoretical and market views. Under perfect collateralization, pricing rules for collateralized trades remain linear, thus the use of (multiple) discount curves. We then show how to deal with partial collateralization, involving haircuts, asymmetric CSA, counterparty risk and funding costs as an extension of the perfect collateralization case. We therefore intend to provide a unified view. Mathematical or legal details are not dealt with and we privilege financial intuition and easy to grasp concepts and tools.

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Introduction

The pricing of collateralized OTC contracts, such as swaps ruled by numerous CSA involved in ISDA master agreements or cleared on LCH.Clearnet, ICE, CME, etc. has become a subject of tremendous importance. It concerns the pricing of new trades, of novation trades and the backload of older trades to a clearinghouse. In the latter cases, the changes in collateral cash-flows may have a positive or adverse impact on the P&L. Also, consistency between the pricing of collateralized and uncollateralized trades is likely to be a topic for a while, thus we need a unified framework for the computation of CVA and collateralized trades[4].

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[4] See Cesari et al. (2010), Gregory (2012) or Ernst & Young (2012) among many references regarding counterparty risk of uncollateralized derivative contracts. The scope instruments and entities concerned by central clearing may vary according to regulators, such as CFTC, ESMA, EBA, EIOPA,

Whatever the contractual or legal framework, it is rather clear, as the settlement of Lehman trades or the paper "ISDA Valuation Cases" by Macey-Dare (2010) have proven, that there is some uncertainty about the cash-flows to be priced. This is a rather uncomfortable situation for mathematical finance as pointed out recently by Brigo and Morini (2010) or Brigo et al. (2011)[5]. Collateral disputes may not only result from the bootstrapping of yield curves, smoothing and second order calendar effects. They reveal divergences about computation methodologies for settlement prices and variations margins. For instance, using a OIS or Libor swap curve, a collateralized or uncollateralized one is not innocuous.

Apart from descriptive documentation issued for instance by clearing houses and some professional presentations (see Ireland (2008), Morini (2008)), there are a number of academic papers that deal with the valuation of collateralized trades under the new financial architecture. Johannes and Sundaresan (2007) can be seen as pioneering while the papers by Fuji et al. provide a deep investigation of many pricing issues. Bianchetti (2010), Bielecki et al. (2011), Brigo et al. (2011), Crepey (2011), Kan and Pedersen (2011), Kenyon (2010), Piterbarg (2010, 2011) also deal with various aspects of the valuation of collateralized trades.

The aim of this paper is to provide an intuitive approach to the main results routinely used, an understanding of the (strong) assumptions involved and, on the other hand, to emphasize the wide range of applications. We also show a number of new results in connection with the recursive approach introduced by Duffie and Huang (1996) and Duffie *et al.* (1996). This field of research is still in progress, a number of issues such as the impact of funding costs, optimal posting of collateral, gap (or slippage) risks and the impact of initial margin may not be matter of consensus.

The paper is organized as follows. Section 1 presents the pricing framework. This includes modeling of savings and collateral accounts, whether different currencies or securities are posted, variation margins as inflows and outflows on collateral accounts, settlement prices and collateralization schemes. Section 2 provides an intuitive approach to the basic pricing and discounting result for collateralized products in the simple framework of perfect collateralization. This is specialized to OIS discounting, Futures' pricing, costless collateral, when posting securities or currencies. For these two latter cases, it is shown that one can reasonably rely upon observable quantities such as repo rapes or interest rate differentials. Eventually, the optimal choice of posted securities is investigated. Section 3 provides a general approach to multiple discount curves in the context of swaps or of forward contracts. Suitable changes of measure are introduced leading to simple pricing formulas and generic convexity adjustments. Standard examples, involving easy to bootstrap collateralized discount factors, are investigated. It is noticeable that, in the above setting, we remain in a linear pricing framework, thus the existence of discount factors. Section 4 provides the

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Joint Committee of the European Supervisory Authorities (2012) or BIS (see Heller and Vause (2012)), through time and obviously given the willing of the parties involved in the trades. We refer to Finger (2012), LCH.Clearnet (2012), Pirrong (2011) for an overview of the issues at hand. Regarding central clearing of repos, we refer to Penney (2011) and technical documentation issued by clearing houses.

[5] One could also look at the short note posted by Tracy Alloway on ftalphaville on January 2011, "When (derivatives) counterparties collapses" which shows the kind of legal uncertainty regarding the claims in case of default.

present value of cash-flows associated with managing collateral accounts and a margining rule in line with banking books approaches. Section 5 broadens the view by considering imperfect collateralization schemes. We firstly deal with of dynamic haircuts, which may be of interest for the pricing of long-term repos or credit linked notes. We show that this can easily be adapted to cope with the case of asymmetric CSA. In such a non linear pricing framework, BSDE tools are quite useful. We provide some results that connect the stochastic discount factor to marginal pricing at book level and easy to compute trade contributions, some topics of great practical importance. Eventually, the same techniques can be applied when introducing counterparty risk and costs of funding. Though the CVA approach used here is not identical to market practice, it can be thought of as a first-order approximation. Above all, it provides an integrated and easy to grasp pricing framework.

1. Pricing framework

1.1. Default free short-term rates and corresponding savings accounts.

$r(t)$ will denote the usual default-free short rate at time t. This is taken as a starting point to keep in line with the standard mathematical finance framework and thus simplifying the exposition. We do not claim that the short-term default-free rate is actually associated with "real trades" or that it is an observable quantity. This is actually rather fortunate since credit and counterparty risk permeates almost all traded contracts.

Overnight "rates" such as Eonia, Sonia or the effective Fed funds rates are related to unsecured interbank loans and should not be confused with the default-free rate. There is a default component in such overnight rates (see Afonso et al. (2011)). Apart from periods of market stress, this default component is likely to be small, though difficult to assess, given that $r(t)$ is not directly observable. Fortunately enough, in a number of important cases, such as interest rate swaps cleared at LCH.Clearnet, the settlement prices and present values will only depend upon observable quantities such as rates of collateralized swaps.

The value of the default-free savings account numéraire will be denoted by $\beta(t)=\exp\int_0^t r(s)ds$ and the corresponding risk-neutral probability by Q^{β} [6]. Whenever currency needs to be specified, we will use notations such as $\beta^{\$},\beta^{€},\beta^{¥},\beta^{£}$ or $r^{\$},r^{€},r^{¥},r^{£}$ for the "G4 currencies". For simplicity, we will consider the US dollar as the base currency (and remove any superscript unless necessary) and denote by $FX^{€/\$}(t)$ the spot exchange rate, i.e. the USD price at time t of 1 EUR. As a consequence, the cash-numéraire account in euros, when expressed in USD is equal to $FX^{€/\$}(t)\times\exp\left(\int_0^t r^{€}(s)ds\right)$.

1.2. Cash in different currencies and securities posted as collateral.

[6] To ease the exposition, Q^{β} is assumed to be given unambiguously. We therefore chose not to discuss the switch from historical to risk-neutral probability. Similarly, we assume that a "default-free" short rate is given, even though it is dealt with difficulty and eventually does not appear in the simplest valuation formulas. We refer to Piterbarg (2011) for an alternative route.

To cope with various netting agreements, we need to enlarge the cash collateral setting. Let us denote by $A(t)$, the price at time t of the posted asset, which may be a bond or some currency. For simplicity, we will assume that $A(.)$ is a positive semimartingale admitting the decomposition $\frac{dA(t)}{A(t\_)} = r_A(t)dt + dM_A(t)$, where M_A is a Q^β - martingale[7]. We assume that the price can actually be observed and that the market for the security A is frictionless, i.e., no bid-ask spread and constant unit price.

Let us consider the case of cash collateral where $c(.)$ denotes the rate (with continuous compounding) paid on the collateral account. In the case of LCH.Clearnet, for an interest rate swap denominated in USD, the cash-collateral account is denominated in USD and earns the effective fed funds rate $c(.) \equiv c^{\$}(.)$ [8].

We define a cash savings account that earns the collateral rate:

$$C(t) = \exp\left(\int_0^t c(s)ds\right).$$

Since a central counterparty is a financial institution, it collects deposits, especially initial margins, and manages various accounts as would do an ordinary bank. As argued in the footnote below, we will neglect the counterparty risk associated with a clearinghouse[9]. Even

[7] The price process may not be continuous, which may be of practical importance, for instance if the posted securities are defaultable. However, in the examples below, we do not account for such a feature and for any kind of gap risk. Therefore, one can think of using a Brownian filtration and Brownian martingales.

[8] This rule applies to the variation margin account and does not extend to the initial margin, where securities are posted. The effective fed funds rate is a member of the family of indices based on overnight unsecured rates on interbank markets. Afonso et al. (2011) report large discrepancies amongst overnight rates paid by borrowers in the Fed funds market. Let us remark that such effects should not occur with predictable default times, as would be the case with simple structural models, in which credit spreads are equal to zero for very short term loans. Even if there remains some amount of credit risk when lending to, say, LCH.Clearnet (given first the safety nets of the clearinghouse and then implicit public support) this is likely to be smaller than the credit risk associated to the effective fed funds rate. However, one cannot construct an arbitrage since the collateral account is tied-up with a trade and one cannot freely go to, say LCH.Clearnet and claim to lend at Eonia.

[9] One should consider the various risks and the way there are dealt with by clearing houses. Regarding variation margins, the interest and any cash-flow involved in the management of variation margin accounts is usually simply passed-through from the collateral accounts of the involved parties. This means that since all variation margin accounts cancel out, there is no interest rate risk at the clearing house level, whatever the chosen rate on variation margin accounts. Regarding the initial margin, the previous line of reasoning does not apply. Initial margin is computed at the portfolio level and is non-negative for all the parties that clear their trades through the central counterparty. When the posted collateral may consist in securities, the secure policy consists in simply holding these

though ICE and LCH.Clearnet tend to be prominent Central Counterparties regarding CDS and interest rate swaps respectively, we might think of introducing different CCP's, with different collateral rates. Such an issue also arises when deciding to backload an OTC interest rate swap, run under an ISDA master agreement with a specific CSA, to a clearing house. As mentioned above for novation trades, this may negatively impact one of the two parties.

Dealing with the rate earned on the collateral account also raises the issue of so-called asymmetric CSA, where for instance only one party will be required to post collateral. This is further discussed in greater detail.

1.3. Variation margins paid on a collateral account.

In an ISDA CSA, variation margins are exchanged between the two parties involved in the CSA. In a LCH.Clearnet trade, or more generally when a central counterparty is involved, variation margins are exchanged between the CCP and, say a clearing member.

The computation of variation margins involves a collateral account. We will denote by $V(s)$ the amount at date s of the collateral account. The amount of the collateral account is the market value of the securities being posted as collateral. The collateral account may be managed by the central counterparty. Cash or posted securities might be or not withdrawn (rehypothecation) from the collateral account, this is not of importance, as far as we are only concerned with the pricing of the collateralized trade and not by liquidity and counterparty risk management of the entities involved in the trade. The collateral account does not need either to be segregated from other accounts, provided that posted securities are actually available in case of default.

At time $s, t \leq s \leq T$, the number of securities posted on the collateral account is equal to $\frac{V(s)}{A(s)}$. The value of the posted collateral at $s+ds$ is then equal to $\frac{V(s)}{A(s)} \times A(s+ds)$.[10]

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securities. Nevertheless, due to operational or gap risks, for instance, clearing houses cannot be fully compared with narrow banks (see Flannery (1994) or Freixas and Rochet (2008)). ICE statements regarding investment and risk management policies are quite explicit about this issue. Given their corporate governance and that clearing house act as "G20 – sponsored entities" and are purposely too interconnected to fail, it is however fair to neglect the probability of default of central counterparties, even though there might be some capital charges regarding the credit risk of cash or securities stored within a clearing house. See EBA (2012) or Basel Committee on Banking Supervision (2012) for further discussion about capital charges for bank exposures to central counterparties.

[10] $\frac{V(s)}{A(s)} \times A(s+ds)$ can be seen as the value of the collateral account, prior to the variation margin payment.

As a consequence, the variation margin for the time interval between s and $s+ds$ is equal to:

$$dVM(s) = V(s+ds) - \frac{V(s)}{A(s)} \times A\left(s+ds\right).$$

We can rewrite the variation margin as:

$$dVM(s) = dV(s) - V(s)\frac{dA(s)}{A(s)}$$

In the case of cash collateral with rate $c\left(.\right)$, $A \equiv C$, $\frac{A(s+ds)}{A(s)} = 1 + c(s)ds$ and the variation margin equals $dVM(s) = dV(s) - V(s)c(s)ds$.

Since $dV(s) = V(s) \times \frac{dA(s)}{A(s)} + dVM(s)$, the term $V(s) \times \frac{dA(s)}{A(s)}$ corresponds to the self-financed part. The variation margin $dVM(s)$ corresponds to the inflow in the collateral account. In the case of a cash account, $dV(s) = c(s)V(s)ds + dVM(s)$.

1.4. Settlement prices for collateralized trades.

We will consider a trade with single maturity T. This readily extends to multiple payment dates as in the case of interest rate swaps. The non-linear case, say with an asymmetric CSA, is discussed further.

The contractual payoff at date T paid in the base currency, say USD, will be denoted by $h(T)$. As stated before, we do not take into account immediately the possibility of default. Apart from interest rate swaps, we could think of forward FX contracts, securities lending and repo transactions[11] as other examples involved in collateralized transactions.

$h(t)$ will denote the settlement price at time t of a collateralized trade with maturity T (in the same base currency) and our purpose is the computation of such a settlement price.

$h(t)$ will play a key role in margining, either for the computation of variation margins or regarding the collateral account amount.

The settlement price will converge at time T to the contractual cash-flow, thus the same notation. For simplicity, we implicitly assume that the contract is cash-settled.

[11] In these examples, $h(T)$ corresponds to the net payment at T. In the case of the repo contract, we will then consider the difference between the market value of the delivered bond at T and the actual cash payment at T.

$h(t)$ can be either positive or negative. Since we want to account for deals that are not traded at par or for the backload of existing collateralized OTC trades to a clearinghouse, we need to deal with the case where $h(t) \neq 0$. Keeping in mind the simple case of an interest rate swap, the fixed rate of an existing trade may not be necessarily at par. Thus an initial payment has to be made, which accounts for past variation margins unpaid to the central counterparty [12].

Before going any further, we need to detail the various cash-flows at hand.

Let us first contrast with the case of a default-free uncollateralized trade, with same terminal cash-flow $h(T)$ as the considered collateralized trade. The time t price of such an uncollateralized contract, denoted by $g(t)$, is provided by: $g(t) = E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T r(s)ds\right)\right]$.

There are only two cash-flows exchanged between the two parties to the contract, $-g(t)$ at time t and $g(T) = h(T)$ at time T [13]. The basic pricing rule is that the sum of the net expected[14] discounted cash-flows, including cash-flow at inception of the trade, equals zero: $E_t^{Q^\beta}\left[-g(t) + g(T)\exp\left(-\int_t^T r(s)ds\right)\right] = 0$.

Now, let us turn back to the same payoff at maturity, $h(T)$, within a collateralized framework. The net cash-flow at inception date t is equal to $-h(t)+V(t)$, where $h(t)$ is the settlement or collateralized price and $V(t)$ is the value of the posted collateral. The net cash-flow at expiration of the contract is $h(T)-V(T)$. The pricing equation differs since we need to take into account the collateral cash-flows, most notably the variation margins paid between t and T.

This can be written as:

$$E_t^{Q^\beta}\left[-h(t) + h(T)e^{-\int_t^T r(s)ds} + V(t) + \int_t^T e^{-\int_t^s r(u)du} dVM(s) - V(T)e^{-\int_t^T r(s)ds}\right] = 0,$$

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[12] If the deal is ruled by an ISDA master agreement and a specific CSA, it may be that the theoretical settlement price $h(t)$ differs from the value of posted collateral.

[13] From the other's party point of view, the cash-flows will be $g(t)$ at time t and $-g(T) = -h(T)$ at time T. This does not make any difference regarding the computation of $g(t)$. Let us remark that since at this stage contracts are default-free, we do not need to distinguish between the two entities involved in the contract to be priced.

[14] Under Q^β.

Where $h(t)$ and $h(T)$ are the cash-flows directly tied to the collateralized trade[15], $V(s)$, $t \le s \le T$ is the value of the collateral account at time s and $dVM(s)$ is the variation margin. Let us remark that that $V(.)$ is not self-financed since there are some inflows or outflows, i.e. the variation margins. The collateral account can involve cash account(s) managed by a third party, as a central bank or (default-free) central counterparty. For simplicity, we allow for negative values, i.e. $V(s)$ is the net value of collateral held by one party[16]. If collateral consists in securities, $V(s)$ is the market value of the securities held. We subsequently assume the market of posted securities to be frictionless[17].

$E_t^{Q^\beta}\left[V(t)+\int_t^T \exp\left(-\int_t^s r(u)du\right)dVM(s)-V(T)\exp\left(-\int_t^T r(s)ds\right)\right]$ is the time t value of the collateral cash-flows. Whenever, this differs from zero, $h(t)$ will differ from $g(t)$, the value of a default-free uncollateralized trade with same terminal payoff $h(T)$. For simplicity, we have left aside the initial margin cash-flows. There are not tied to a specific contract with payoff $h(T)$ but to the portfolio involved in the CSA or cleared with a given central counterparty. If such a portfolio is frozen or managed on a run-off basis, we could envisage the modeling of future initial margins. Recent evidence show that rules governing initial margins paid to clearing houses may change from time to time. This leads to some uncertainty in the involved cash-flows. More importantly, when a portfolio is actively managed, new trades will be involved in the initial margins computations. At inception, it is then difficult to isolate the contribution of a portfolio to the stream of initial margin cash-flows.

1.5. Collateralization schemes.

A collateralization scheme relates settlement prices $h(t)$ and the value of the collateral account $V(t)$. For instance, in the so-called perfect collateralization case, at any point in time, the value of the collateral is equal to the settlement price[18]: $V(s)=h(s)$, for all $s, t \le s \le T$. For an asymmetric CSA, we would have $V(s)=\max\left(h(s),0\right)$ or $V(s)=\min\left(h(s),0\right)$.

[15] In an uncollateralized world, the inception and termination payments are $g(t)$ and $g(T)$.

[16] Hence, the net value from the other party is $-V(s)$.

[17] We do not address here the rehypothecation issue, one could either think of cash or securities actually held in the collateral account (no rehypothecation) or consider $V(s)$ as the value of a notional account.

[18] Thus, there are no extra-collateralization features, such as haircuts or margin ratios, initial margin or independent amount and no under-collateralization (thresholds, minimum transfer amounts, …). For instance, the initial payment $V(t)$ corresponds to the market value (or settlement price). We do not consider either risks related to unwinding the trades, known as “slippage effects” and leading to extra counterparty risk.

With perfect collateralization, the variation margins paid throughout the life of the trade perfectly match the potential loss in case of default of one of the parties. This holds whatever we precisely mean by "loss in case of default": replacement cost, market quotation, loss method, replacement cost including CVA, as agreed between the parties or by court ruling. Basically, we will assume that in case of a counterparty defaulting, there will be no close-out amount (i.e. no overcollateralization issues) or legal uncertainty relative to the closing of the deal and the availability of the collateral. We do not either enter in the details of credit events, dealing with restructuring or other early termination clauses. This depends upon the chosen master agreement, the CSA or rules that apply to a specific clearinghouse[19] and is likely to change over time. Similarly, we will assume that there is no gap risk when unwinding the trade in case of early termination[20]. Thus, there will be no independent amount or initial margin or initial deposit in our setup. We are aware of the roughness of such assumptions, especially given the liquidity issues related to downgrading triggers, initial margin multipliers and eligible collateral for such initial margin[21]. Going on with the simplifying assumption of perfect collateralization, no haircuts related to eligible collateral will be assumed in the perfect collateralization scheme. This assumption will be further relaxed. We will assume that, whatever the collateral posted, there is no ambiguity about the price, i.e. the two parties agree about the value of the transferred collateral, which apparently is a minor assumption, compared with collateral disputes related to the pricing of a trade[22].

Regarding the posting of securities, instead of cash, perfect collateralization means that there are no haircuts and no difficulty in assessing the market value of the posted securities. Introducing some haircuts can result in over collateralization, thus one of the parties is subject to counterparty risk.

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[19] This remark might sound a bit self-provocative, but we could see a large and well capitalized derivatives house, with sound risk management policies as a central counterparty. Some of them would be more efficient in the aftermath of the crisis, given the increased concentration that will lead to better netting and given their expertise regarding the pricing of non standardized, custom made swaps. The management of tri-party repos and the leading role played by The Bank of New York Mellon and JP Morgan clearly shows that large banks could actually be considered as Central Counterparties.

[20] The issue may be dealt with differently when dealing with interest rate or currency swaps on one hand and CDS on the other hand. Gap risk is much larger in the case of CDS and is actually a challenge when dealing with large individual exposures.

[21] The likely increased fragmentation of the market following new regulations on clearing of OTC trades might also contribute to magnify liquidity buffers and the wholesale transaction demand for money.

[22] Modeling operational risks associated with the transfer of collateral, segregated or omnibus accounts, connections between clearing members and the clearinghouse is beyond the scope of this paper.

2. Pricing results with perfect collateralization.

2.1. The basic discounting result with perfect collateralization.

Let us consider the case where the deals stops at time τ, $\tau \leq T$. τ may or not be associated with a credit event of one of the two parties, in the former case, we do not consider any "gap risk" or possible discontinuous effects at this stage. In a futures market, τ would simply be the time at which a party wishes to exit the trade[23]. Then the exiting party receives the settlement price $h(\tau)$, gives back the collateral amount $V(\tau)$ and no further cash-flows are exchanged. Let us remark that we assume perfect liquidity in the collateralized market, i.e. the parties can actually exit the trade at settlement price $h(\tau)$ and there is no settlement risk either; the collateral assets can be cash-settled without risk or liquidity costs[24]. Cancellation of the deal before contractual maturity depends upon the context. It may no be feasible formally within an ISDA Master Agreement[25], while it would be easier on a Future's market or when a central counterparty is involved. The pricing equation of the contractual maturity T can be replaced by τ which leads to:

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[23] As mentioned above, we do not consider slippage effects when exiting the trade. An obvious but important feature of perfect collateralization pricing is that the default quality of the contracting parties is irrelevant for the computation of the settlement price. Thus, all possible contracting parties agree about the same settlement price. The settlement price can then be considered as an unambiguous market price, depending only upon h and collateral management features. As an example, we can compute the settlement price of a collateralized trade assuming that the two parties are default-free. Of course, on economic grounds, these two parties would not need a collateralized trade. Nevertheless, this shows that, even in a default-free context, the settlement price of a collateralized trade can be different from the price of an uncollateralized trade with same terminal payoff.

[24] Assuming that the trade can be terminated at settlement price $h(\tau)$ is related to market liquidity in the collateralized market. This may or may not be a reasonable assumption depending on the size of the trades, contract's category, features of the collateralized markets. One could consider a number of future's exchanges and interest rate swaps on major currencies collateralized through LCH.Clearnet to fulfil these liquidity assumptions. On the other hand, OTC exotic trades collateralized through ISDA CSA are likely to be illiquid. This point is quite important with respect to the funding cost issue. Basically, it shows that funding costs of the parties are irrelevant to the computation of the settlement price whenever early termination of the collateralized trade can be achieved at a zero cost. For instance, we could think of τ being the first date at which one of the parties would have some difficulties in funding the variation margins. The same reasoning applies to initial margins or independent amounts, say related to downgrade triggers. Of course, early termination of trades, say related to a downgrade trigger, may entail indirect costs, such as breaking down hedges, increasing overall market exposures and the need for regulatory capital.

[25] One could then think of entering a reciprocal transaction with some other counterparty, neglecting "imperfections" such as minimum amounts, independent amounts that would add-up and the issue of re-hypothecating posted securities.

$$E_t^{Q^\beta}\left[-h(t)+h(\tau)e^{-\int_t^\tau r(s)ds}+V(t)+\int_t^\tau e^{-\int_t^s r(u)du}dVM(s)-V(\tau)e^{-\int_t^\tau r(s)ds}\right]=0.$$

Using that $h(t)=V(t)$ and $h(\tau)=V(\tau)$, we end up with the equation:

$$E_t^{Q^\beta}\left[\int_t^\tau \exp\left(-\int_t^s r(u)du\right)dVM(s)\right]=0,$$

which means that the expected discounted value of the variation margins paid while the position is held equals zero. Informally, let us take $\tau=t+dt$ and neglect second order terms, we then have: $E_t^{Q^\beta}\left[dVM(t)\right]=0$. This equation has quite an intuitive meaning since when entering a collateralized deal at time t and exiting this deal at time $t+dt$, one has only to pay the variation margin $dVM(t)$ at $t+dt$. Since $dVM(t)=dh(t)-h(t)\frac{dA(t)}{A(t)}$, $E_t^{Q^\beta}\left[dh(t)\right]-h(t)E_t^{Q^\beta}\left[\frac{dA(t)}{A(t)}\right]=0$. Thus, $E_t^{Q^\beta}\left[dh(t)\right]=r_A(t)h(t)dt$. This leads to the basic discounting equation:

$$h(t)=E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T r_A(s)ds\right)\right]$$ [26].

The settlement price is obtained by discounting the terminal payoff at the risk-neutral expected rate of return of the collateral. This contrasts with using the default-free rate for uncollateralized trades. Let us remark that the funding costs of the entities involved in exchanging cash-flows are (hopefully) not involved in the computation of the settlement price of a perfectly collateralized traded. This will no longer hold for CSA with thresholds or partial collateralization schemes, as discussed below.

2.2. OIS discounting.

A simple case corresponds to the use of USD cash collateral. We then have: $\frac{dA(t)}{A(t)}=\frac{dC^\$(t)}{C^\$(t)}=c^\$(t)dt$, where $c^\$(t)$ is the effective fed funds rate and the USD settlement price is provided by $h(t)=E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T c^\$(s)ds\right)\right]$, i.e. the so-called "OIS discounting".

$h(T)=1$ corresponds to the commitment to pay one unit of base currency at date T. For simplicity cash collateral in the base currency is being used. Let us consider the case where the collateral rate $c(.)$ is deterministic. To guarantee the above payment, a cash amount $h(t)=\exp\left(-\int_t^T c(s)ds\right)$ has to be posted at date t on the (default-free) collateral account.

[26] A more rigorous proof, going into the lines of the papers by Fujii, Shimada and Takahashi, is provided in Appendix A.

This will compound to 1 at date T, guaranteeing the terminal payment even if the party having posted the cash collateral has defaulted meanwhile. Let us remark that there are no variation margins to be paid except at inception of the collateralized trade.

2.3. Futures' pricing.

The previous pricing equation also extends already known results for futures contracts (see Duffie (1989)). Actually, the pricing of futures contracts corresponds to cash collateral and collateral rate set to zero: $c(.) \equiv 0$ [27]. The settlement price is then a Q^{β} - martingale.

2.4. Costless collateral and uncollateralized prices.

Let us remark that when $r_A = r$ and $A(t) = \beta(t) = \exp\left(\int_0^t r(s)ds\right)$, there is no difference with the ordinary pricing equation for default-free uncollateralized trades: In both cases is done at the short-term default-free rate $h(t) = E_t^{Q^{\beta}}\left[h(T)\exp\left(-\int_t^T r(s)ds\right)\right]$. This corresponds to "costless collateral" according to Johannes and Sundaresan (2007) terminology[28]. Thus, when the collateral rate equals the short-rate, uncollateralized prices are equal to (perfectly) collateralized prices,.

2.5. Posting securities.

Another example corresponds to $h(T) = A(T)$ and security A being used as collateral. To guarantee the delivery of unit of security A at date T, one simply needs to post one unit of security A at date t in the collateral account and stay still until maturity date[29]. As a consequence, in such a case, $h(t) = A(t)$.

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[27] It may be worth comparing this rule, used for long in futures markets, and the no interest rate policy applied to compulsory central bank reserves (seigniorage effect). Nowadays, the Fed and the ECB pay some interest on such reserves. For instance, reserve holdings at the ECB are remunerated at the Eurosystem's rate on its main refinancing operations. It is not surprising therefore that clearing houses such as LCH.Clearnet also pay some interest related to money markets on the collateral accounts.

[28] Let us remark that in the representation of the dynamics of the security posted as collateral, we assumed that the bounded variation part was smooth. Thus, the existence of a drift term r_A. This assumption can be relaxed leading to slightly more general expressions.

[29] If the collateral account is managed by a clearing house, we keep on using the outstanding assumption of such clearing house or custodian bank to be default-free. Then, rehypothecation of the posted security is not an issue. No variation margin needs to be paid except at inception of collateralized trade. If the party that has posted the security defaults before maturity date, the security being held safely in a segregated account will still be delivered timely.

It should be noted that the settlement price should refer to the posted collateral. For instance in the above case, we should write $h_{C^\$}(t)$ and more generally $h_A(t)$. Thus, we have as many settlement prices, given a terminal payoff, as posted securities (or portfolios of posted securities).

2.6. Posting bonds.

In the repo market, a large set of securities can be posted, quite often bonds. Under Q^β, the expected rate of return of a bond should be equal to the risk-free rate r unless the bond is special[30]. This is rather inconvenient since r is not observed. In bond markets, a common practice is to consider repo rates.

We show below that "the" short-term repo rate can be under some assumptions to be detailed and discussed be seen as the expected rate of return of the bond under Q^β. This is a key point in subsequent analysis related to optimal choice of collateral and discounting results. The core point is that a short-term repo contract with no haircut can also be seen as an "instantaneous" forward bond contract and the repo rate is then related to a default-free forward bond price.

Let us consider a short-term repo contract, say between dates t and $t+dt$. As above $A(t)$ denotes the bond price at date t and no coupon is to be paid between t and $t+dt$. Let us assume that no haircut is applicable[31]. Thus, the repo contract turns out to be a (short-term) forward contract, with no payment at inception and a payment equal to $A(t+dt)-\left(1+\text{repo}_A(t)dt\right)A(t)$. If the payment is made for sure, i.e. none of the parties is in default at date $t+dt$, we have $E_t^{Q^\beta}\left[A(t+dt)-\left(1+\text{repo}_A(t)dt\right)A(t)\right]=0$ and thus:

$$r_A(t)dt = E_t^{Q^\beta}\left[\frac{dA(t)}{A(t)}\right] = \text{repo}_A(t)dt.$$

If defaults can occur but are predictable, then one can state whether or not the parties will be in default at date $t+dt$. Clearly, there is no reason why a trade would occur with one of parties going to default with certainty at $t+dt$. If none of the parties are to default "locally", then $\left(1+\text{repo}_A(t)dt\right)A(t)$ appears as a default-free forward price of the bond. Stated in other words, this forward price is exempt from CVA/DVA adjustments.

[30] For a defaultable bond, there is a seeming extra-return, conditionally on no default, which vanishes when taking into account defaults.

[31] One could think of Treasury GC as a proxy. In our simplified setting, we could think of the haircut as an independent amount or an initial margin, that are not dealt with in this paper, even though it is not innocuous when pricing a collateralized trade.

There are a number of cases where defaults could occur, yet $\left(1+\text{repo}_A(t)dt\right)A(t)$ can still be seen as a default-free forward price.

The first one corresponds to cancellation[32] of the deal at default. We denote by $\lambda(t)dt$, the probability of the deal to be cancelled, due to default of one of the parties. The pricing equation becomes:

$$E_t^{Q^\beta}\left[\left(A(t+dt)-\left(1+\text{repo}_A(t)dt\right)A(t)\right)\times\left(1-\lambda(t)dt\right)\right]=0.$$

Thus we still have $r_A(t)dt = E_t^{Q^\beta}\left[\frac{dA(t)}{A(t)}\right]=\text{repo}_A(t)dt$ [33].

If the payment to the non defaulted party is of the form $\delta\max\left(A(t+dt)-\left(1+\text{repo}_A(t)dt\right)A(t),0\right)$, where δ is a recovery rate and provided that the two parties have the same recovery rate and default probability, we would still have: $E_t^{Q^\beta}\left[\frac{dA(t)}{A(t)}\right]=\text{repo}_A(t)dt$. This symmetry between the parties would hold for instance if "tier-one" or "prime" banks are to trade together.

A further relaxation consists in the assumption of no "gap risks" at default. If we assume that A is driven by a Brownian motion or that its jump component is independent of the default counting processes of the counterparties, then a simple expansion shows that $E_t^{Q^\beta}\left[\frac{dA(t)}{A(t)}\right]=\text{repo}_A(t)dt$ still holds and that CVA/DVA terms can be neglected when computing the instantaneous forward price of a bond.

Let us remark that, in our simplified setting regarding counterparty and operational risks, the repo rate $\text{repo}_A(t)$ is only related to security A, without consideration of the parties identity.

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[32] In such a framework, the claim on default for a repo contract would differ from the claim associated with other OTC trades, such as interest rate swaps. We may relate cancellation to fails-to-deliver without penalty.

[33] Thus in the above, default times of the counterparties are totally inaccessible and $\lambda(t)$ is the intensity of the first to default-time. As can be seen from above, even if the custodian bank is considered as default-free (say we are considering a tri-party repo), there remains a credit exposure of $A(t+dt)-\left(1+\text{repo}_A(t)dt\right)A(t)$, before the variation margin is being paid. This is the purpose of the haircut. If A is driven by a Brownian motion, and collateral is posted continuously, there is no need of a positive haircut. Loosely speaking, the CVA terms will involve quantities or order of magnitude smaller than dt and can thus be neglected with respect to the pricing. One can actually speak of "the" repo rate of bond A, without refereeing to the parties, if CVA/DVA terms can be neglected.

A short-term repo contract with no haircut is a perfectly collateralized trade. We are considering buying a security deliverable at date $t+dt$, the transaction being secured with cash and the cash collateral rate being equal to $\mathrm{repo}_A(t)$. In other words, the repo rate can be seen as a cash collateral rate paid by/to the custodian bank. The term $A(t+dt)-(1+\mathrm{repo}_A(t)dt)A(t)$ corresponds to the variation margin to be paid on an open repo[34].

Unsurprisingly, the repo rate is tied to instantaneous forward price of a bond. We emphasize that we directly considered a forward market and related forward prices without refereeing to lending cash in between t and $t+dt$. This is purposeless since there are no cash-flows being exchanged at date t and no need for transferring funds across time. As a consequence, the default-free forward price is primitive and is not derived from a borrowing and lending cash market as in the standard cash and carry approach.

2.7. Posting foreign currencies.

In the case of cash-collateral paid in the base currency (say USD), the collateral account typically earns the effective fed funds rate, thus $r_A \equiv c^{\$}$ and the settlement price of a collateralized trade is computed by discounting the terminal cash-flow (expressed in USD) at the collateral rate under the usual risk-neutral probability, $Q^{\beta^{\$}}$ associated with the USD savings account numéraire. This is, for example, the case of an LCH.Clearnet interest rate swap contract in USD, where the settlement price $h(t)$ is obtained by discounting at the effective fed funds rate. The above result also corresponds to equation (7) in Piterbarg (2010), corresponding to our case where the derivatives contract is fully collateralized.

We may need to consider collateral accounts in different currencies. As an example, we could consider a cash-collateral account in a foreign currency, for instance $A^{€}(t)=FX^{€/\$}(t)\times\exp\int_0^t c^{€}(s)ds$, where $c^{€}(t)$ corresponds to the Eonia at time t. As for bonds, we will have: $\frac{dA^{€}(t)}{A^{€}(t_-)}=r_{A^{€}}(t)dt+dM_{A^{€}}(t)$, $M_{A^{€}}$ being a $Q^{\beta^{\$}}$ martingale, and we need to determine $r_{A^{€}}(t)$. For this purpose, we can go along the same lines as for bonds and repo contracts. Let us introduce an instantaneous forward FX contract that is assumed to be default-free and denote by $FX^{€/\$}(t)\left(1+\mathrm{ird}^{\$/€}(t)dt\right)$, the forward exchange rate, where $\mathrm{ird}^{\$/€}(t)$ is the interest differential between USD and EUR. As for the repo contract, we have:

$$E_t^{Q^{\beta^{\$}}}\left[FX^{€/\$}(t+dt)-FX^{€/\$}(t)\left(1+\mathrm{ird}^{\$/€}(t)dt\right)\right]=0,$$

[34] If the bond A is special, the corresponding repo rate is lower.

which leads to: $E_t^{Q^{\beta^\$}}\left[\frac{dFX^{€/\$}(t)}{FX^{€/\$}(t)}=\right]=\text{ird}^{\$/€}(t)dt$.

Thus, $r_{A^€}(t)dt=E_t^{Q^{\beta^\$}}\left[\frac{dA^€(t)}{A^€(t)}\right]=\left(c^€(t)+\text{ird}^{\$/€}(t)\right)dt$. Let us remark that $c^€(t)$ and $\text{ird}^{\$/€}(t)$ are observable quantities[35].

Since the above instantaneous forward FX contract consists in exchanging 1 EUR against $FX^{€/\$}(t)\left(1+\text{ird}^{\$/€}(t)dt\right)$ USD, up to a scaling factor it also consists in exchanging $\frac{1}{FX^{€/\$}(t)\left(1+\text{ird}^{\$/€}(t)dt\right)}$ EUR against 1 USD. Using $\frac{1}{FX^{€/\$}(t)}=FX^{\$/€}(t)$ and $\frac{1}{1+\text{ird}^{\$/€}(t)dt}=1-\text{ird}^{\$/€}(t)dt$, we end-up with $\text{ird}^{\$/€}(t)=-\text{ird}^{€/\$}(t)$. Thus, when EUR is used as the base currency, we get: $E_t^{Q^{\beta^€}}\left[\frac{dFX^{\$/€}(t)}{FX^{\$/€}(t)}=\right]\equiv\text{ird}^{€/\$}(t)dt=-\text{ird}^{\$/€}(t)dt$.

The counterparty risk issues associated with the instantaneous forward FX contract are the same as the ones already discussed in the repo case. In the simplest cases of no gap risk, the CVA/DVA terms can be neglected, as if the instantaneous forward FX contract was default-free.

While $C^\$(t)=\exp\left(\int_0^t c^\$(s)ds\right)$, where $c^\$(t)$ is the effective fed funds rate is “risk-free”, this is not anymore the case with $A^€(t)$, due the martingale term “$dM_{A^€}(t)\neq 0$”.

2.8. Optimal choice of posted securities.

We assume that a set of eligible securities and cash currencies $\{1,\ldots,I\}$ can be posted in the collateral account[36]. To ease the exposition, let us assume that $h(t)>0$ and that the two parties commit to enter the transaction at time t. We leave aside funding constraints for a while. For instance we may assume that the party posting collateral holds amounts of eligible securities greater than $h(t)>0$, so there is no need to borrow cash or posted securities. Since these securities (or cash) can be got back at $t+dt$ by getting out the deal, there is no liquidity commitment after $t+dt$. According to standard rules, the party posting collateral chooses the posted security. The variation margin that has to be paid at $t+dt$ by

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[35] In the standard mathematical framework, we would have: $\text{ird}^{\$/€}(t)=r^\$(t)-r^€(t)$, where $r^\$(t),r^€(t)$ are the instantaneous USD and EUR default-free rates.

[36] We assume that the posted securities are liquid enough so that no haircut has to be applied to the market value. This point is investigated further, when dealing with partial collateralization.

the posting collateral party equals $dVM_i(t) = dh(t) - h(t)\frac{dA_i(t)}{A_i(t)}$ if collateral of type i is posted. Since the deal can be cancelled out at $t+dt$, $-dVM_i(t)$ is the PV impact if collateral i is posted. The optimal choice of collateral posting results in minimizing the expected variation margin payment $E_t^{Q^\beta}\left[dVM_i(t)\right] = E_t^{Q^\beta}\left[dh(t) - h(t)\frac{dA_i(t)}{A_i(t)}\right]$ [37]. Since $h(t) > 0$, the posted security will have the highest (risk-neutral) expected return $E_t^{Q^\beta}\left[\frac{dA_i(t)}{A_i(t)}\right]$ among eligible securities $i \in \{1,\dots,I\}$ [38]. The pricing equation becomes $\max_{i\in\{1,\dots,I\}} E_t^{Q^\beta}\left[dVM_i(t)\right] = 0$ and leads to:

$$E_t^{Q^\beta}\left[\frac{dh(t)}{h(t)}\right] = \max_{i\in\{1,\dots,I\}} E_t^{Q^\beta}\left[\frac{dA_i(t)}{A_i(t)}\right] = \max_{i\in\{1,\dots,I\}} r_{A_i}(t)dt\ .$$

As a consequence the previous discounting result is extended to:

$$h(t) = E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T\left(\max_{i\in\{1,\dots,I\}} r_{A_i}(s)\right)ds\right)\right].$$

Let us remark that we simply need to substitute the discount rate and use $\max_{i\in\{1,\dots,I\}} r_{A_i}(s)$. The optionality in the collateral choice magnifies the discount rate but the pricing rule relating terminal cash-flow $h(T)$ to settlement price $h(t)$ remains linear. A CSA (or another collateral agreement as in a tri-party repo) provides the set of assets that can be posted. The settlement price is specific to the CSA and there are thus many potential settlement prices depending upon the collateral pools. If we assume that $h(t) = 0$ for a given CSA, enlarging the pool of deliverable assets may either lead to positive or negative settlement prices. Thus novation trades or the backload of ISDA CSA trades to LCH.Clearnet, which usually involve changes in the collateral pool are rather intricate.

Let us now deal with two simple, practical and illustrative examples.

The first one corresponds to choosing between two currencies, assuming cash-collateral is to be posted. We could think of a FX USD/EUR contract, the choice being between posting USD

[37] Equivalently, one may consider maximizing the PV impact.

[38] If $h(t) < 0$, the collateral choice turns out to maximise the expected variation margin to be received, $E_t^{Q^\beta}\left[dh(t) - h(t)\frac{dA_i(t)}{A_i(t)}\right]$, also leading to choosing security i which maximizes $r_{A_i}(t) = E_t^{Q^\beta}\left[\frac{dA_i(t)}{A_i(t)}\right]$. Let us also remark that since we assume that the deal can be cancelled out at any time, it implies that collateral can be substituted at any time. This does not apply for a number of CSA within standard ISDA Master Agreements.

and EUR[39]. Using the previous notations, the rate of return of USD cash collateral is $c^{\$}(t)$ and the $Q^{\beta^{\$}}$ expected rate of return when posting EUR cash collateral is equal to $r_{A^{€}}(t) = c^{€}(t) + \mathrm{ird}^{\$/€}(t)$. If $c^{€}(t) + \mathrm{ird}^{\$/€}(t) > c^{\$}(t)$ or equivalently if $c^{€}(t) - c^{\$}(t) > \mathrm{ird}^{\$/€}(t)$, the parties would use euro as collateral currency.

We recall that an idealized world, we would have: $\mathrm{ird}^{\$/€}(t) = r^{\$}(t) - r^{€}(t)$. Euro would be posted if $c^{€}(t) - r^{€}(t) > c^{\$}(t) - r^{\$}(t)$. $c^{€}$ corresponds to EONIA and $c^{\$}$ to the effective fed funds rate. Since the clearing house is assumed to be default-free, by lending in euros at $c^{€}$, the cash lender earns an extra return above the risk-free rate $r^{€}$ equal to $c^{€}(t) - r^{€}(t)$. The currency choice corresponds to maximizing the extra return. $c^{€}(t) - r^{€}(t)$ is related to the average short-term credit spread of banks operating in the eurozone. As a consequence, when credit quality of European banks is lower compared to their overseas counterparts, it is more likely that euro would be posted. In such a case, constraining the use of USD would be detrimental to the collateral posting party.

Let us consider the case where one of the parties' accounts is denominated in euros instead of US dollars.

Using the same notational framework as above, we have $C^{€}(t) = \exp\left(\int_0^t c^{€}(s)ds\right)$ and $A^{\$}(t) = FX^{\$/€}(t) \times \exp\int_0^t c^{\$}(s)ds$.

Since $E_t^{Q^{\beta^{€}}}\left[\frac{dFX^{\$/€}(t)}{FX^{\$/€}(t)}\right] \equiv \mathrm{ird}^{€/\$}(t)dt = -\mathrm{ird}^{\$/€}(t)dt$, $E_t^{Q^{\beta^{€}}}\left[\frac{dA^{\$}(t)}{A^{\$}(t)} = \right] = \left(c^{\$}(t) - \mathrm{ird}^{\$/€}(t)\right)dt$ and $E_t^{Q^{\beta^{€}}}\left[\frac{dC^{€}(t)}{C^{€}(t)}\right] = \frac{dC^{€}(t)}{C^{€}(t)} = c^{€}(t)dt$. Thus, the EUR investor would post EUR if $c^{€}(t) > c^{\$}(t) - \mathrm{ird}^{\$/€}(t)$, leading to the same collateral choice as the USD investor.

Another typical example consists in choosing between posting Treasury GC and cash (in the same currency, say USD). In the idealized framework depicted above, the posting party should choose cash rather than bonds whenever the effective fed funds rate is above the overnight Treasury GC rate[40]. Given that a special bond will be associated with a lower repo rate, it should never be delivered in an unconstrained collateral setting.

[39] If the two parties involved in the FX contract are, say UK banks, they could find it more convenient to post GBP. We restrict ourselves to USD and EUR to ease the exposition.

[40] We do not account for funding constraints such as a restricted access to fed funds by a number of market participants.

3. Multiple curves.

An important application of the above techniques is the pricing of interest rate swaps or forward FX contracts under the perfect collateralization scheme. The terminal payoff $h(T)$ is then specialized. We first introduce some useful changes of measures, associated with convenient discount factors.

3.1. Changes of measure and further notations.

We will define a set of measures, equivalent to Q^{β}, denoted as Q_T^A associated with the collateral account by their density:

$$\frac{dQ_T^A}{dQ^{\beta}} = \frac{\exp\left(-\int_0^T r_A(s)\right)}{E^{Q^{\beta}}\left[\exp\left(-\int_0^T r_A(s)\right)\right]}.$$

Q_T^A is named "collateralized forward measure" by Fujii et al. (2010)[41].

The density process of Q_T^A (with respect to Q^{β}) is defined as usual by: $\xi_T^A(t) = E_t^{Q^{\beta}}\left[\frac{dQ_T^A}{dQ^{\beta}}\right]$.

Let us further denote:

$$B_A(t,T) = E_t^{Q^{\beta}}\left[\exp\left(-\int_t^T r_A(s)ds\right)\right].$$

$B_A(t,T)$ is the date t - settlement price of a cash-flow of 1 paid at date T collateralized with A. It can then be considered as a A-collateralized discount bond price. For instance, $B_{C^\$}(t,T) = E_t^{Q^{\beta}}\left[\exp\left(-\int_t^T c^\$(s)ds\right)\right]$ corresponds to the settlement price of a zero-coupon collateralized with cash remunerated at fed funds rate.

We can write the density process as:

$$\xi_T^A(t) = \frac{\exp\left(-\int_0^t r_A(s)ds\right) \times B_A(t,T)}{B_A(0,T)}.$$

The usual forward measure is Q_T^{β} defined as:

[41] Let us remark that the density of Q_T^A with respect to Q^{β} only involves the (expected) rate of return of the collateral. Thus, we depart from the change of numéraire and notational setup used for instance by Geman, El Karoui and Rochet (1995) or Schroder (1999).

$$\frac{dQ_T^\beta}{dQ^\beta} = \frac{\exp\left(-\int_0^T r(s)\right)}{E^{Q^\beta}\left[\exp\left(-\int_0^T r(s)\right)\right]} = \frac{1}{\beta(T)\times B(0,T)},$$

under the standard notation: $B(t,T) = E_t^{Q_T^\beta}\left[\exp\left(-\int_t^T r(s)ds\right)\right]$. If the default-free savings account is being used as collateral, then $r_A(.) \triangleq r(.)$ and $Q_T^A = Q_T^\beta, \forall T$.

When the expected rate of return on the posted collateral, r_A is equal to zero, then $Q_T^A = Q^\beta$, which is typically the case of a futures market.

From the above expression, we can write the settlement price $h_A(t)$ as:

$$h_A(t) = B_A(t,T)\times E_t^{Q_T^A}\left[h(T)\right]$$

Since we have considered different collaterals, it is interesting to switch from one to another. Let us denote by A, B two kinds of collateral assets and r_A, r_B the corresponding Q^β-expected rates of return. We then have:

$$\frac{dQ_T^B}{dQ_T^A} = \frac{\exp\left(-\int_0^T (r_B - r_A)(s)\right)}{E^{Q_T^A}\left[\exp\left(-\int_0^T (r_B - r_A)(s)\right)\right]}$$

We denote by $\xi_T^{B,A}(t) = E_t^{Q_T^A}\left[\frac{dQ_T^B}{dQ_T^A}\right]$ the density process of Q_T^B (with respect to Q_T^A). It can be shown that $\xi_T^{B,A}(t) = \frac{\xi_T^B(t)}{\xi_T^A(t)}$.

3.2. OIS contracts with cash collateral.

We will first focus on the simplest payoff, a stylized (with continuous compounding) overnight indexed swap, where:

$$h(T) = \exp\left(y(T - t_0)\right) - \exp\left(\int_{t_0}^T c(s)ds\right).$$

Where t_0 is the trade date, T the maturity date of the trade. y is the (continuously compounded) fixed rate of the collateralized overnight indexed swap, while $c(t)$ is the reference rate, in the contract, such as Eonia or the effective Fed funds rate. As discussed above, such reference rate in the above contract may not correspond to actual lending rates in the interbank market. Eonia and effective Fed funds rates mix lending rates at different intraday time points and of borrowing banks with heterogeneous credit quality. Within the eurozone, one has to keep in mind that the panel of borrowers is not the same as the panel of contributing banks. Eventually, there is some endogenous selection of contributors, since banks with poorer credit quality might use central bank lending facilities[42] or try to find cash by other means.

As above, we denote the effective fed funds rate, which is also the collateral rate by $c^{\$}$.

The settlement price, $h(t)$, $t_0 \le t \le T$, is provided by:

$$h(t) = E_t^{Q^{\beta\$}}\left[h(T)\exp\left(-\int_t^T c^{\$}(s)ds \right) \right].$$

It is worth noting that the compounding and discounting rates are equal leading to simplifications. Under our standing notations, $B_{C^{\$}}(t,T) = E_t^{Q^{\beta}}\left[\exp\left(-\int_t^T c^{\$}(s)ds \right) \right]$. Thus,

$$h(t) = \exp\left(y\left(T - t_0\right)\right) B_{C^{\$}}(t,T) - \exp\left(\int_{t_0}^{t} c^{\$}(s)ds \right).$$

It can be seen that $B_{C^{\$}}(t,T)$ acts as a discount factor.

Let us consider a par trade at inception, i.e. $t = t_0$ and settlement price equal to 0. Then, the par rate, at date t, of such a collateralized OIS contract with maturity date T, $y^{\$}\left(t,T\right)$, is derived from:

$$\exp\left(-y^{\$}(t,T)\times\left(T-t\right)\right) = B_{C^{\$}}\left(t,T\right).$$

Let us remark that the discount factors $B_{C^{\$}}\left(t,t+\delta\right)$ are directly related to market observables, i.e. par rates of collateralized OIS. As a consequence, we can readily write settlement prices of collateralized OIS contracts in a model free setting (up to interpolation schemes).

The analysis readily extends to forward trades and discrete compounding.

3.3. Libor swaps.

Let us denote by $L\left(T,T+\delta\right)$ the Libor (or Euribor rate) at time T for maturity δ. The case of a Libor FRA contract is such that $h(T+\delta) = \delta\times\left(FRA - L(T,T+\delta)\right)$ where δ is the coverage factor and FRA is the contractual FRA rate. We then have:

$$h_A\left(t\right) = \delta B_A\left(t,T+\delta\right)\times\left(FRA - E_t^{Q^A_{T+\delta}}\left[L\left(T,T+\delta\right)\right]\right).$$

The par rate $L_A\left(t,T,T+\delta\right)$ is the FRA rate such that the settlement price at date t equals zero. $L_A\left(t,T,T+\delta\right)$ is a forward collateralized (with A) Libor rate. We have:

42 This does not necessarily mean that such banks will actually choose to use such central bank facilities. One has also to keep in mind that the gap between the rate on central bank lending facility and its target rate, let us say the rate on MRO for the ECB, may change over time, such changes are obviously related to bank credit worthiness.

$$L_A\left(t,T,T+\delta\right)=E_t^{Q_{T+\delta}^A}\left[L\left(T,T+\delta\right)\right].$$

The forward Libor curve depends upon the collateral choice. For instance, one might think of using LCH.Clearnet type contracts, where cash-collateral is posted in the same currency as the one of the interest rate swap contract and earns the corresponding reference overnight rate. We would then have $L_{C^{\$}}\left(t,T,T+\delta\right)=E_t^{Q_{T+\delta}^{C^{\$}}}\left[L\left(T,T+\delta\right)\right]$. If we were to think of Eurodollar futures, the (collateralized) forward Libor rate would be different, since the latter case, the collateral rate equals zero and not the reference overnight rate.

We can then write the settlement price of a collateralized FRA contract as:

$$h_A\left(t\right)=\delta B_A\left(t,T+\delta\right)\times\left(FRA-L_A\left(t,T,T+\delta\right)\right),$$

which involves market observables or quantities that can be derived from market observables. From this and the linearity property of settlement prices, we can compute the present value of Libor swaps collateralized at overnight rates of the LCH.Clearnet type.

3.4. Collateralized forward FX contracts.

The contractual payoff is such that: $h(T)=FX^{€/\$}(T)-K$ where, for simplicity, we state that:

$$FX^{€/\$}(T)=FX^{€/\$}(t)\exp\left(\int_t^T \mathrm{ird}^{\$/€}(s)ds\right)\times\exp\left(-\frac{\left(\sigma^{€}\right)^2}{2}(T-t)+\sigma^{€}\left(W^{€}(T)-W^{€}(t)\right)\right).$$

Let us assume that the collateral is posted in cash USD and earns the effective fed funds rate $c^{\$}$. Then, applying the pricing equation: $h(t)=E_t^{Q^{\beta^{\$}}}\left[h(T)\exp\left(-\int_t^T c^{\$}(s)ds\right)\right]$, we obtain:

$$h(t)=FX^{€/\$}(t)E_t^{Q^{\beta^{\$}}}\left[\exp\left(-\int_t^T\left(c^{\$}(s)-\mathrm{ird}^{\$/€}(s)\right)ds\right)\right]-K\times E_t^{Q^{\beta^{\$}}}\left[\exp\left(-\int_t^T c^{\$}(s)ds\right)\right].$$ [43]

Let us remark that the discount factors $E_t^{Q^{\beta^{\$}}}\left[\exp\left(-\int_t^T c^{\$}(s)ds\right)\right]=B_{C^{\$}}\left(t,T\right)$ can be obtained from the quotes on the USD interest swaps collateralized at effective fed funds rate (LCH.Clearnet type contracts). Thus, from the quotes of the collateralized forward FX contracts (with cash collateral in USD), we can derive more discount factors. More precisely, let us denote by $FX_{\$}^{€/\$}\left(t,T\right)$, the collateralized (in cash USD) forward price of one euro. It is such that: $0=FX^{€/\$}(t)E_t^{Q^{\beta^{\$}}}\left[\exp\left(-\int_t^T\left(c^{\$}(s)-\mathrm{ird}^{\$/€}(s)\right)ds\right)\right]-FX_{\$}^{€/\$}\left(t,T\right)B_{C^{\$}}\left(t,T\right)$. Thus, we can compute the discount factor $E_t^{Q^{\beta^{\$}}}\left[\exp\left(-\int_t^T\left(c^{\$}(s)-\mathrm{ird}^{\$/€}(s)\right)ds\right)\right]$ from quoted forward FX contracts. As for a non par collateralized contract, with contractual payoff

[43] This illustrative example, where $W^{€}$ is a Brownian term independent of other involved quantities and the volatility is assumed to be constant can readily be extended. This simply involves different expressions of discount factors. Switching to a € risk-neutral measure is suitable in our setting.

$h(T)=FX^{€/\$}(T)-K$, the settlement price at time t is simply provided by $B_{C^\$}(t,T)\times\left(FX_\$^{€/\$}(t,T)-K\right)$.

3.5. Generic convexity adjustments.

We can think of comparing two derivatives assets, with same terminal payoff, perfect collateralization, but different collaterals. A typical example would involve a Eurodollar future (collateral rate equal to zero) and a FRA collateralized at effective Fed funds rate.

For a given payoff $h(T)$ collateralized with A, its T- forward price $h_{A,T}(t)$ is equal to $h_{A,T}(t)=E_t^{Q_T^A}\left[h(T)\right]$. Indeed, $0=B_A\left(t,T\right)\times E_t^{Q_T^A}\left[h(T)-h_{A,T}(t)\right]$. Thus we are to compare $E_t^{Q_T^A}\left[h(T)\right]$and $E_t^{Q_T^B}\left[h(T)\right]$. This involves a covariance term between the payoff and the spread between (expected) rates of return on the two collaterals. In a Brownian filtration setting, it can be shown that:

$$E_t^{Q_T^B}\left[h(T)\right]=E_t^{Q_T^A}\left[h(T)\right]E_t^{Q_T^A}\left[\varepsilon\left(\int_t^T\sigma_h\left(s,T\right)dW_A^{h,T}(s)\right)\varepsilon\left(\int_t^T\sigma_B\left(s,T\right)dW_B^{A,T}(s)-\sigma_A\left(s,T\right)dW_A^{A,T}(s)\right)\right]$$

We refer to the Appendix B for proofs, computations and further details.

3.6. Linear pricing rules and (perfectly collateralized) discount factors.

Let us first remind that within a perfect collateralization framework as defined above, there is no gap risk when unwinding a trade, thus no independent amount or initial margin based on non linear measures of risk and thus no resulting portfolio effect.

In a number of cases, the expected rate of return of the posted collateral, $r_A(.)$ is not independent of $h(T)$. This may lead to a number of quanto effects. However, this does not change the important point that in a perfect collateralization framework, the pricing rule remains linear. If one considers the pricing of a collateralized terminal cash-flow $h_1(T)+h_2(T)$, then the cash-flows associated with the variation margins will be the sum of those associated with $h_1(T)$ and $h_2(T)$ considered independently. As a consequence, after discounting these variation-margin cash-flows at the default-free short rate under Q^β, we will have that the settlement price at time t of the collateralized terminal payoff $h_1(T)+h_2(T)$, is simply $h_1(t)+h_2(t)$. Similarly, the settlement price at time t of the collateralized terminal payoff $\alpha\times h(T)$, where α is a real number, is $\alpha\times h(t)$.

If we consider square integrable payoffs, from Riesz representation theorem, we can write $h(0)=E^{Q^\beta}\left[X_T\times h(T)\right]$ for some square-integrable random variable X_T. Taking $h(T)=1$, we get $B_A(0,T)=E^{Q^\beta}\left[X_T\right]$. Then, we can define a new probability measure Q_T^A from its density: $\frac{dQ_T^A}{dQ^\beta} = \frac{X_T}{E^{Q^\beta}\left[X_T\right]}$ [44]. As a consequence, the pricing rule $h(0) = B_A\left(0,T\right)E^{Q_T^A}\left[h(T)\right]$ is not surprising but a direct consequence of the linear structure of cash-flows in perfect collateral schemes. In that framework, and once again neglecting gap risks and initial margin effects, linearity means the possibility of trade per trade PV computations, without the need to consider netting effects within portfolios of trades.

Linearity also holds for terminal payoffs associated with different maturities. We can notice that $B_A(t,T)$ acts as a "collateralized discount factor".

4. Present value of cash-flows associated with the collateral account.

It is interesting to consider the present value of the variation margins in isolation. This provides a margin based interpretation of the present value adjustment due to variation margins.

Using the above results, the time t present value of cash-flows associated with managing the collateral account can be written as:

$$E_t^{Q^\beta}\left[V(t) + \int_t^T e^{-\int_t^s r(u)du}\left(dV(s) - V(s)\frac{dA(s)}{A(s)}\right) - V(T)e^{-\int_t^T r(u)du}\right].$$

This can also be seen as the "present value of the CSA". When cash-flows associated with the collateral account are paid or received by a default-free clearing house, there is no need to account for CVA/DVA terms.

The present value of collateral cash-flows can be written as[45]:

$$E_t^{Q^\beta}\left[\int_t^T \left(r(s) - r_A(s)\right)V(s)\exp\left(-\int_t^s r(u)du\right)ds\right].$$

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[44] We have assumed without any further checking that $X_T > 0$, Q^β - almost surely and thus Q_T^A is actually a probability measure equivalent to Q^β.

[45] Define $\zeta(s) = V(s)e^{-\int_t^s r(u)du}$. $d\zeta(s) = e^{-\int_t^s r(u)du}dV(s) - r(s)\zeta(s)ds$. Thus, the present value of the collateral cash-flows equals: $E_t^{Q^\beta}\left[\zeta(t) + \int_t^T d\zeta(s) + \left(r(s)ds - \frac{dA(s)}{A(s)}\right)\zeta(s)ds - \zeta(T)\right]$, which can be simplified as: $E_t^{Q^\beta}\left[\int_t^T\left(r(s)ds - \frac{dA(s)}{A(s)}\right)V(s)e^{-\int_t^s r(u)du}\right]$, which leads to the stated expression.

In the case of cash-collateral, with collateral rate $c(.)$, $\frac{dA(s)}{A(s)} = c(s)ds$ and the latter expression simplifies to $E_t^{Q^\beta}\left[\int_t^T \big(r(s)-c(s)\big)V(s)\exp\left(-\int_t^s r(u)du\right)ds\right]$.

Let us remark that though the clearing house is assumed to be default-free (in our theoretical setting), clearing members are lending to that clearing house at a different rate. From the point of view of clearing members, $r(s)-c(s)$ is the convenience yield associated with the collateral account and $\big(r(s)-c(s)\big)V(s)ds$ is simply the margin associated with the management of the collateral account, between s and $s+ds$. Unsurprisingly, the present value of collateral cash-flows equals the present value of the margins (accrued interest point of view) as is standard with any cash account. Obviously, in the case where the collateral rate $c(.)$ equals the risk free rate $r(.)$, the present value of collateral cash-flows is equal to zero since there is no net margin associated with the collateral amount ("costless collateral").

The present value of cash-flows associated with the management of the collateral account is the additive correcting term to switch from an uncollateralized PV to a collateralized PV. It is worth noting that this holds for all collateralization schemes as only the value $V(t)$ of the collateral account is involved and not the settlement price $h(t)$.

5. Pricing with partial collateralization.

We first consider the case of haircuts. We account for possible changes of haircut ratios through time. It will eventually be shown that this includes the case of asymmetric CSA pricing which is dealt with in a second step.

5.1. Locally proportional haircuts.

When a security is being posted, a proportional, though not constant through time, some haircut is usually applied. This is a form of partial collateralization. For instance, if the posting party is required to post more bonds (in USD value) than the settlement price, default of the other party might be costly if the bonds in excess are not returned on default. In the case of securities lending, the amount of cash usually exceeds the USD value of lent securities. Thus, we may need to deal with haircut ratios that are above or below 1.

Let us focus on the changes in the variation margins with partial collateralization features, leaving aside the CVA/DVA effects. Let us denote by $\frac{1}{\alpha_A(s)}$ the haircut ratio at time s associated with security A. Given potential "runs on repos", it makes sense to deal with time varying haircuts. At time $s, t \le s \le T$, the amount of collateral account is then equal to

$V(s) = \alpha_A(s)h(s)$ and the number of posted securities is given by $\frac{\alpha_A(s)h(s)}{A(s)}$. Using the ongoing notations, the variation margin $dVM(t)$ becomes:

$$dVM(t) = V(t+dt) - \frac{\alpha_A(t)h(t)}{A(t)} A(t+dt) = \alpha_A(t+dt)h(t+dt) - \alpha_A(t)h(t)\frac{A(t+dt)}{A(t)}$$

$$dVM(t) = d\left(\alpha_A(t)h(t)\right) - \alpha_A(t)h(t)\frac{dA(t)}{A(t)}$$

Let us consider a trade initiated at time t and unsettled at $t+dt$. The net cash-flow at time t is given by: $V(t) - h(t) = \left(\alpha_A(t) - 1\right)h(t)$. The net cash-flow at $t+dt$ is provided by:

$$\begin{aligned} h(t+dt) - V(t+dt) + dVM(t) &= h(t+dt) - V(t+dt) + V(t+dt) - \frac{\alpha_A(t)h(t)}{A(t)} A(t+dt) \\ &= h(t+dt) - \alpha_A(t)h(t) - \alpha_A(t)h(t)\frac{dA(t)}{A(t)} \\ &= \left(1-\alpha_A(t)\right)h(t+dt) + \alpha_A(t)\left(dh(t) - h(t)\frac{dA(t)}{A(t)}\right) \end{aligned}$$

In the simplified setting where no defaults occurs at $t+dt$, the previous cash-flow is paid for sure and the pricing equation becomes:

$$E_t^{Q^\beta}\left[\left(\alpha_A(t)-1\right)h(t) + \left(1-r(t)dt\right)\times\left(\left(1-\alpha_A(t)\right)h(t+dt) + \alpha_A(t)\left(dh(t) - h(t)\frac{dA(t)}{A(t)}\right)\right)\right] = 0 .$$

This leads to the pricing equations (not accounting for CVA/DVA effects, i.e. default-free counterparties entering a collateralized trade):

$$E_t^{Q^\beta}\left[dh(t)\right] = \left(\left(1-\alpha_A(t)\right)r(t) + \alpha_A(t)r_A(t)\right)h(t)\,dt .$$

An intuitive way to look at this drift restriction is to consider that $\alpha_A(t)h(t)$ is invested in the collateral account with expected rate of return $r_A(t)$. The remaining part $\left(1-\alpha_A(t)\right)h(t)$ corresponds to a cash-flow invested in the market at expected rate of return $r(t)$. This leads to the simple discounting formula:

$$h(t) = E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T \left(\left(1-\alpha_A(s)\right)r(s) + \alpha_A(s)r_A(s)\right)ds\right)\right].$$

The discount rate that used to be $r_A(t)$ with perfect collateralization (i.e. $\alpha_A(t) = 1$) has to be adapted in an easy to understand way. The most common market feature consists in using haircut ratios below 1 (overcollateralization) and is associated with $\alpha_A(t) > 1$. This magnifies the importance of the Q^β- expected rate of return on the collateral asset. $0 \le \alpha_A(t) < 1$ corresponds to under-collateralization, i.e. the amount of collateral is only a fraction of the value of the settlement price (or PV) of the trade. If $\alpha_A(t) = 0$, no collateral is being posted and provided that the counterparties are default-free, we are led back to the standard discounting at the default-free rate $r(.)$, $h(t)=E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T r(s)ds\right)\right]$. Let us remark that the default-free rate is explicitly back in the discount rate formula.

When dealing with multiple bonds with different haircuts, one has to account for the haircut adjusted expected rate of return associated with collateral A: $\left(1-\alpha_A(t)\right)r(t)+\alpha_A(t)r_A(t)$. Let us consider for a while that we deal with a set of bonds, say within a Treasury GC pool. If we assume that $r_A(t)=\text{repo}_A(t)$ does not depend on the posted Treasury and that the same haircut is applicable to all bonds, fortunately the optionality cancels out, the choice of delivered bond is out of purpose and we deal with a unique discount rate. If for such Treasury GC bonds, $\alpha_A(t)$ remains quite close to one, using the repo rate as the discounting rate is a fair approximation.

When the collateral pool is enlarged, the applicable haircuts (and their dynamics) become critical in the choice of posted collateral. We have to consider $r(t)+\alpha_A(t)\left(r_A(t)-r(t)\right)$ for various assets. One has thus to post the collateral that will maximize $\alpha_A(t)s_A(t)$, where $s_A(t)=r_A(t)-r(t)$ is the stochastic basis associated with collateral A. However, provided that the haircut does not depend upon the settlement price, the mapping between terminal cash-flows and settlement prices remains linear.

5.2. Asymmetric CSA, CSA with thresholds.

There are a number of cases where linearity is no longer valid. This may be the case with minimum or with independent amounts. This also applies with LCH.Clearnet initial margins and requires a pricing at the portfolio level. However, if we consider a well-balanced dealers book, the price impact of the initial margin (which for a new deal can be negative or positive whether the deal is risk-reducing or risk – enhancing) is equal to zero in average[46].

In a typical asymmetric CSA framework, no collateral is to be posted if the settlement price is above zero while the value of posted collateral equals the settlement price is negative if it is negative. Thus, $V(t)=\min\left(0,h(t)\right)=\left(0\times 1_{\{h(t)>0\}}+1\times 1_{\{h(t)\le 0\}}\right)h(t)$. This corresponds to a haircut being set to $\alpha_A(t)=0$ if $h(t)>0$ and $\alpha_A(t)=1$, if $h(t)\le 0$. Thus, the applicable discount rate is $r(t)1_{\{h(t)>0\}}+r_A(t)1_{\{h(t)\le 0\}}$ and we obtain the pricing formula for default-free counterparties and asymmetric CSA as a special case of pricing with haircuts:

[46] This would not hold with, say, a corporate or a retail banking institution with directional swap portfolios macro-hedging interest rate risks tied to business driven exposures.

$$h(t)=E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T\left(r(s)1_{\{h(s)>0\}}+r_A(s)1_{\{h(s)\le 0\}}\right)ds\right)\right]$$

For a CSA with threshold, we would have $V(t)=\min\left(h(t)-a,0\right)$ for some real number a. This leads to:

$$h(t)=E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T\left(r(s)1_{\{h(s)>a\}}+r_A(s)1_{\{h(s)\le a\}}\right)ds\right)\right].$$

Let us remark that the default-free rate $r(.)$ is now explicitly involved in the pricing formula. Moreover, the pricing equation is recursive since future settlement prices $h(s), s\ge t$ are involved in the discount rate.

Such equations are well known in finance. They have been considered in a CVA context in Duffie and Huang (1996) and Duffie *et al.* (1996) and more recently by Crépey (2011), Fujii and Takahashi (2012) and Henry-Labordère (2012)[47]. It is interesting to go back to our starting point, i.e. looking at the drift term of $h(t)$ under Q^β:

$$E_t^{Q^\beta}\left[dh(t)\right]=\left(r(t)\max\left(0,h(t)\right)+r_A(t)\min\left(0,h(t)\right)\right)dt.$$

This corresponds to a BSDE (backward stochastic differential equation) with terminal value $h(T)$ corresponding to contractual payoff and generator $g(h)=rh^+ + r_A h^-$, with $h^+=\max(h,0)$ and $h^-=\min(h,0)$. In a Brownian filtration setting, such BSDE is a standard mathematical finance tool (El Karoui et al. (1997)) and admits a unique solution. $h(t)$ is known as the conditional $g-$expectation of $h(T)$ (see Peng (2004)). Let us remark that the collateralization scheme is embedded in the shape of the generator of the BSDE. Moreover, properties of the generator translate to the price functional (Peng (2006)). For instance, in an asymmetric CSA, the generator is positively homogeneous. Thus, the price functional is also positively homogenous: for a positive real number λ, the price at time t of $\lambda h(T)$ is equal to $\lambda h(t)$. If $r\ge r_A$, the generator is concave and thus the functional associating the contractual payoff (terminal value) $h(T)$ to current settlement price $h(t)$ is also concave[48]. Non linearity is associated with portfolio effects: under the assumption that $r\ge r_A$, the present value of a portfolio of two trades $h_1(T)+h_2(T)$ is greater than the sum of the present values of the two trades considered in isolation.

[47] We also refer to Cesari *et al.* (2010), Burgard and Kjaer (2011) for related approaches.

[48] Actually, if $r\ge r_A$, the generator is super-additive and the price functional, associated with an asymmetric CSA, too. Conversely, if $r\le r_A$, the generator is sub-additive and the price functional is sub-additive and thus convex, thanks to positive homogeneity. Concavity or convexity are useful properties, since when the price functional is not differentiable with respect to terminal payoff, we can still consider a non-empty subdifferential and deal with left and right Gâteaux derivatives, which stills allows for simple one-sided first order expansions.

In such a non-linear context, one needs to consider pricing at portfolio level and the settlement price has to be understood as the present value of a portfolio of trades against a given counterparty (possibly a CCP) and under a specific CSA.

Two issues need to be addressed, marginal pricing and trade contributions. As will be shown these two issues are closely connected and are quite intuitive on a financial point of view.

Regarding marginal pricing, we consider the present value impact of a small trade with payoff εZ, $\varepsilon \in \mathbb{R}$ being a scaling factor. Thus the portfolio payoff translates from $h(T)$ to $h(T)+\varepsilon Z$. Let us denote by $(h+\varepsilon Z)(t)$ the settlement price after inception of the new trade. The generator $g(h) = rh^{+} + r_A h^{-}$ is differentiable with respect to h, except for $h=0$, which actually deserves a specific treatment[49]. Thanks to proposition 2.4 of El Karoui et al. (1997), $\varepsilon \to (h+\varepsilon Z)(t)$ is differentiable with derivative at $\varepsilon = 0$ is given by:

$$E_t^{Q^\beta}\left[Z \times \exp\left(-\int_t^T \left(r(s) 1_{\{h(s)>0\}} + r_A(s) 1_{\{h(s)>0\}} \right) ds \right) \right].$$

The previous term is known as the directional derivative of the price functional around the current portfolio in the direction of the new trade Z. Marginal pricing is based on the first order expansion:

$$(h+\varepsilon Z)(t) = h(t) + \varepsilon E_t^{Q^\beta}\left[Z \times \exp\left(-\int_t^T \left(r(s) 1_{\{h(s)>0\}} + r_A(s) 1_{\{h(s)>0\}} \right) ds \right) \right] + o(\varepsilon).$$

The deflator (or stochastic discount factor) $\exp\left(-\int_t^T \left(r(s) 1_{\{h(s)>0\}} + r_A(s) 1_{\{h(s)>0\}} \right) ds \right)$ is the functional derivative (or Fréchet derivative) of the price with respect to terminal payoff. The important point is that the first order term is linear in Z, thus leading to standard pricing methods.

We have already noticed that, in the case of an asymmetric CSA, the settlement price is positively homogenous with respect to terminal payoff. Thus, Euler's equation holds and can be written as: $h(t) = E_t^{Q^\beta}\left[h(T) \exp\left(-\int_t^T \left(r(s) 1_{\{h(s)>0\}} + r_A(s) 1_{\{h(s)\le 0\}} \right) ds \right) \right]$, where $\exp\left(-\int_t^T \left(r(s) 1_{\{h(s)>0\}} + r_A(s) 1_{\{h(s)\le 0\}} \right) ds \right)$ is the derivative of the price. By connecting the already stated pricing formula to Euler's equation, we can use standard risk allocation techniques to compute the trade contributions. Let the portfolio payoff $h(T)$ be the sum of

[49] When $h=0$, the first-order expansion is readily derived and leads to different left and right derivatives.

J individual trades $h(T)=h_1(T)+\cdots+h_J(T)$. Then according to Euler's allocation rule, the trade contribution of $h_j(T)$ to portfolio present value $h(t)$ is provided by:

$$E_t^{Q^\beta}\left[h_j(T)\exp\left(-\int_t^T\left(r(s)1_{\{h(s)>0\}}+r_A(s)1_{\{h(s)\le 0\}}\right)ds\right)\right].$$

This is not to be confused with the standalone present value:

$$E_t^{Q^\beta}\left[h_j(T)\exp\left(-\int_t^T\left(r(s)1_{\{h_j(s)>0\}}+r_A(s)1_{\{h_j(s)\le 0\}}\right)ds\right)\right].$$

Euler's allocation rule is totally additive, i.e. the sum of trade contributions is equal to the portfolio present value. Unlike the standalone present values, the trade contributions involve a standard (linear) expectation.

It can be seen that the trade contributions and marginal prices are computed quite similarly. Let us once again emphasize that the collateralization scheme is embedded in the shape of the generator and thus in the stochastic discount factor $\exp\left(-\int_t^T\left(r(s)1_{\{h(s)>0\}}+r_A(s)1_{\{h(s)\le 0\}}\right)ds\right)$.

In the perfect collateralization case, we introduced the suitable collateralized forward measures. This can readily be extended to locally proportional haircuts and therefore to asymmetric CSA. We can introduce a CSA and portfolio specific change of measure:

$$\frac{dQ_T^{h,CSA}}{dQ^\beta}=\frac{\exp\left(-\int_0^T\left(r(s)1_{\{h(s)>0\}}+r_A(s)1_{\{h(s)\le 0\}}\right)ds\right)}{E\left[\exp\left(-\int_0^T\left(r(s)1_{\{h(s)>0\}}+r_A(s)1_{\{h(s)\le 0\}}\right)ds\right)\right]}.$$

Let us further denote by $B_{h,CSA}(t,T)=E_t^{Q^\beta}\left[\exp\left(-\int_t^T\left(r(s)1_{\{h(s)>0\}}+r_A(s)1_{\{h(s)\le 0\}}\right)ds\right)\right]$, the collateralized discount bond price. Then, the trade contribution or the marginal price of $h_j(T)$ can simply be written as $B_{h,CSA}(t,T)E_t^{Q_T^{h,CSA}}\left[h_j(T)\right]$.

5.3. Counterparty risks and asymmetric CSA

In the perfect collateralization case (collateral amount $V(t)$ equals settlement price $h(t)$), and neglecting gap risks (jump in the collateral amount or the settlement price at default) or operational and legal risks, counterparty risk is completely eliminated. Default of one of the parties is simply an early termination of the contract, without loss and we actually took that costless early termination to compute the basic discounting formulas.

In an asymmetric CSA case, only one party is protected against the default of the other, thanks to posted collateral. One could either consider that the party that is not posting collateral is default-free and the pricing formula stated in previous sub-section is valid. However, in most cases, one would need to take into account the price impact of counterparty risk.

As outlined by Brigo and Morini (2010), setting-up the closeout payment in case of default is an intricate issue. The easiest way to go is to compute so called "risk-free"[50] present values, either by discounting at overnight rates, as in the computation of the settlement price of a perfectly collateralized trade, collateralized at overnight rate or by using an uncollateralized swap curve. The previous approach is questionable in our asymmetric CSA case. Using a recovery based upon settlement price in the spirit of Duffie and Huang is simpler and provides useful guidance: we can derive a settlement price coping both with CVA and collateral valuation adjustments.

First, let us recall that we only deal with the default risk of one party due to one one-side counterparty risk protection. Thus, depending on the party's point of view, the settlement price is free of CVA or DVA. We can take the point of view of a dealer, typically an investment bank, involved in trades with a sovereign counterparty and thus exposed to that sovereign default. We do not detail the resolution process of sovereign default on derivatives contracts and assume that sovereign default intensity process, λ and recovery rate δ are given. We chose here a fractional recovery of market value approach, as in Duffie and Huang (1996), a route which is not privileged by market practitioners, but which is meaningful in our context. Then, the loss in case of default, which occurs between t and $t+dt$ with conditional probability $\lambda(t)dt$ is simply equal to $(1-\delta)h(t)1_{\{h(t)>0\}}$. By considering, as before, the expected change in the settlement price, prior to default, we readily get:

$$E_t^{Q^\beta}\left[dh(t)\right]=\Big(\big(r+\lambda(1-\delta)\big)(s)1_{\{h(s)>0\}}+r_A(s)1_{\{h(s)\le 0\}}\Big)h(t)dt\,.$$

As above, we focused on the generator of the BSDE, which now takes the form $g(h)=\big(r+\lambda(1-\delta)\big)h^{+}+r_A h^{-}$ and therefore remains piecewise linear. The pre-default portfolio value is simply written as:

[50] "Risk-free" means that an uncollateralized swap curve or overnight rates in the interbank market are to be used in the discounting process. While this can be understood from an historical perspective, when the default component in effective fed funds rates or EONIA was neglected or when swaps contracts were deemed default-free and CVA/DVA effects could be neglected, the market terminology is misleading. "Recovery of swap", similarly to "recovery of Treasury" would be more explicit.

$$h(t)=E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T\left(\left(r+\lambda(1-\delta)\right)(s)1_{\{h(s)>0\}}+r_A(s)1_{\{h(s)\le 0\}}\right)ds\right)\right],$$

Where the term $\lambda(1-\delta)$ accounts for the counterparty risk adjustment.

5.4. Cost of funding and asymmetric CSA

It is worth remembering that the risk-free rate is not involved in the stochastic discount factor when considering a perfectly collateralized contract. Since the trades are financed through the collateral account, only the collateral rate (or expected rate of return) is involved. This is fortunate, since it is easier to deal with good proxies of instantaneous rates of return by using overnight interbank rates or repo rates, than with the more abstract instantaneous default-free rate.

However, in an asymmetric CSA case, the default free short-term rate is back in the stochastic discount term. This is because part of the deal is to be funded on the market. For a non defaultable party, without limitation on borrowed quantities, the usual abstract default-free rate might be considered even though market practitioners would rather use some funding rate. Let us remark that the default-free rate $r(.)$ is a free parameter in the discounting formula.

Shifting $r(.)$ is far from being innocuous for the price functional involved in an asymmetric CSA and thus from a trading and risk management point of view. Let us recall that depending on the sign of $r(.)-r_A(.)$, the price functional will be sub-additive (resp. super-additive) and concave (resp. convex), leading to strikingly different portfolio effects.

Regarding different borrowing and lending rates, it is worth, first, to recall theoretical results in a default-free setting. Let us denote by r_b, r_l the borrowing and lending short rates. Then, from Cvitanić and Karatzas (1993), Bergman (1995), Korn (1995), El Karoui et al. (1997), we know that the replication price[51] of an uncollateralized contract is given by:

$$h(t)=E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T\left(r_b(s)1_{\{h(s)>0\}}+r_l(s)1_{\{h(s)\le 0\}}\right)ds\right)\right].$$

This comes from solving the BSDE $E_t^{Q^\beta}\left[dh(t)\right]=\left(r_b(s)1_{\{h(s)>0\}}+r_l(s)1_{\{h(s)\le 0\}}\right)h(t)dt$, where once again the generator $g(h)=r_bh^+ + r_lh^-$ is a key tool.

Using the same pricing ideas as before, we readily obtain a simple pricing and discounting formula that accounts for CVA and funding costs for a portfolio of contracts with asymmetric collateralization.

[51] Different default-free borrowing and lending rates do not break perfect replication.

$$h(t)=E_t^{Q^\beta}\left[h(T)\exp\left(-\int_t^T\left(\left(r_b+\lambda(1-\delta)\right)(s)1_{\{h(s)>0\}}+r_A(s)1_{\{h(s)\le 0\}}\right)ds\right)\right].$$

Thus incorporating counterparty risk and funding cost in the computation of settlement prices under an asymmetric CSA simply involves a shift from the default-free rate r to $r_b+\lambda(1-\delta)$. From a practical perspective, this does not seem to change anything, since we substitute a free parameter to another. However, one might think it would be easier to calibrate the latter quantity, since it is related to the funding cost of the party that does not post collateral. As discussed earlier in the paper, we can compute the marginal impact of a new trade and trade contributions to book P&L in a linear setting.

Conclusion

In this paper, we kept as closely as possible to mainstream mathematical finance. It has been shown that well known concepts and tools can be used with minor adaptation to the collateralized pricing issues. A striking result is that the same generic discounting approach is applicable to a wide range of contracts, including collateralized and uncollateralized swaps.

Let us for example, restrict to the case of two OIS parties (see Duffie and Singleton (1997)), i.e. two parties funding consistently at the index overnight rate throughout the life of the contract. Under the CVA/DVA model of Duffie and Huang (1996), the present value of such a defaultable swap is obtained by discounting cash-flows at the index overnight rate. Thus, the predefault present value of an uncollateralized contract is equal to the settlement price of an otherwise identical collateralized contract on LCH.Clearnet (say). This shows that "systemic credit spread risk", as expressed as the difference between the unsecured overnight interbank rate is involved in very similar ways in the uncollateralized and collateralized markets. Of course, one would have to take into account the effect of initial margins, close-out conventions and idiosyncratic credit risk components. Nevertheless, these connections are not innocuous given the regulatory and accounting agendas.

Though we dealt comprehensively with various kinds of collateral and collateralization schemes, we left aside the pricing impacts of initial margins. This is an important topic but the rules governing the computations of initial margins are not yet stabilized. We did not investigate the numerical issues associated with solving BSDE in the asymmetric CSA case and the connections with the standard approaches to the computations of CVA. This is left for future research.

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Appendix A: Present value of a collateralized cash-flow.

This is the sum of the present value of the collateral account cash-flows and the present value of the promised cash-flow at payment date T (the second part is simply the value of the forward contract).

$$h(t) = E_t^{Q^\beta}\left[\int_t^T \left(r(s)ds - \frac{dA(s)}{A(s_-)}\right)h(s)e^{-\int_t^s r(u)du} + h(T)e^{-\int_t^T r(u)du}\right].$$

Solving this functional equation and the pricing methodology has been dealt with in a series of papers by Fujii, Shimada and Takahashi (see references). For the paper to be self-contained, we recall the solving of previous equation along the lines of the quoted papers.

Define $X(t) = e^{-\int_0^t r(s)ds}h(t) + \int_0^t e^{-\int_0^s r(u)du}\left(r(s)ds - \frac{dA(s)}{A(s_-)}\right)h(s)$ and substitute $h(t)$ as defined in equation (1.1). This yields:

$$X(t) = E_t^{Q^\beta}\left[e^{-\int_0^T r(u)du}h(T) + \int_0^T \left(r(s)ds - \frac{dA(s)}{A(s_-)}\right)h(s)e^{-\int_0^s r(u)du}\right]$$

It is worth noting that the term between brackets is simply the discounted (with the savings account) payoff of the collateralized deal, the first part corresponding to the forward payoff and the second to variation margins. Since the quantity between brackets does not depend upon t, $X(t)$ is a Q^{β} - martingale. Going back to the definition of $X(t)$ yields: $dX(t) = e^{-\int_0^t r(s)ds} dh(t) - e^{-\int_0^t r(s)ds} r(t)h(t)dt + e^{-\int_0^t r(s)ds}\left(r(t)dt - \frac{dA(t)}{A(t_{-})}\right) h(t)$, or equivalently:

$dX(t) = e^{-\int_0^t r(s)ds} dh(t) - e^{-\int_0^t r(s)ds} h(t)\frac{dA(t)}{A(t_{-})}$. As a consequence, we obtain the dynamics of $h(t)$ the present value of the (perfectly) collateralized trade: $dh(t) = h(t)\frac{dA(t)}{A(t_{-})} + \beta(t)dX(t)$. Let us remark that, provided that $h(t) \neq 0$, we can write:

$$\frac{dh(t)}{h(t)} = \frac{dA(t)}{A(t_{-})} + \frac{\beta(t)}{h(t)} dX(t) = r_A(t)dt + dM_A(t) + \frac{\beta(t)}{h(t)} dX(t).$$

Thus, $E_t^{Q^{\beta}}\left[\frac{dh(t)}{h(t)}\right] = E_t^{Q^{\beta}}\left[\frac{dA(t)}{A(t_{-})}\right] = r_A(t)dt$.

In the case of cash-collateral, $\frac{dA(t)}{A(t_{-})} = c(t)dt$ where $c(.)$ is the collateral rate associated with the cash-collateral account.

Define $\vartheta(t) = h(t)\mathrm{e}^{-\int_0^t r_A(s)ds}$. It can be seen that $\vartheta(t)$ is a Q^{β} - martingale. Since $\vartheta(t) = E_t^{Q^{\beta}}\left[\vartheta(T)\right]$, we obtain the basic pricing equation for the computation of the settlement price of a collateralized derivatives contract:

$$h(t) = E_t^{Q^{\beta}}\left[h(T)e^{-\int_t^T r_A(s)ds} \right].$$

Appendix B: Convexity adjustments

$E_t^{Q_T^B}\left[h(T)\right] = E_t^{Q_T^A}\left[\frac{dQ_T^B}{dQ_T^A}h(T)\right] \Big/ E_t^{Q_T^A}\left[\frac{dQ_T^B}{dQ_T^A}\right]$. Thus,

$$E_t^{Q_T^B}\left[h(T)\right] = E_t^{Q_T^A}\left[\frac{\exp\left(-\int_t^T (r_B - r_A)(s)ds\right)}{E_t^{Q_T^A}\left[\exp\left(-\int_t^T (r_B - r_A)(s)ds\right)\right]} h(T) \right]$$

It can easily be shown that: $E_t^{Q_T^A}\left[\exp\left(-\int_t^T (r_B - r_A)(s)ds\right)\right] = \frac{B_B(t,T)}{B_A(t,T)}$. Thus,

$$E_t^{Q_T^B}\left[h(T)\right] = E_t^{Q_T^A}\left[h_A(T)\right] + \frac{B_A(t,T)}{B_B(t,T)}\mathrm{Cov}_t^{Q_T^A}\left(\exp\left(-\int_t^T (r_B - r_A)(s)ds\right), h(T)\right)$$

which involves a covariance term between the payoff $h(T)$ and the stochastic spread terms $r_B - r_A$.

Let us consider the special case associated with a Brownian filtration. Since ξ_T^A is a positive Q^β - martingale, we have: $\frac{d\xi_T^A(t)}{\xi_T^A(t)} = \sigma_A(t,T)\, dW_A^\beta(t)$, $\xi_T^A(t) = \varepsilon\left(\int_0^t \sigma_A(s,T)\, dW_A^\beta(s)\right)$ where W_A^β is a Q^β - Brownian motion. Since $\xi_T^{B,A}(t) = \frac{\xi_T^B(t)}{\xi_T^A(t)}$ is a Q_T^A - martingale and using Girsanov theorem, we have:

$$\xi_T^{B,A}(t) = \varepsilon\left(\int_0^t \sigma_B(s,T)\, dW_B^{A,T}(s) - \sigma_A(s,T)\, dW_A^{A,T}(s)\right),$$

Where $W_B^{A,T}, W_A^{A,T}$ are Q_T^A - Brownian motions.

$$E_t^{Q_T^B}\left[h(T)\right] = E_t^{Q_T^A}\left[\frac{\xi_T^{B,A}(T)}{\xi_T^{B,A}(t)} h(T)\right] = E_t^{Q_T^A}\left[\varepsilon\left(\int_t^T \sigma_B(s,T)\, dW_B^{A,T}(s) - \sigma_A(s,T)\, dW_A^{A,T}(s)\right) h(T)\right].$$

$\sigma_A(t,T)$ is related to collateralized discount bond prices since from $B_A(t,T) = B_A(0,T)\xi_T^A(t)\exp\left(\int_0^t r_A(s)ds\right)$, we get $\frac{dB_A(t,T)}{B_A(t,T)} = r_A(t)dt + \sigma_A(t,T)\, dW_A^\beta(t)$.

Let us assume that $h(T) > 0$, thus $h_{A,T}(t) > 0$. $\frac{dh_{A,T}(t)}{h_{A,T}(t)} = \sigma_h(t,T)\, dW_A^{h,T}(t)$, where $W_A^{h,T}$ is a Q_T^A- martingale and $h(T) = E_t^{Q_T^A}\left[h(T)\right]\varepsilon\left(\int_t^T \sigma_h(s,T)\, dW_A^{h,T}(s)\right)$. This leads to the following relation between forward prices under collaterals A, B:

$$E_t^{Q_T^B}\left[h(T)\right] = E_t^{Q_T^A}\left[h(T)\right] E_t^{Q_T^A}\left[\varepsilon\left(\int_t^T \sigma_h(s,T)\, dW_A^{h,T}(s)\right)\varepsilon\left(\int_t^T \sigma_B(s,T)\, dW_B^{A,T}(s) - \sigma_A(s,T)\, dW_A^{A,T}(s)\right)\right]$$