Sovereign recovery schemes:

Discounting and risk management issues

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Abstract

We consider some pricing and risk management issues related to defaultable bonds, in the context of sovereign debt default and restructuring. Standard recovery schemes such as fractional recovery of market value, of Treasury and of face value are investigated: we discuss their consistency with market practice both from a pricing and a risk management perspective. We also pay attention to the tradable basic instruments such as defaultable discount bonds or IOs/POs that are the building blocks of traded level coupon bonds. Model-free pricing formulas are provided. Whatever the recovery framework, bond pricing formulas involve similar ingredients, such as par rates and defaultable level annuities. We also show that the fractional recovery of par, our preferred approach from an economical point of view, involves two discount curves, one for principal payments and one for coupon payments, a departure from the simplest bootstrapping and pricing engines.

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Introduction

Let us consider that we are given a set of traded defaultable level coupon bonds issued by a given sovereign. Our interest is to provide a consistent pricing framework: the model to be used is calibrated to market prices[2] and leads to arbitrage free prices of new bonds of the same issuer. These new bonds may involve different payment schedules, different maturities and/or strips of existing bonds. We are concerned with the relative prices provided by different pricing engines as the coupon rate varies. The purpose of this paper is to assess the consequences of recovery mechanisms on bond pricing, especially given that there is not always a clear best choice in the case of sovereign debt default and restructuring[3].

Four approaches will be considered. While, obviously, scheduled bond cash-flows are paid until default, what differs between the approaches is the cash-flow description in case of default.

- The first approach ("street approach") focuses upon the scheduled (or contractual) bond cash-flows and does not deal explicitly with default dates or recovery mechanisms. Though it is quite simple, it has a certain number of drawbacks, such as the possibility of inconsistent prices.
- In the second approach, at default time, the bond holder receives a fraction of the pre-default market value. This approach is known as "fractional recovery of market value". Even though, such an approach has been criticized for being unrealistic, it keeps some flavour due to its simplicity in discounting computations.
- The third approach is known in the academic literature as "fractional recovery of Treasury" though we will often use the terminology "fractional recovery of cash-flows" which is more in line with our analysis. Up to default time, the scheduled cash-flows are being paid while at default time, a proportional haircut is applied on all post-default payments, coupon or principal payments[4]. A potential example of such a restructuring mechanism would be a forced conversion of outstanding debt denominated in euro to debt denominated in a new and less valuable currency.
- The fourth approach is the well-known "fractional recovery of face value[5]". At default time, the claim is based on the nominal value of the bond[6], irrespectively of the coupon and maturity of the defaulted bond.

In order to conduct a cross-sectional analysis of defaultable bonds, a few preliminary basic assumptions are required. The considered default date and associated recovery mechanism

[2] Dealing with relevant calibration market prices is intricate, especially in the context of debt restructuring. One has to deal with market liquidity which can be scarce in the context of debt restructuring.
[3] While we focus upon standard recovery mechanisms, one could extend the scope of analysis. For instance, the 2003 Uruguay bond swap was associated with a lengthening of bond maturities, the coupon rate of bonds being unchanged.
[4] In the fractional recovery of Treasury, the bondholders receive a fraction of the present value of the post-default cash-flows where the discount rates are derived from a Treasury curve. This is equivalent to fractional recovery of cash-flows provided that Treasury is default-free. Duffie *et al.* (1996) use the terminology "fractional recovery of a default-free version of the same security".
[5] This approach is often referred to as fractional recovery of par as in Duffie [1998] or as fractional recovery of nominal. Guha [2003], Madan *et al.* [2006] use the "fractional recovery of face value" terminology.
[6] Plus potentially accrued interest.

have to be the same for all the considered bonds. When considering a CDS triggering event, assuming "Old R" would apply as is the case for Western European sovereigns, all bonds would have to be considered. However, if a selective restructuring process is involved, such that only bonds within a given range of maturities are to be restructured, then we should consider those bonds separately[7].

Fortunately enough, it is worth noting that the main results in the paper do not depend upon an arbitrary choice of recovery parameter. Connections between basic building blocks such as defaultable discount bonds and traded level coupon bonds are essentially model-free.

The key results of our paper are as follows:

- While the pricing formulas under fractional recovery of market values and fractional recovery of cash flows are different, once the model is calibrated to market prices they are reconciled. This means that a newly introduced bond will have the same price and risk under both methods. These methods are also consistent with the street approach.
- The situation is different under fractional recovery of face value. In this case a different discount curve has to be used for coupon and principal flows with principal flows more valuable than coupon flows. As a result, even after calibration, the prices of new bonds will be different from the ones computed under the two other methods: IOs will be cheaper and POs will be dearer.

The paper is organized as follows:

- Section 1 recalls the "street approach" to pricing and risk managing defaultable bonds.
- Section 2 connects the street approach to the fractional recovery of market value scheme.
- Section 3 discusses the applicability of the fractional recovery of Treasury method. It is shown that after calibration to quoted bond prices, it provides similar results to those obtained in the former approaches
- Eventually, Section 4 deals with the fractional recovery of par (or face value) approach.

1) "Street methods".

Further assumptions are made to avoid unnecessary notational burden. We recall that F will denote the face value, C the coupon paid at dates $t_1 = 1, \ldots, t_n = n$. We restrict the set of scheduled coupon payment dates to integers. $c = \frac{C}{F}$ will be the coupon rate. $T = t_n = n$ will denote the maturity date of the bond. Today's date will be denoted as t_0 or t and for simplicity, we may have $t_0 = 0$ [8].

[7] Andritzky [2006] provides a comprehensive review of issues in sovereign bond restructuring. Duffie *et al.* [2003] illustrate the specificities of sovereign distress focusing on Russian debt, different default treatments leading to possibly different discount factors.

[8] The pricing methodologies studied below can readily be extended to account for a larger set of payment dates (either discrete or continuous tenors) and amortization schemes or step-up coupons at the price of notational burden.

Though quite standard, the above cash-flow description of a bond is incomplete, since we have only specified the payments in the no-default case. To conduct a rigorous pricing and risk analysis of bonds, we should also have a view about payments at default time. This is a bit problematic since this cannot be stated unambiguously. Either for that practical reason, either because the possibility of default was neglected, market practices have conducted analyzes based only upon contractual cash-flows.

For instance, if P denotes today's price of a traded bond, the corresponding yield to maturity y is computed through: $P = \sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n}$[9].

Let us by $A(y,n) = \sum_{i=1}^{n} \frac{1}{(1+y)^i}$, the present value of an annuity of maturity $T = n$ discounted at y. Thanks to $1 - \frac{1}{(1+y)^n} = \sum_{i=1}^{n} \frac{y}{(1+y)^i}$, we can write:

$$P = F + (C - yF)A(y,n),$$

or $\frac{P}{F} = 1 + (c - y)A(y,n)$. These equations only relate a given bond price P and its yield to maturity y. However, if we were given another bond with different coupon C and maturity T, we could think of applying the same discounting yield y, which sounds reasonable provided that coupon rates and maturity are not too different from the traded bond and the traded prices are reliable. Then,

$$P_0(C,F,T) = F + (C - yF)A(y,T),$$

or $\frac{P_0(c,T)}{F} = 1 + (c - y)A(y,T)$ become actual pricing formulae where y is derived from the reference quoted bond[10]. It is even possible to leave aside any connection to quoted bonds and state that the yield to maturity to be applied to bonds is given exogenously. This was for instance the approach proposed by the IIF, in the case of the Greek bond swap, to value a stream of coupon payments (i.e. $F = 0$), with $y = 9\%$ or $y = 12\%$ and, in the case of a level coupon, the price being given by: $C \times A(y,T)$.

This elementary pricing approach can be extended to account for a set of bonds. A standard bootstrapping procedure consists in looking for non negative discount factors $B^*(0,i)$, $i = 1,2,\ldots$ such that the traded prices of bullet bonds P_T (with corresponding coupons C_T) for maturities $T = 1,2,\ldots$ fulfil :

[9] The computation of yields to maturities for bonds departs from the computation of an internal rate of return used elsewhere in finance. When assessing the internal rate of return of an investment involving risky cash-flows, one would rather use the expected cash-flows, rather than the cash-flows in the most favourable case, in our case, the no-default case. Clearly, this approach was designed in a low default environment and is likely to be challenged when being used under the current market environment

[10] By construction, these pricing formulae are consistent with the price P of the quoted bond (i.e. calibrated to the unique market price). In the following, for notational simplicity, we will omit the reference to, say coupon rate c or maturity T, but it should be made clear that P_0 is a function, thus the "pricing formula" terminology, not to be confused with market prices of traded bonds, that will be calibrating inputs. The same distinction will apply to the various recovery mechanisms studied below.

$$P_T = \sum_{i=1}^{T} C_T \times B^*(0,i) + F \times B^*(0,T)$$

This provides a simple pricing scheme for bonds, with non standard maturities or coupon rates. The price of a bond with face value F, coupon C and maturity T is given by:

$$P_1 = C \times \left(\sum_{i=1}^{T} B^*(0,i) \right) + F \times B^*(0,T).$$

The par rate for maturity T is defined as the coupon rate such $y_{1,T}$ that: $F = \sum_{i=1}^{T} y_{1,T} F \times B^*(0,i) + F \times B^*(0,T)$. We can thus write:

$$P_1 = F + (C - y_{1,T} F) \times \left(\sum_{i=1}^{T} B^*(0,i) \right),$$

or $\frac{P_1}{F} = 1 + (c - y_{1,T}) \times \left(\sum_{i=1}^{T} B^*(0,i) \right)$, $\sum_{i=1}^{T} B^*(0,i)$ being the value of an annuity with maturity T in that pricing context. This has the same flavour as the first pricing formula, $\frac{P_0}{F} = 1 + (c - y) A(y,T)$.

The formula $P_1 = C \times \left(\sum_{i=1}^{T} B^*(0,i) \right) + F \times B^*(0,T)$ is far from being innocuous. Let us consider two bonds, with same maturity T, same face value F and different coupons, C and C^*. Then, their prices will differ by $(C^* - C) \times \left(\sum_{i=1}^{T} B^*(0,i) \right)$. The prices will only be equal if $\sum_{i=1}^{T} B^*(0,i) = 0$ or equivalently $B^*(0,i) = 0$, thus all bond prices should be equal to zero. When looking at Greek bonds on the 23rd of November, 2011, we can notice that two bonds, one with coupon rate equal to 3.7% maturing on 20/07/2015 and one with coupon rate equal to 6.1%, maturing almost at the same time, on 20/08/2015, have almost the same clean price of 29% of face value. This provides a clear indication of the difficulties in using without caution risky discount factors in bond pricing. This point has been stressed by Andritzky [2005], distressed bonds being traded on a price basis rather than a spread basis.

Let us expand briefly about the above setting and calibration issues and consistency with market quotes. Given the simplified yearly time scale and assuming that maturities of traded bonds span that time scale, we can compute the discount factors by solving a triangular system of linear equations. For instance, if we assume that the one year bond is traded (with yearly coupon), we will have directly $B^*(0,1)$. We should then be able to compute $B^*(0,2)$ from $B^*(0,1)$ and the price of the two year bond $P_2 = C_2 \times \left(\sum_{i=1}^{2} B^*(0,i) \right) + F \times B^*(0,2) = C_2 \times B^*(0,1) + (F + C_2) \times B^*(0,2)$.

The above analysis can be used to assess what would the par rate $y_{1,T}$, i.e. the coupon rate of a new bond, with maturity T, issued at par, given a possibly distressed traded price P_T, associated with a level coupon C_T: $y_{1,T} = \frac{C_T}{F} + \frac{1}{\sum_{i=1}^{T} B^*(0,i)} \times \left(\frac{F - P_T}{F} \right)$. We can remark that, in the previous equation, the only quantity which is not directly observed from market quotes is the annuity $\sum_{i=1}^{T} B^*(0,i)$.

In our simplified setting, the set of payment dates for traded bonds coincides with the set of tenor dates involved in interest rate risk management [11]. The basic equation $P_1 = C \times \left(\sum_{i=1}^{T} B^*(0,i) \right) + F \times B^*(0,T)$ can be seen as a linear model involving latent factors $B^*(0,i)$. Only discount factors with maturities corresponding to payment dates are involved. A further step consist in splitting the discount factor $B^*(0,i)$ as $B^*(0,i) = B(0,i) \times S(0,i)$. We recall that $B(0,i)$ is the "default-free discount factor"[12].

2) Connection between the "street approach" and default modelling.

As can be seen from above, we were able to discuss the pricing of defaultable bonds, not only without using a mere notion of probability, but furthermore without any consideration of the recovery mechanism in case of default.

However, it is worth wondering whether there is some rationale behind the above approach: It is quite standard and well-studied in the default-free case. It is not that obvious that it can be extended to the defaultable case, whether we can leave aside the description of cash-flows in the case of default or whether we can associate the above discount factors to say, defaultable discount bond prices.

To go further in the pricing and risk management analysis, we need a proper description of cash-flows of defaultable bonds. This obviously includes the description of contractual cash-flows, coupons and principal in the case where no default or restructuring occurs before maturity, but also the recovery payment (or secondary claim) at default, if default occurs before maturity[13]. We will subsequently consider three recovery mechanisms, extensively studied in the academic literature, with well-known pro and cons, namely, fractional recovery

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[11] It is worth noting that the pricing scheme is incomplete since we would only know about the discount factors for tenor dates and we would be unable to price a new bond with payment dates not corresponding to the stated tenor dates. For this purpose, we need a second layer in order to map discount factors with discrete tenors to discount factors with continuous tenor dates: $B^*(0,i)$, $i = 1, \ldots, n \rightarrow B^*(0,t)$, $t \geq 0$. Once this interpolation or smoothing procedure has been specified, the pricing scheme for bonds, with non standard maturity and/or coupon payment dates is complete.

[12] This is to be understood as the discount factor associated with a base curve, usually derived for swap quotes or from a benchmark Treasury curve. Provided that no negative basis effect occurs, $S(0,i) < 1$. However, it may be that the base swap rate curve goes above the sovereign curve. Then, $S(0,i) > 1$, and we cannot think of *S* to be a survival function.

[13] Not so long ago, it was a common assumption to neglect the possibility of default in the case of sovereign bonds. For instance, in Elton and Green [1998], the starting point is the statement "*cash-flows of non-callable treasury securities are fixed and certain, simplifying the pricing of these assets to a present value calculation*".

of market value, of Treasury and of face value. This is not intended to deal with all practical cases, but provides good benchmarks for further analyses. For simplicity, we will assume that the default date is the same economical and mathematical object, whatever the recovery mechanism and τ will subsequently denote the default date of the bond.

A second step involves identifying relevant building blocks, streams of coupons, defaultable discount bonds[14], such that already traded bonds or tradable assets that could be obtained from stripping are linear combinations of the building blocks. We will subsequently deal with a frictionless market and thus with linear pricing rules. What is needed is a procedure that allows extracting the prices of the building blocks, consistently with prices of traded and liquid bonds. One may either think of a bootstrapping procedure as suggested above, when describing the "street approach". Actually, the pricing formulas shown below, though they account for default, do not require any probabilistic modelling.

Formal dynamical models of default will provide further structure regarding the pricing of the building blocks. The standard mathematical finance framework associated with reduced-form modelling involves a short-term default-free rate r, a default intensity λ[15] and a pricing probability measure Q[16]. Some extra technical restrictions are involved in order to have a simple connection with standard discounting techniques[17]. It should be made clear that the main results of the paper do not rely upon such modelling assumptions.

3) Fractional recovery of market value method

Under the fractional recovery of market value approach, the bond holders receive, at default time, a fraction, equal to the recovery rate δ, of the pre-default market value of the bond. Subsequently, δ will be a deterministic parameter[18]. Let us remark that the defaultable

[14] Regarding discount bonds, depending upon the context, we may have to deal with continuous tenors or remain in a discrete tenor setting.

[15] In this framework, default date is a totally inaccessible stopping time. Moreover, the compensator of the default indicator function is assumed to be absolutely continuous with respect to the Lebesgue measure. One may refer to λ as a pre-default intensity. The fractional recovery of market value approach is coupled here with the reduced-form approach (see Jarrow and Turnbull [1995], Jarrow *et al.* [1997]). When considering stochastic models of default, we will remain in the above setting throughout the paper.

[16] This risk-neutral measure is associated with the savings account numeraire. Existence of a default-free short rate is postulated.

[17] See the articles by Duffie *et al.* [1996], Duffie and Singleton [1999] and Collin-Dufresne *et al.* [2004]. In this approach, the arrival of information, i.e. the market filtration is given and includes observation of default date. Similar results have been obtained in a slightly different setting by Blanchet-Scalliet and Jeanblanc [2004], Bielecki *et al.* [2004] and the book by Bielecki and Rutkowski [2010]. In this the latter approach, the market filtration results from a progressive enlargement of a background filtration with the observation of default arrival. The above approaches are rather abstract and general. One can think of using more restrictive but easier to grasp modelling, for instance Cox processes as in Lando [1998, 2004], Duffie and Singleton [2003]. Such a framework, thanks to the conditional independence of default time and state variables, guarantees that the basic technical assumptions of the general approaches are fulfilled. When dealing with fractional recovery of market value, there is an extra degree of mathematical involvement. The payment in case of default involves the pre-default price, resulting in a recursive valuation problem and the need to solve for an integral equation.

[18] See Merrick [2001], Madan *et al.* [2006], Pan and Singleton [2008], Das and Hanouna [2009] for a relaxation of this constraint in a pricing framework.

discount bond prices have a rather tricky pattern since their price is scaled down by δ at default time. As was mentioned in the introduction, the fractional recovery of market value approach leads to easy to deal with computations, even though its economic significance and practical use have been questioned. Under the fractional recovery of market value approach, defaultable bonds can actually be seen as portfolios of defaultable discount bonds, which will be the basic tradable instruments.

The usual outcome of the above stochastic model of default and recovery is the writing of risky discount factors as: $B^*(0,i) = E^Q\left[\exp\left(-\int_0^i \left(r(s)+\lambda(s)\times(1-\delta)\right)ds\right)\right]$. It is worth noting that these discount factors are the pre-default prices of defaultable discount bonds with the same fractional recovery of par mechanism as level coupon bonds. Let us also emphasize that the recovery rate does not depend upon maturity of the defaultable discount bond and that, regarding default and recovery, only the process $\lambda(s)\times(1-\delta)$ is involved[19]. We can define the (risk-neutral) survival probabilities, i.e. the probabilities of not having defaulted up to some time t, as $S(i) = Q(\tau > i) = E^Q\left[\exp - \int_0^i \lambda(s)ds\right]$. The default-free discount factors are provided by $B(0,i) = E^Q\left[\exp - \int_0^i r(s)ds\right]$. It is often assumed that default-free rates and default intensities are independent. In the new global financial context, this simplifying assumption may be challenged but it provides further simplification in the computation of discount factors and risk analysis. From that latter point of view, interest rate and default (or spread) risks become somehow “orthogonal”. Assuming the recovery rate to be constant, we can write $B^*(0,i) = B(0,i)\times S(i)^{1-\delta}$, thus the spread term $S(0,i) = \frac{B^*(0,i)}{B(0,i)} = S(i)^{1-\delta}$ only involves quantities that are directly related to the default date and the recovery rate. Under the independence assumption between default-free short rates and default-intensities, a shift in default intensities is consistent with a shift in “zero-coupon spreads” (with continuous compounding). Thus, there is some kind of consistency between market practices, such as DV01 computations to credit spreads and stochastic modelling approaches.

Let us stress that the decomposition of defaultable discount bonds with positive recovery $B^*(0,i) = B(0,i)\times S(i)^{1-\delta}$ is model dependent and that defaultable discount bonds with zero recovery are artefacts, which may not be the case with other recovery mechanisms.

Though the above computations could be useful, for instance in assessing recovery risk, the building of defaultable discount bonds and thus their pricing from level coupon bonds is model free. Let us go back to cash-flow analyses and the example depicted above. We are given a zero coupon defaultable bond scheduled to mature in one year, and a two year defaultable level coupon bond with coupon C_2. It can readily be seen that holding the

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[19] Since only the process $\lambda(s)\times(1-\delta)$ is involved, one cannot disentangle recovery and the arrival of defaults. From that perspective, having a time dependent recovery rate is irrelevant. The assumption that all bonds, with same seniority deserve the same recovery seems almost obvious and in accordance with priority rules. However, sovereign default and restructuring is not governed by standard rules applicable to corporate entities ruled by, say, New York or English law. Regarding the Private Sector Involvement in Greek debt restructuring, different treatments according to bond maturity had been considered. Then the assumption of a common recovery rate applicable to all bonds has to be relaxed.

previous level coupon bond and being short C_2 units of one year defaultable bond (with face value being equal to 1) synthesizes a two year defaultable discount bond (with face value $F + C_2$) and the same fractional recovery of market value mechanism at default as the two constituent traded bonds. In the depicted static replication procedure, only defaultable bonds are involved and there is no need for default-free bonds to be traded. Thus, the discount factors $B^*(0,i)$ do correspond to prices of defaultable discount bonds and their prices can be computed from traded level coupon bonds in an algebraic and thus essentially model-free way.

4) Fractional recovery of Treasury method.

The recovery mechanism corresponds to a proportional haircut applied to all cash-flows after default. The cash-flows are scaled by δ, which is the recovery rate. We can then depict the effective cash-flows on a given defaultable bond as: $F_i 1_{\tau > i} + \delta F_i 1_{\tau \le i}$, where $F_i = C$ for $i = 1, \ldots, n-1$ and $F_T = F + C$. The fractional recovery of treasury mechanism has been considered by Jarrow and Turnbull [1995] or Jarrow *et al.* [1997] among others.

Regarding sovereign default, that recovery mechanism can be associated with a forced conversion of bonds issued, say in euros, to a new domestic currency, δ being thus the exchange rate at the time of the forced exchange.

Let us remark that at this elementary stage of cash-flow analysis and regarding recovery mechanism, one important, if not the most important assumption is the constancy of the recovery rate, at default time, across defaulted bonds, i.e. the recovery rate does not depend upon the maturity or the coupon rate.

Remaining at cash-flow analysis, the recovery rate needs to be actually known before the scheduled payment date following default, for the cash-flows to be sensibly defined. Thus, recovery rate neither needs to be known at pricing time nor to be given a specific probability distribution.

The cash-flows of coupon bonds can be seen as linear combinations of elementary cash-flows of the form $1_{\tau > i} + \delta 1_{\tau \le i}$ paid at dates $i = 1, \ldots, n$.

Conversely, let us go back once again to the two year level coupon bond example. It can be seen that the two year defaultable discount bond, paying $1_{\tau > 2} + \delta 1_{\tau \le 2}$ can be synthesized exactly the same way as in the fractional recovery of market value. Thus, in the fractional recovery of cash-flows, the discount factors $B^*(0,i)$ also correspond to prices of defaultable discount bonds and their prices can be computed from traded level coupon bonds in an algebraic and thus essentially model-free way. As in the recovery of market value approach, the replication only involves defaultable bonds and no default-free bonds need to be involved. However, it must be understood that, even though we use the same terminology “defaultable discount bond”, we do not speak of the same assets in the recovery of market value and in the recovery of cash-flow cases: the cash-flows of discount bonds and associate price processes do not coincide.

We have been able to give the basic equation $P_1 = C \times \left(\sum_{i=1}^{T} B^*(0,i) \right) + F \times B^*(0,T)$ some economic content and to relate the defaultable discount factors to the prices of some well-defined defaultable discount bonds.

To go further into pricing and risk analysis comparisons, we can make for a while the assumption that the recovery parameter is known from inception. Since we wrote the cash-flows of the defaultable bond as $\delta F_i + (1-\delta)F_i 1_{\tau > i}$, $i = 1, \dots, n$, they can be split into two streams, one which is default free, while the other one involves defaultable cash-flows $(1-\delta)F_i$ with zero recovery.

Let us deal first with the default-free component of the bond cash-flows. At this stage we assume that the default-free discount bonds for the relevant maturities $i = 1, \dots, n$ are tradable assets, with prices $B(0,i)$, $i = 1, \dots, n$. $A_{RF}(T) = \sum_{i=1}^{n} B(0,i)$ will denote the value of a default-free annuity of maturity $T = n$. The default-free bond par rate, denoted by r_T is such that $1 = r_T A_{RF}(T) + B(0,T)$.Today's price of the stream of default-free cash-flows δF_i, $i = 1, \dots, n$ is given by: $\delta \times \left(\sum_{i=1}^{n} CB(0,i) + FB(0,T) \right)$. This can also be written as: $\delta \times \left(F + (C - r_T F) A_{RF}(T) \right)$.

Let us now consider the stream of defaultable cash-flows. The building blocks and tradable assets to be considered are defaultable discount bonds with zero-recovery; their cash-flows are $1_{\tau > i}$ paid at $i = 1, \dots, n$. Let us denote by $\overline{B}(0,i)$ the corresponding prices and by $\overline{A}(T) = \sum_{i=1}^{n} \overline{B}(0,i)$, the value of a defaultable annuity with zero recovery of maturity $T = n$. We will also consider the par rate of a defaultable bond with zero-recovery, $\overline{r}_T$, being such that $1 = \overline{r}_T \overline{A}(T) + \overline{B}(0,T)$. Today's price of the stream of default-free cash-flows $(1-\delta)F_i 1_{\tau > i}$, $i = 1, \dots, n$ is then given by: $(1-\delta) \times \left(\sum_{i=1}^{n} C\overline{B}(0,i) + F\overline{B}(0,T) \right)$ and can also be written as: $(1-\delta) \times \left(F + (C - \overline{r}_T F) \overline{A}(T) \right)$. Thus, the bond price is provided by:

$$P_2 = F + \delta (C - r_T F) A_{RF}(T) + (1-\delta)(C - \overline{r}_T F) \overline{A}(T),$$

or $\frac{P_2}{F} = 1 + \delta (c - r_T) A_{RF}(T) + (1-\delta)(c - \overline{r}_T) \overline{A}(T)$. The bond price is a weighted average of a Treasury and a defaultable bond with zero recovery.

The par rate $y_{2,T}$ is defined by: $F = F + \delta (y_{2,T} - r_T) F A_{RF}(T) + (1-\delta)(y_{2,T} - \overline{r}_T) F \overline{A}(T)$. Thus:

$$\delta A_{RF}(T)(r_T - y_{2,T}) + (1-\delta)\overline{A}(T)(\overline{r}_T - y_{2,T}) = 0.$$

By combining the equations providing P_2 and the par rate $y_{2,T}$, we obtain the simple pricing expression:

$$P_2 = F + (C - y_{2,T} F) \times \left(\delta A_{RF}(T) + (1-\delta)\overline{A}(T) \right),$$

or equivalently:

$$\frac{P_2}{F} = 1 + (c - y_{2,T}) \times \left(\delta A_{RF}(T) + (1-\delta)\overline{A}(T) \right),$$

to be compared with $\frac{P_1}{F} = 1 + \left(c - y_{1,T}\right) \times \left(\sum_{i=1}^{T} B^*\left(0,i\right) \right)$. The pricing formula providing P_2 allows assessing a change in the coupon C on the bond price. A scaling factor of $\delta A_{RF}\left(T\right) + \left(1-\delta\right)\overline{A}\left(T\right)$ is involved to be compared with $\sum_{i=1}^{T} B^*\left(0,i\right)$ in the market practice approach. Let us remark that $\delta A_{RF}\left(T\right) + \left(1-\delta\right)\overline{A}\left(T\right) = \sum_{i=1}^{n} \delta B\left(0,i\right) + \left(1-\delta\right)\overline{B}\left(0,i\right)$

Calibration only involves the defaultable discount factors $B^*\left(0,i\right)$. Since calibrated discount factors are such that:

$$B^*\left(0,i\right) = \delta B\left(0,i\right) + \left(1-\delta\right)\overline{B}\left(0,i\right), \quad i = 1,\ldots,n\,,$$

prices of zero recovery defaultable discount bonds, $\overline{B}\left(0,i\right)$ do depend upon δ.

It is worth writing the pricing formula, associated with recovery of cash-flows, as:

$$P_2 = \sum_{i=1}^{n} C \times \overline{B}\left(0,i\right) + F \times \overline{B}\left(0,n\right) + \delta \times \left(\sum_{i=1}^{n} C \times \left(B\left(0,i\right) - \overline{B}\left(0,i\right)\right) + F \times \left(B\left(0,n\right) - \overline{B}\left(0,n\right)\right) \right).$$

Here, we have split the bond price in two parts. $\sum_{i=1}^{n} C \times \overline{B}\left(0,i\right) + F \times \overline{B}\left(0,n\right)$ is the bond price under zero-recovery while $\delta \times \left(\sum_{i=1}^{n} C \times \left(B\left(0,i\right) - \overline{B}\left(0,i\right)\right) + F \times \left(B\left(0,n\right) - \overline{B}\left(0,n\right)\right) \right)$ is the value of the recovery.

Let us have a closer look at the pricing and discounting formulas under fractional recovery of market value and of cash-flows. To ease the computations, we will assume that default-free rates and default intensities are independent. Under the fractional recovery of market value and of cash-flows, respectively, we can write:

$$\begin{cases} P_1 = \sum_{i=1}^{n} C \times B\left(0,i\right) S\left(i\right)^{1-\delta} + F \times B\left(0,n\right) S\left(n\right)^{1-\delta} \\ P_2 = \sum_{i=1}^{n} C \times B\left(0,i\right)\left(\delta + \left(1-\delta\right)S\left(i\right)\right) + F \times B\left(0,n\right)\left(\delta + \left(1-\delta\right)S\left(n\right)\right) \end{cases}.$$

Let us think that these are used for the sake of calibration to a set of traded defaultable bullet bonds. We can think of the default-free discount bonds being computed elsewhere and from these, a standing recovery rate and market prices of defaultable bond prices, the survival probabilities $S\left(i\right)$ are derived. Clearly, we will not get the same survival probabilities in the two approaches. Also, once the survival probabilities are calibrated, a shift in default intensity does not have the same impact on prices, depending upon the chosen recovery mechanism.

It is worth stressing that the building blocks involved in defaultable bonds with fractional recovery of cash-flows are defaultable discount bonds with the same recovery mechanism. Zero-recovery defaultable discount bonds can be split out of the latter only under stringent assumptions: availability of truly default-free discount bonds and knowledge of recovery parameter at pricing date.

Fortunately enough, we found two mechanisms, recovery of market value and of cash-flows leading to discounting contractual bond cash-flows with a set of risky discount factors as commonly done under market practice. The building blocks of bullet bonds are defaultable discount bonds in the two cases, even though the dynamics of such defaultable discount bonds are not the same. Also, bond formulas $\frac{P_1}{F}=1+\left(c-y_{1,T}\right)\times\left(\sum_{i=1}^{T}B^*\left(0,i\right)\right)$ and $\frac{P_2}{F}=1+\left(c-y_{2,T}\right)\times\left(\delta A_{RF}\left(T\right)+\left(1-\delta\right)\overline{A}\left(T\right)\right)$ do not involve the same defaultable level annuities. However, after calibration to a set of traded bonds, these two models are formally equivalent. The lead to the same prices and rate sensitivities of newly issued bonds.

5) Fractional recovery of face value.

In the fractional recovery of face value approach, the bond holder receives $\delta\times F$ at default time τ, i.e. a fraction of the face value[20]. Let us remind that, in the fractional recovery of cash-flows approach, the bond holder receives at default time τ [21], a fraction δ of the present value of the bond, where the discounting is based on default-free rates. This is the road followed by Merrick [2001], Andritzky [2005] or Vrugt [2011] in the context of sovereign bond pricing.

Let us first provide some qualitative insights and hints aimed at comparing the two recovery mechanisms with respect to pricing and risk management. In both approaches, we can depict the stream of cash-flows as the sum of two components:

- The first one corresponds to pre-default cash-flows, which are simply the scheduled cash-flows times an indicator of survival of the issuer at scheduled default time: $F_i 1_{\tau>i}$, where $F_i=C$ for $i=1,\ldots,n-1$ and $F_T=F+C$. We have already analysed the valuation of this stream of cash-flows, associated with a defaultable bond with zero-recovery. The replication of this component and the required building blocks are the zero-recovery defaultable discount bonds. The corresponding value is $C\times\left(\sum_{i=1}^{n}\overline{B}\left(0,i\right)\right)+F\overline{B}\left(0,n\right)$.
- While the pricing formulas under the “fractional recovery of cash-flows” share the same pre-default component, the default payment, i.e. the recovery components obviously differ[22]. In the fractional recovery of par, we have to consider a payment of the recovery rate at default time τ, provided that defaults occurs before maturity date T [23]. Unless in special and unrealistic cases, the building block associated with the

[20] For simplicity, we will not account for the promised running coupon. The subsequent stripping procedure between coupon and principal payments obviously still holds without that assumption.
[21] For simplicity and ease of comparison, we will assume for a while that the default date is the same under the two approaches and that only the recovery mechanism differs.
[22] Of course, this holds unless the recovery rate is equal to zero.
[23] For such a cash-flow to be properly defined, we only need for the recovery rate to be well defined as default date. In the fractional recovery of par framework, we need the existence of a proper secondary market of bonds at default time, which is not granted for loans, but makes sense for sovereign bonds. Then, the recovery rate is, by construction, the unique market price (per unit of nominal) of all distressed bonds. The building block associated to this cash-flow is quite similar to the default leg of a credit default swap. For practical purpose, given that relevant default times may differ on the credit default swap and the bond markets and given the magnitude of basis effects between the two markets,

default payment under fractional recovery of par cannot be readily replicated by using defaultable and default-free discount bonds. Thus, we think of the payment at default time, conditionally on default occurring before T being a new tradable asset[24].

Let us pinpoint some differences between the fractional recovery of cash-flows and the fractional recovery of face value approaches.

At default date τ, with fractional recovery of par, the bond holder receives $\delta \times F$ plus possibly some recovery on the running coupon. With fractional recovery of cash-flows, the bond holder receives δ times a level coupon bond with coupon C, face value F and payment dates corresponding to the remaining scheduled payment dates. Given the definition of recovery of cash-flows, there is no further default risk on the level coupon bond. Its value can then be computed as $\delta \times \left(\sum_{i>n} C \times B(\tau, i) + FB(\tau, n) \right)$. This shows that the sensitivity of the bond with fractional recovery of par, with respect to a shift in default-free rates (DV01), is smaller than the sensitivity of bond with fractional recovery of cash-flows. Indeed, with fractional recovery of par, the bond holder receives a given amount of cash at default date, while the bond holder is left with a bond position with fractional recovery of cash-flows.

When $\delta = 1$, there is no difference between the cash-flows of a default-free bond and of a defaultable bond under a fractional recovery of cash-flows. This is no longer the case with the recovery of face value mechanism: default becomes formally equivalent to a prepayment. If the coupon rate is above the default-free par rate, this will result in a loss for the bond holder. The converse is true. Thus, for bonds with small coupons (compared with default-free rates), default will be good news, provided that the recovery rate is high enough. With fractional recovery of face value, we may have the price of a defaultable bond above the price of a default-free bond with same scheduled cash-flows.

Another difference between recovery of face value and recovery of cash-flows is the schedule of payments. With recovery of cash-flows, the set of payment dates can be seen as the set of scheduled payments; in case of default, there is a haircut applicable at the scheduled payment dates. This leads to an essentially discrete tenor discounting. On the other hand, recovery of face value can lead to a cash payment at any date prior to default. Of course, we can restrict the set of default dates to bond payment dates, so that only the same discrete set of tenor dates, as in the recovery of cash-flows case, needs to be considered. This is somehow artificial and recovery of face value involves a continuous tenor modelling.

Even though payments up to default are the same in the two approaches, the pre-default pricing formula $C \times \left(\sum_{i=1}^{n} \overline{B}(0, i) \right) + F\overline{B}(0, n)$ is misleading from a risk management and a model risk perspective. We show below that in a number of cases, zero-recovery defaultable discount bonds can actually be synthesized from defaultable coupon bonds under the recovery

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reliance upon the CDS market for pricing and risk management on the bond market should be done with great caution, especially in the case of sovereign issuers.

[24] One could think of stripping a given defaultable bond into two assets: a stream of cash-flows equal to committed bond payments until default date, at which the bond contract is cancelled (zero recovery) and another stream of cash-flows, where some investors receive the recovered part of the bond at default time.

of face mechanism. On the other hand, one would not be able to split a zero-recovery defaultable discount bond in the recovery of cash-flows framework: Even in the hypothetical case where default-free discount bonds were to be traded, one would need to know the recovery rate from inception in order to disentangle the zero-recovery part.

To ease the computations and the analysis, let us turn to the case of a constant recovery δ and default date independent of default-free rates under the pricing measure Q. The value of all payments prior to default date is given by: $\sum_{i=1}^{n} C \times B(0,i)S(i) + FB(0,n)S(n)$, where S stands for the survival function of τ.

As for the default payment, we need to compute $E^{Q}\left[\delta \times FB(0,\tau)1_{\tau \le n}\right] = -\delta \times F \int_{0}^{n} B(0,t)dS(t)$, neglecting the price impact of the recovery on the running coupon. On the other hand, the value of the default payment with fractional recovery of cash-flows can be written as:

$$\delta \times \left(\sum_{i=1}^{n} C \times B(0,i)(1 - S(i)) + FB(0,n)(1 - S(n)) \right).$$

Up to a scaling factor δ, the difference between the two pricing formulas can be written as:

$$C \times \left(\sum_{i=1}^{n} B(0,i) \times (1 - S(i)) \right) + F \times \left(B(0,n) \times (1 - S(n)) + \int_{0}^{n} B(0,t)dS(t) \right).$$

This corresponds to the present value (with discounting at default-free rates) of an amortizing bond with outstanding nominal at time t equal to $F \times (1 - S(t))$ for $0 \le t < n$, maturity date n and coupon rate $\frac{C}{F}$. Since $S(0) = 1$, the initial outstanding nominal equals zero[25]. The par rate of the above amortization structure is the coupon rate such that the present value is equal to the nominal, i.e. 0. Let us remark that this par rate is a default-free rate[26]. If the coupon rate is equal to the par bond rate, then recoveries of cash-flows and of face value lead to the same price. If the coupon rate $\frac{C}{F}$ is higher than the previous par rate, then the price of the bond will be lower when considering the recovery of face value mechanism rather than recovery of cash-flows (using the same recovery rate).

The above formula also allows to assess the differences between fractional recovery of cash-flows and fractional recovery of face value, regarding the risk exposures to the default-free discount factors $B(0,t)$ and the spread terms $S(t)$.

Unlike in the previous recovery mechanisms, coupons and principal payments deserve different treatments.

In the fractional recovery of face value approach there is no claim on coupons after default[27]. Thus, the recovery rate will not be involved in the valuation of the stream of coupons. Stated otherwise, the value of the stream of coupons can be done with a zero recovery assumption.

[25] We have negative amortization up to maturity date.
[26] If the default-free rates do not depend upon maturity, then the par rate equals the unique default-free rate.
[27] Apart from the running coupon.

The replication of the stream of coupons thus only involves zero-recovery defaultable discount bonds.

Regarding the valuation of principal payments, in the fractional recovery of face value approach, the bond holder receives a fraction δ of the face value at default date, corresponding to principal acceleration while, when considering fractional recovery of cash-flows, the bond holder receives a fraction δ of the face value *at maturity date*. Thus the value of the principal payment is smaller in the case of fractional recovery of cash-flows (assuming non-negative default-free rates) than in the case of fractional recovery of face value.

Unsurprisingly, since in case of fractional recovery of face value, the bondholders have different claims on principal and coupons, this will result in different discounting treatments. Even though, in our approach, the pricing rules remains linear whatever the recovery mechanism, the discount factors to be applied to the contractual coupon and principal payments will differ leading to two different discounting curves. Such a feature draws a clear line between the pricing methods under the two recovery mechanisms.

Given these preliminary remarks, let us go back to a more formal approach and denote by $\hat{B}(0,T)$ today's price of a contract paying the recovery rate δ on the defaultable bonds, at default time τ, provided that τ is smaller that $T = n$. Then, the price of the bond can be written as:

$$P_3 = \sum_{i=1}^{n} C \times \overline{B}(0,i) + F \times \left(\overline{B}(0,n) + \hat{B}(0,n)\right).$$

As was stated qualitatively, different discount factors are involved for coupons, $B^C(0,t) = \overline{B}(0,t)$, and principal payments, $B^P(0,t) = \overline{B}(0,t) + \hat{B}(0,t)$ [28]. The discount factors related to principal payments are always higher than the discount factors applicable to coupon payments [29]. The coupons are priced with a zero-recovery assumption. The absence of arbitrage opportunities imply that $B^C(0,t) \le B(0,t)$ and $B^P(0,t) \le B(0,t)$.

At first sight, the pricing formula:

$$P_3 = \sum_{i=1}^{n} C \times B^C(0,i) + F \times B^P(0,n),$$

with $B^C(0,i) \le B^P(0,i)$ looks inconsistent with the linearity of the pricing rule. Actually, it is not. Linearity is still valid when applied to actual cash-flows received by bond holders. It is only the use of contractual coupon and principal payments in the pricing formula that lead to this seemingly inconsistency.

As was done for the two former recovery mechanisms, let us consider the calibration to traded level coupon bonds. Let us consider two one year maturity defaultable bonds, with face value F, one with coupon C and the other with coupon $C' \neq C$, with respective prices P and P'. Being long the first bond and short the second one, creates an exposure to coupon payment only and $B^C(0,1) = \dfrac{P' - P}{C' - C}$, i.e. the increase rate in bond prices of given maturity with coupon

[28] This is related to the structure of default payments and should not be confused with tax effects which are, for instance, discussed in McCulloch [1975] or Elton and Green [1998].

[29] $\hat{B}(0,t) > 0$ is the price associated with a positive cash-flow.

rate. Using the same procedure with two year level coupon bonds allows to synthesize a two year IO and derive $B^C(0,2)$. Given the prices of IOs and the prices of level coupon bonds, we readily synthesize and prices POs[30]. Clearly, the calibration is more involved since two sets of level coupon bonds with different coupon rates are required. As for the two former recovery mechanisms, the bootstrap calibration procedure makes clear the constituents of defaultable level coupon bonds.

Let us further investigate the pricing formula associated with the recovery of face value approach. The par rate $y_{3,T}$ is the coupon rate such that $F = \sum_{i=1}^{T} y_{3,T} F \times B^C(0,i) + F \times B^P(0,T)$.

By combining the equation providing P_3 and the definition of the par rate $y_{3,T}$, we obtain:

$$P_3 = F + (C - y_{3,T} F) \times \left(\sum_{i=1}^{T} B^C(0,i) \right),$$

or equivalently:

$$\frac{P_3}{F} = 1 + (c - y_{3,T}) \times \left(\sum_{i=1}^{T} B^C(0,i) \right),$$

which is to be compared with the pricing formulas associated with the recovery of market value and the recovery of Treasury approaches: $\frac{P_1}{F} = 1 + (c - y_{1,T}) \times \left(\sum_{i=1}^{T} B^*(0,i) \right)$ and: $\frac{P_2}{F} = 1 + (c - y_{2,T}) \times \left(\delta A_{RF}(T) + (1-\delta)\overline{A}(T) \right)$. Let us recall that after calibration to level coupon bonds, $\sum_{i=1}^{T} B^*(0,i) = \delta A_{RF}(T) + (1-\delta)\overline{A}(T)$ for all T and thus $y_{1,T} = y_{2,T}$. However, $y_{3,T}$ is different from $y_{1,T}$ and $y_{2,T}$ unless a par bond is actually traded for this maturity, which would be the case at origination.

Then, the par rate becomes a market observable for that maturity, denoted by y_T. In that case, calibration of models imply that $y_{1,T} = y_{2,T} = y_{3,T} = y_T$. We can then study the effect of a change of coupon rate on the prices for maturity *T*: $\frac{P_1}{F} = 1 + (c - y_T) \times \left(\sum_{i=1}^{T} B^*(0,i) \right)$, $\frac{P_2}{F} = 1 + (c - y_T) \times \left(\delta A_{RF}(T) + (1-\delta)\overline{A}(T) \right)$ and $\frac{P_3}{F} = 1 + (c - y_T) \times \left(\sum_{i=1}^{T} B^C(0,i) \right)$. These pricing formulas are quite simple and readily extend the well-known swap formulas. They only differ from one to another by the expression of the "risky level", i.e. the price of defaultable annuity of maturity *T*. For instance, $\sum_{i=1}^{T} B^C(0,i)$ is the price of a zero-recovery risky level annuity associated with the bond market[31].

[30] We do not deal here with practical issues such as fungibility, see Tuckman and Serrat [2011].

[31] This will usually differ from the risky level involved in credit default swap pricing since, for instance, CDS and bond default dates may differ.

Conclusion

It has been shown that the simplest discounting approaches to defaultable level coupon bonds could be made consistent with standard recovery mechanisms such as recovery of market value or recovery of cash-flows. More precisely, the defaultable discount factors can be seen as the prices of defaultable discount bonds with positive recovery[32]. Such defaultable discount bonds can be obtained by static replication from traded defaultable level bonds. It is worth noticing that no default-free bond is required in such analysis, which is good news given the difficulty of stating what should be a default-free discount curve. No assumption about recovery rates or about independence between default date and default-free rates is required for the discounting scheme to apply. Of course, it is possible to go along this way, at the price of model dependency.

Unfortunately, the preferred recovery mechanism, recovery of face value, is inconsistent with the assumption of a unique discounting curve. One has to consider principal and coupon payments separately. This leads to different calibration approaches from level coupon bonds. The bond building blocks involve defaultable discount bonds with zero-recovery on one hand and a specific instrument associated with payment of recovery at default time. As for the recovery of cash-flows, no assumption about recovery rates, default dates, default-free rates is required for the pricing scheme to apply.

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