A note on the risk management of CDOs

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Purpose of the paper
- To provide a framework for the risk management of CDO tranches on large indices
  - iTraxx, CDX
- In a conditionally independent upon default intensities framework

Thought provocative result
- Concentrate on the dynamic hedging of credit spread risk
  - Idiosyncratic and parallel credit spreads movements
- Default risk is statically hedged by diversification
  - Since default events are conditionally independent upon credit spreads
  - Insurance idea
A note on the risk management of CDOs

• Default risk
  – Default bond price jumps to recovery value at default time.
  – Drives the CDO cash-flows
  – Possibility of multiple defaults

• Credit spread risk
  – Changes in defaultable bond prices prior to default, due to shifts in credit quality or in risk premiums.
  – Changes in the marked to market of tranches
  – Increase or decrease the probability of future defaults
  – Changes in the level, the dispersion of credit spreads, the correlation between credit spreads

• Recovery risk
  – Magnitude of aggregate loss jumps is random
A note on the risk management of CDOs

• Overall view of the presentation
  – Learning curve for credit modelling, pricing and hedging
  – Risks and hedging issues as seen from different models
    ➢ Structural models
    ➢ Contagion or copula models
    ➢ Market practitioner’s approach
    ➢ Multivariate Poisson
    ➢ Aggregate loss models
    ➢ Intensity models
  – First to Default Swap example
  – Hedging large portfolios in intensity models
A note on the risk management of CDOs

- Learning curve for credit modelling
- Parallel with equity derivatives
  - Building of a gothic cathedral
  - Bachelier, Samuelson \( \mu \) instead of \( r \)
    - no notion of risk neutral measure, no notion of duplication cost
  - Black-Scholes (prior to Merton)
    - Risk-neutral measure thanks to the use of CAPM, not a perfect hedge
  - Black-Scholes (final version)
    - Local approach to the hedging
      - Eventually, Harrison-Kreps, Harrison-Pliska, martingale representation theorem, Girsanov and so on…
Learning curve for credit modelling…
- Start with a “risk-neutral” or pricing probability
  - Compute expectations of payoffs
  - Assumption of perfect markets
  - Pricing disconnected from hedging
- Use of intensity or reduced-form models:
  - Lando, Jarrow, Lando & Turnbull, Duffie & Singleton
- One step backward
  - Copula models:
    - static approach
    - Default intensities are deterministic between two default times
A note on the risk management of CDOs

- Learning curve for credit modelling (cont)
  - One further step backward
    - 1F Gaussian copula + base correlation
    - Not a probabilistic model, Arbitrage opportunities
    - Fall of the Roman Empire

- Incomplete market approaches?
  - Not used by investment banks

- Market practitioners’ approach
  - Take some copula model (boo!)
  - “bump” the marginal credit curves
  - Compute CDS credit deltas
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- Dynamic hedging of basket credit derivatives
  - Bielecki, Jeanblanc & Rutkowski [2006], Frey & Backhaus [2006]
    - Credit spreads are driven by defaults
    - Martingale representation theorem under the natural filtration of default times
      - Jacod (1975) or Brémaud, chapter III
    - Hedging instruments: credit default swaps
    - Complete markets under the assumption of no simultaneous defaults
- Static replication of basket credit derivatives with first to default swaps
  - Brasch [2006]: “A note on efficient pricing and risk calculation of credit basket products”
- Super-replication: Walker [2005]
- Mean-variance hedging (local minimization): Elouerkhaoui [2006]
Risk within CDO tranches

- Risks as seen from different models
  - Aggregate loss or collective models
    - Hedge with the index (iTraxx or CDX) only
  - Name per name or individual models
    - Hedging using the set of underlying CDS
      - Important issue for the hedging of equity tranches

- Hedging in different models
  - Structural models
  - Copula and contagion models
  - Practitioner’s approach
  - Multivariate Poisson models
  - Aggregate loss models
  - Econometric approach
  - Intensity models
• **Structural models:**
  – defaultable bonds seen as equity barrier options

• **Multiname credit derivatives can be perfectly hedged in a Black-Cox framework**
  – Defaults are predictable
  – **Not very realistic:**
    ➢ perfect correlation between equity returns and credit spreads
    ➢ Term-structure of credit spreads
  – **Huge numerical issues**
  – **Not so far from copula models:**
    ➢ Hull, Pedrescu & White
Models for multivariate credit risk analysis

- Contagion models or Copula models
  - Credit spreads are deterministic between two default dates
- Multivariate Poisson models
  - Allow for multiple defaults
- Aggregate loss models
  - Direct specification of loss dynamics
  - CDO tranches only involve European options on aggregate loss
  - Aggregate loss: Marked Point Process
- Intensity models
  - Cox or doubly stochastic Poisson processes, conditionally independent defaults
  - Defaults are not informative
  - No jumps in credit spreads at default times
Models for multivariate credit risk analysis

- Contagion models (interacting intensities)
  - Jumps of credit spreads of survival names at default times
  - Jarrow & Yu, Yu, Frey & Backhaus

- Copula models
  - Starting point: copula of default times
  - Copula specification states the dependence between default times
  - Marginal default time distributions are self-calibrated onto credit spread curves
  - Intensities in copula models
  - Related to partial derivatives of the copula
  - May be difficult to compute
  - Default intensities are deterministic between two default times
  - Jump at default times
  - Contagion effects in copula models
Hedging in different modeling framework

• Copula and contagion models: theory
  – Default intensities are only related to past defaults
  – In other words, credit spread risk derives from default risk

• Smooth copula precludes simultaneous defaults
  – In previous models, perfect hedge of multiname credit derivatives with single name CDS
  – Complete markets
  – Representation theorems for multivariate point processes
  – Only default risk, no “true” credit spread risk
  – Work in progress

➢ Bielecki, Jeanblanc & Rutkowski
Copula models: practice very different from theory
- Practical implementation of hedging strategies
- **Focus on credit spread risk only**
- Price of a CDO tranche depends upon marginal credit curves and the copula
- Compute CDS hedge ratio by bumping the marginal credit curves and compute the CDO price increment
- Local sensitivity analysis
  - Model dependent
  - No guarantee that local hedging leads to a correct global hedge
  - Does gamma effects offset theta effects?
• Credit deltas within a stochastic correlation model
  - Burstchell et al
Copula models: gamma effects

Homogeneous portfolio

- Gamma matrix of a CDO tranche (wrt credit spreads)
  \[
  \begin{pmatrix}
    I & B & B & B & B \\
    B & I & B & B & B \\
    B & B & I & B & B \\
    B & B & B & I & B \\
    B & B & B & B & I
  \end{pmatrix}
  \]

- \((s_1, \ldots, s_n)\) change in credit spreads
  
  - Assume credit delta hedging with CDS
  - First order change in PV are equal to zero
Copula models: gamma effects

- Assume $s_2 = \cdots = s_n = 0$

  ➢ Change in PV $\frac{I}{2} s_1^2$ idiosyncratic gamma effect

- Assume $s_1 = \cdots = s_n = s$

  ➢ Change in PV $\frac{n}{2} (I + (n - 1)B)s^2$ parallel gamma

Homogeneous portfolio

- Credit spread covariance matrix

$$
\sigma^2 \Delta t 
\begin{pmatrix}
1 & \rho & \rho & \rho & \rho \\
\rho & 1 & \rho & \rho & \rho \\
\rho & \rho & 1 & \rho & \rho \\
\rho & \rho & \rho & 1 & \rho \\
\rho & \rho & \rho & \rho & 1
\end{pmatrix}
$$

- Expected gamma P&L

$$
= n \frac{\sigma^2 \Delta t}{2} \left( (1 - \rho)I + \rho(I + (n - 1)B) \right),
$$

- $(n - 1)B \rho$ high spread correlation sensitivity
Hedging CDO tranches in the base correlation approach
  - Tranchelets on standard indices
  - Bespoke portfolios
Correlation depends upon the expected loss of the tranche
Change in credit spreads changes the marginal credit curves and the implied correlation parameter
  - Sticky deltas
Still main focus upon credit spread hedging
  - Still dispersion risk (idiosyncratic gamma) and parallel spread risk
Hedging CDO tranches with liquid tranches
- Case of tranchelets on iTraxx or CDX
- Not the same hedging instruments

Entropic calibration
- Perfect copula type approach
- Start from some specification of conditional default probabilities
- $g_0$ a priori density function of conditional default probabilities
- Look for some a posteriori density function of cdp:

$$
\min_g \int g(p) \ln \frac{g(p)}{g_0(p)} dp
$$

- consistency constraints with liquid tranches prices

$$
\int_0^1 (p-k_i)^+ g(p) dp = \pi_i
$$

$$
g(p) = g_0(p) \exp \left( \lambda + \sum_{i=0}^{I} \lambda_i (p-k_i)^+ \right)
$$

- Hedge ratios: compute partial derivatives of tranchelets wrt $\pi_i$
• Multivariate Poisson models
  – Shock models
  – Default indicators are driven by a multivariate Poisson model
    ➢ Lindskog & McNeil, Elouerkhaoui, Duffie & Singleton
  – Common and idiosyncratic shocks
    ➢ Common shocks can be fatal or non fatal
    ➢ A name can survive a non fatal shock
  – Armageddon risk
    ➢ possibly large values for senior tranches
  – Intensities are deterministic between two shocks
    ➢ Not really any credit spread risk
• Multivariate Poisson models
  – Possibility of simultaneous defaults
    ➢ Name 1 and 2 may default altogether
    ➢ Name 1 and 3 may default altogether
    ➢ Name 2 and 3 may default altogether
    ➢ Name 1, 2, and 3 may default altogether
  – This drives the dependence
  – **High degree of default risk incompleteness**
    ➢ \(2^n\) states of the world
    ➢ \(n\) hedging instruments (single name CDS)
Models for multivariate credit risk analysis

- **Aggregate loss models**
  - Increasing Market Point Process
  - Aggregate loss intensity = sum of name default intensities
  - Magnitude of jumps = 1 – recovery of defaulted name

- **Markovian models**
  - SPA, Schönbucher
  - Markov chain (or more general) processes for the aggregate loss

- **Non Markovian**
  - Giesecke & Goldberg
  - Self-exciting processes, Hawkes, ACD type
  - Loss intensity only depends upon past losses
  - Top-down approach?
Hedging in different modeling framework

- Hedging in aggregate loss models
  - No notion of idiosyncratic gamma
  - Individual credit spreads are perfectly correlated
  - Jumps in aggregate loss process (default risk)
  - Change in loss intensity: parallel Gamma
- Hedging on a name per name basis
- Or based upon the index: same hedge ratios for all names
  - Hedging equity tranche with an aggregate loss model can become problematic
  - High sensitivity to heterogeneity between credit spreads
  - Hedge ratios for riskier names are likely to be higher
  - Does not take into account idiosyncratic gamma
Models for multivariate credit risk analysis

- Econometric approach to credit spread hedging
- Hedging liquid tranches with the index
  - iTraxx or CDX
  - Look for historical data on tranche premiums and index credit spread
  - Try to relate through some regression analysis changes in tranche premiums to changes in spreads
  - Check the hedging performance of different models
    - Houdain & Guegan
    - Similar ideas in equity derivatives markets
    - Baskhi, Cao & Chen
• **Intensity models**
  – Default arrivals are no longer **predictable**
  – Model conditional local probabilities of default $\lambda(t)\ dt$
  – $\tau$: default date, $\lambda(t)$ **risk intensity** or **hazard rate**
    
    $$\lambda_i(t)\ dt = P[\tau_i \in [t, t + dt] | \tau_i > t]$$
    
    – **Marginal default intensity**

• **Multivariate case: no simultaneous defaults**
  – Model starts from specifying default intensities

• **Multivariate Cox processes**
  – Credit spreads do not jump at default times
  – Duffie Singleton, Lando, …
Consider a basket of $M$ defaultable bonds
- multiple counterparties
• First to default swaps
  - protection against the first default
• Hedging and valuation of basket default swaps
  - involves the joint (multivariate) modeling of default arrivals of issuers in the basket of bonds.
  - Modeling accurately the dependence between default times is a critical issue.
Hedging **First to default swaps**: introduction

- **Hedging Default Risk in Basket Default Swaps**
- **Example:** first to default swap from a basket of two risky bonds.
  - If the first default time occurs before maturity,
  - The seller of the first to default swap pays the non recovered fraction of the defaulted bond.
- **Assume that the two bonds cannot default simultaneously**
  - We moreover assume that default on one bond has *no effect* on the credit spread of the remaining bond.
- **How can the seller be protected at default time?**
  - The only way to be protected at default time is to hold **two** default swaps with the *same nominal* than the nominal of the bonds.
  - The *maturity* of underlying default swaps *does not matter*. 
Hedging First to default swaps: introduction

• Some notations:
  – \( \tau_1, \tau_2 \) default times of counterparties 1 and 2,
  – \( \mathcal{H}_t \) available information at time \( t \),
  – \( P \) historical probability,
  – \( \lambda_1, \lambda_2 \) : (historical) risk neutral intensities:
    \[
    P\left[ \tau_i \in [t, t + dt \mid \mathcal{H}_t] = \lambda_i dt, \ i = 1, 2 \right.
    \]

• Assumption: « Local » independence between default events
  – Probability of 1 and 2 defaulting altogether:
    \[
    P\left[ \tau_1 \in [t, t + dt \mid \mathcal{H}_t] , \tau_2 \in [t, t + dt \mid \mathcal{H}_t] = \lambda_1 dt \times \lambda_2 dt \text{ in } (dt)^2 \right.
    \]
  – Local independence: simultaneous joint defaults can be neglected
Building up a tree:
- Four possible states: \((D,D), (D,ND), (ND,D), (ND,ND)\)
- Under no simultaneous defaults assumption \(p_{(D,D)}=0\)
- Only three possible states: \((D,ND), (ND,D), (ND,ND)\)
- Identifying (historical) tree probabilities:

\[
\begin{align*}
\lambda_1 dt & \quad (D, ND) \\
\lambda_2 dt & \quad (ND, D) \\
1 - (\lambda_1 + \lambda_2) dt & \quad (ND, ND)
\end{align*}
\]

\[
\begin{cases}
p_{(D,D)} = 0 \Rightarrow p_{(D,ND)} = p_{(D,D)} + p_{(D,ND)} = p_{(D,\cdot)} = \lambda_1 dt \\
p_{(D,D)} = 0 \Rightarrow p_{(ND,D)} = p_{(D,D)} + p_{(ND,D)} = p_{(\cdot,D)} = \lambda_2 dt \\
p_{(ND,ND)} = 1 - p_{(D,\cdot)} - p_{(\cdot,D)}
\end{cases}
\]
Hedging First to default swaps: introduction

- Cash flows of (digital) CDS on counterparty 1:
  - $\lambda_1 \phi_1 \, dt$ CDS premium, $\phi_1$ default risk premium

\[
\begin{align*}
\lambda_1 dt & \quad 1 - \lambda_1 \phi_1 dt \quad (D, ND) \\
\lambda_2 dt & \quad -\lambda_1 \phi_1 dt \quad (ND, D) \\
1 - (\lambda_1 + \lambda_2) dt & \quad -\lambda_1 \phi_1 dt \quad (ND, ND)
\end{align*}
\]

- Cash flows of (digital) CDS on counterparty 1:

\[
\begin{align*}
\lambda_1 dt & \quad -\lambda_2 \phi_2 dt \quad (D, ND) \\
\lambda_2 dt & \quad 1 - \lambda_2 \phi_2 dt \quad (ND, D) \\
1 - (\lambda_1 + \lambda_2) dt & \quad -\lambda_2 \phi_2 dt \quad (ND, ND)
\end{align*}
\]
Hedging First to default swaps: introduction

- Cash flows of (digital) first to default swap (with premium $p_F$):
  
  $\lambda_1 dt \quad 1 - p_F dt \quad (D, ND)$

  $\lambda_2 dt \quad 1 - p_F dt \quad (ND, D)$

  $1 - (\lambda_1 + \lambda_2) dt \quad -p_F dt \quad (ND, ND)$

- Cash flows of holding CDS 1 + CDS 2:

  $\lambda_1 dt \quad 1 - (\lambda_1 \phi_1 + \lambda_2 \phi_2) dt \quad (D, ND)$

  $\lambda_2 dt \quad 1 - (\lambda_1 \phi_1 + \lambda_2 \phi_2) dt \quad (ND, D)$

  $1 - (\lambda_1 + \lambda_2) dt \quad - (\lambda_1 \phi_1 + \lambda_2 \phi_2) dt \quad (ND, ND)$

- Absence of arbitrage opportunities imply:
  - $p_F = \lambda_1 \phi_1 + \lambda_2 \phi_2$
  - Perfect hedge of first to default swap by holding 1 CDS 1 + 1 CDS 2
Three possible states: \((D, ND)\), \((ND, D)\), \((ND, ND)\)

Three tradable assets: CDS1, CDS2, risk-free asset

The market is still « complete »

Risk-neutral probabilities

- Used for computing prices
- Consistent pricing of traded instruments
- Uniquely determined from CDS premiums

\[
\begin{align*}
p_{(D,D)} &= 0, \quad p_{(D,ND)} = \lambda_1 \phi_1 dt, \quad p_{(ND,D)} = \lambda_2 \phi_2 dt, \quad p_{(ND,ND)} = 1 - (\lambda_1 \phi_1 + \lambda_2 \phi_2) dt 
\end{align*}
\]
• **hedge ratios** for first to default swaps
• Consider a first to default swap associated with a basket of two defaultable loans.
  – Hedging portfolios based on standard underlying default swaps
  – Hedge ratios if:
    ➢ simultaneous default events
    ➢ *Jumps* of credit spreads at default times
• Simultaneous default events:
  – If counterparties default *altogether*, holding the *complete* set of default swaps is a conservative (and thus expensive) hedge.
  – In the *extreme* case where default *always* occur altogether, we only need a single default swap on the loan with largest nominal.
  – In other cases, holding a *fraction* of underlying default swaps *does not hedge default risk* (if *only one* counterparty defaults).
Hedging First to default swaps: introduction

• Default hedge ratios for first to default swaps and contagion
• What occurs if there is a jump in the credit spread of the second counterparty after default of the first?
  – default of first counterparty means bad news for the second.
  – Contagion effects
• If hedging with short-term default swaps, no capital gain at default.
  – Since PV of short-term default swaps is not sensitive to credit spreads.
• This is not the case if hedging with long term default swaps.
  – If credit spreads jump, PV of long-term default swaps jumps.
• Then, the amount of hedging default swaps can be reduced.
  – This reduction is model-dependent.
Hedging First to default swaps: introduction

- Default hedge ratios for first to default swaps and stochastic credit spreads

- If one uses short maturity CDS to hedge the FTD?
  - Sell protection on FTD
  - Buy protection on underlying CDS
  - Short maturity CDS: no contagion
  - But, roll-over the hedge until first to default time
  - Negative exposure to an increase in CDS spreads

- If one uses long maturity CDS to hedge the FTD
  - unknown cost of unwinding the remain CDS
  - Credit spreads might have risen or decreased
Hedging credit spread risk for large portfolios

- When dealing with the risk management of CDOs, traders concentrate upon credit spread and correlation risk.
- What about default risk?
  - For large indices, default of one name has only a small effect on the aggregate loss.
- Model framework
  - Given probability $Q$ such that:
    - Defaultable bond prices are martingales
    - Default times follow a multivariate Cox process
    - $Q$ equivalent to historical probability $P$
    - Bounded risk premiums
Hedging credit spread risk for large portfolios

- Notations and model framework
  - $\tau_1, \ldots, \tau_n$ default times
  - $N_i(t) = 1_{\{\tau_i \leq t\}}, i = 1, \ldots, n$ default indicators
  - $H_t = \bigvee_{i=1, \ldots, n} \sigma(N_i(s), s \leq t)$ natural filtration of default times
  - $F_t$ background (credit spread filtration)
  - $G_t = H_t \lor F_t$ enlarged filtration, $P$ historical measure
  - $l_i(t, T), i = 1, \ldots, n$ time $t$ price of an asset paying $N_i(T)$ at time $T$
Assumption 1 There exists a probability $Q$ equivalent to $P$ such that:

1. for $i = 1, \ldots, n$, the price processes of defaultable claims $l^i(., T)$ are $(Q, \mathcal{G})$ martingales:

$$
l^i(t, T) = E^Q[N_i(T) \mid \mathcal{G}_t], \quad (2.1)$$

for $0 \leq t \leq T$.

2. the default times follow a multivariate Cox process:

$$
\tau_i = \inf \left\{ t \in \mathbb{R}^+, U_i \geq \exp \left( - \int_0^t \lambda_{i,u} du \right) \right\}, \quad i = 1, \ldots, n \quad (2.2)
$$

where $\lambda_1, \ldots, \lambda_n$ are strictly positive, $F$ - progressively measurable processes, $U_1, \ldots, U_n$ are independent random variables uniformly distributed on $[0,1]$ under $Q$ and $F$ and $\sigma(U_1, \ldots, U_n)$ are independent under $Q$.

3. $E^Q \left[ \left( \frac{dP}{dQ} \right)^2 \right] < \infty^5$. 
Remarks
- For notational simplicity default-free rates are equal to zero
- Existence of $n$ hedging defaultable bonds
  - Could be $n$ CDS as well
- Existence of (non unique) martingale measure $Q$ from perfect arbitrage free markets
- Equivalence between:
  - $Q$-multivariate Cox process
  - Or “conditionally independent defaults”
  - Or “no contagion effects”
  - Or “defaults are non informative”
  - Or $(H)$ hypothesis holds
Lemma 2.1 Every $(Q, F)$ square integrable martingale is also a $(Q, G)$ square integrable martingale.

- No contagion effects
  - credit spreads drive defaults but defaults do not drive credit spreads
  - For a large portfolio, default risk is perfectly diversified
  - Only remains credit spread risks

Remark 2.2 While $(\tau_1, \ldots, \tau_n)$ is a multivariate Cox process under $Q$, it may not be a Cox process under $P$. For instance, we may have some contagion effects under $P$ (see Kusuoka [18]).

Remark 2.3 The joint survival function is such that $S(t_1, \ldots, t_n) = E^Q [\prod_{i=1}^n \exp(-\Lambda_i, t_i)]$ for $t_1, \ldots, t_n \in \mathbb{R}^+$. Since it is continuous, we must have $Q(\tau_i = \tau_j) = 0$ for $i \neq j$ which precludes simultaneous defaults.

- No simultaneous defaults
  - Otherwise market would be incomplete
Hedging credit spread risk for large portfolios

- **Purpose:** hedging of stylized CDOs, i.e. options on the aggregate loss

We will further consider payoffs of the type \((l_n(T) - K)^+ = \left(\frac{1}{n} \sum_{i=1}^{n} N_i(T) - K\right)^+\), for some \(K \in [0, 1]\) corresponding to so-called **zero-coupon CDOs**. Though zero-coupon CDOs are actually traded in the market, the most commonly traded CDOs involve more complex loss payments (see Laurent and Gregory [21]). To illustrate the risk management approach, we think it is more suitable to deal with simple payoffs.

- **Practical hedge is extremely tricky**
  - Need to hedge both default and credit spread risk
  - Recall that traders focus mainly on credit spread risk
  - Since default risk is already partly diversified at the index level
  - Forget about default? back to the \(F\) filtration
Hedging credit spread risk for large portfolios

- Construction of the hedging strategy
- Step 1: consider some pseudo defaultable bonds
  - i.e. project defaultable bond prices on the $F$ filtration
  - shadow bonds similar to well diversified portfolios
    
    Björk & Naslund, de Donno
    - shadow market only involves credit spread risk
- Step 2: approximate the CDO tranche payoff
  - Replace CDO payoff by its smoothed projection on the $F$ filtration
Hedging credit spread risk for large portfolios

• Step 3: compute perfect hedge ratios
  ➢ With respect to pseudo defaultable bonds 1, …, n
  ➢ Assume that Shadow market is complete
    – This can be relaxed (see paper)
  ➢ Numerical issues are left aside
    – High dimensionality
    – Markovian
    – Use of semi-analytical techniques
    – Not detailed in the paper

• Step 4: apply the hedging strategy to the true defaultable bonds
Main result

- Bound on the hedging error following the previous hedging strategy
- **When hedging an actual CDO tranche with actual defaultable bonds**
- Hedging error decreases with the number of names

- Default risk diversification

Provides a hedging technique for CDO tranches

- Known theoretical properties
- Good theoretical properties rely on no simultaneous defaults, no contagion effects assumptions
Hedging credit spread risk for large portfolios

• Technical background
  – Projection of default indicators on the information generated by credit spreads
  – Smooth projection of the aggregate loss
  – No default risk in the market with incomplete information

Definition 3.1 We denote by $p^i(.)$, the default-free running loss process associated with name $i \in \{0, \ldots, n\}$, which is such that for $0 \leq t \leq T$:

$$p^i(t) \triangleq E^Q[N_i(t) \mid F_t] = Q(\tau_i \leq t \mid F_t) = 1 - \exp(-\Lambda_{i,t}). \quad (3.1)$$

Definition 3.2 The default free $T$ forward loss process associated with name $i \in \{0, \ldots, n\}$, denoted by $p^i(., T)$ is such that for $0 \leq t \leq T$:

$$p^i(t, T) \triangleq E^Q[p^i(T) \mid F_t] = E^Q[N_i(T) \mid F_t] = Q(\tau_i \leq T \mid F_t). \quad (3.2)$$
Definition 3.5 default-free aggregate running loss process The default free aggregate running loss at time $t$ is such that for $0 \leq t \leq T$:

$$p_n(t) \Delta \frac{1}{n} \sum_{i=1}^{n} p^i(t).$$

(3.7)

Assumption 2 There exists some bounded $\mathcal{F}$-predictable processes $\theta_1(\cdot), \ldots, \theta_n(\cdot)$ such that:

$$(p_n(T) - K)^+ = E^Q [(p_n(T) - K)^+] + \frac{1}{n} \sum_{i=1}^{n} \int_0^T \theta_i(t) d p^i(t, T) + z_n,$$

(4.2)

where $z_n$ is $\mathcal{F}_T$-measurable, of $Q$-mean zero and $Q$-strongly orthogonal to $p^1(\cdot, T), \ldots, p^n(\cdot, T)$.

Remark 4.3 The previous equation is simply the $(Q, \mathcal{F})$ Galtchouk - Kunita - Watanabe decomposition of $(p_n(T) - K)^+$. $\theta_1(\cdot), \ldots, \theta_n(\cdot)$ correspond to the optimal $(Q, \mathcal{F})$ mean-variance hedging strategy based upon the abstract forward price processes $p^1(\cdot, T), \ldots, p^n(\cdot, T)$. 
Remark 4.5 The key point in Assumption (2) is the boundedness of the $\theta_i$’s. Let us remark that the individual credit deltas are equal to $\frac{\theta_i(t)}{n}$. For simplicity, we will thereafter assume that $0 \leq \theta_i(.) \leq 1$ for $i = 1, \ldots, n$. This boundedness assumption is related to the propagation of convexity property. We refer to Bergenthum and Rüschendorf [1], Ekström and Tysk [11] and the references therein for some discussion in a multivariate jump diffusion setting.

**Proposition 1** Under Assumptions (1) and (2), the hedging error $\varepsilon_n$ defined as:

$$
\varepsilon_n = (ln(T) - K)^+ - E^Q [(ln(T) - K)^+] - \frac{1}{n} \sum_{i=1}^{n} \int_0^T \theta_i(t) dt^j(t, T),
$$

(4.4)

is such that $E^P [|| \varepsilon_n ||]$ is bounded by:

$$
\frac{1}{\sqrt{2n}} \left( 1 + \left( E^Q \left[ \left( \frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \right)^{1/2} + \frac{1}{n} \left( E^Q \left[ \left( \frac{dP}{dQ} \right)^2 \right] \right)^{1/2} (\sum_{i=1}^{n} (Q(\tau_i \leq T) + E^Q [B_i|T])]^{1/2} + E^P [|| z_n ||].
$$

(4.5)
Remark 4.6 The terms $E^Q [[B_i]_T]$ are related to the riskiness associated with credit spreads. The smaller the "volatility" associated with the credit spreads, the better the approximation hedge will be. Provided that the $E^Q [[B_i]_T]$ are uniformly bounded, that the risk premium term $E^Q \left[ (dF/dQ)^2 \right]$ also remains bounded and that the credit spread market is complete, the previous proposition states that the $L^1(P)$ norm of the hedging error tends to zero at the speed $n^{-1/2}$ as $n$ tends to infinity.
Hedging credit spread risk for large portfolios

• Standard valuation approach in derivatives markets
  ➢ Complete markets
  ➢ Price = cost of the hedging/replicating portfolio

• Mixing of dynamic hedging strategies
  – for credit spread risk

• And diversification/insurance techniques
  – For default risk

• Thought provocative
  – To construct a practical hedging strategy, do not forget default risk
  – Equity tranche [0,3%]
  – iTraxx or CDX first losses cannot be considered as smooth