A note on the risk management of CDOs

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A note on the risk management of CDOs

- **Purpose of the paper**
  - **To provide a framework for the risk management of CDO tranches on large indices**
    - iTraxx, CDX
  - **In a conditionally independent upon default intensities framework**
- **Thought provocative result**
  - **Concentrate on the dynamic hedging of credit spread risk**
    - Idiosyncratic and parallel credit spreads movements
  - **Default risk is statically hedged by diversification**
    - Since default events are conditionally independent upon credit spreads
    - Insurance idea

A note on the risk management of CDOs

- Default risk
  - Default bond price jumps to recovery value at default time.
  - Drives the CDO cash-flows
  - Possibility of multiple defaults
- Credit spread risk
  - Changes in defaultable bond prices prior to default, due to shifts in credit quality or in risk premiums.
  - Changes in the marked to market of tranches
  - Increase or decrease the probability of future defaults
  - Changes in the level, the dispersion of credit spreads, the correlation between credit spreads
- Recovery risk
  - Magnitude of aggregate loss jumps is random

A note on the risk management of CDOs

- **Overall view of the presentation**
  - **Learning curve for credit modelling, pricing and hedging**
  - **Risks and hedging issues as seen from different models**
    - Structural models
    - Contagion or copula models
    - Market practitioner's approach
    - Multivariate Poisson
    - Aggregate loss models
    - Intensity models
  - **First to Default Swap example**
  - **Hedging large portfolios in intensity models**

A note on the risk management of CDOs

- Learning curve for credit modelling
- Parallel with equity derivatives
  - Building of a gothic cathedral
  - Bachelier, Samuelson μ instead of r
    - ➢no notion of risk neutral measure, no notion of duplication cost
  - Black-Scholes (prior to Merton)
    - ➢Risk-neutral measure thanks to the use of CAPM, not a perfect hedge
  - Black-Scholes (final version)
    - ➢Local approach to the hedging
  - Eventually, Harrison-Kreps, Harrison-Pliska, martingale representation theorem, Girsanov and so on…

A note on the risk management of CDOs

- **Learning curve for credit modelling…**
  - **Start with a "risk-neutral" or pricing probability**
    - ➢Compute expectations of payoffs
    - ➢Assumption of perfect markets
    - ➢Pricing disconnected from hedging
  - **Use of intensity or reduced-form models:**
    - ➢Lando, Jarrow, Lando & Turnbull, Duffie & Singleton
  - **One step backward**
    - ➢ Copula models:
      - **static approach**
      - **Default intensities are deterministic between two default times**

A note on the risk management of CDOs

- Learning curve for credit modelling (cont)
  - **One further step backward**
    - 1F Gaussian copula + base correlation
    - Not a probabilistic model, Arbitrage opportunities
    - Fall of the Roman Empire
- Incomplete market approaches?
  - Not used by investment banks
- Market practitioners’ approach
  - Take some copula model (boo!)
  - “bump” the marginal credit curves
  - Compute CDS credit deltas

A note on the risk management of CDOs

- Dynamic hedging of basket credit derivatives
  - Bielecki, Jeanblanc & Rutkowski [2006], Frey & Backhaus [2006]
    - ➢ Credit spreads are driven by defaults
    - ➢ Martingale representation theorem under the natural filtration of default times
      - Jacod (1975) or Brémaud , chapter III
    - ➢ Hedging instruments: credit default swaps
    - ➢ Complete markets under the assumption of no simultaneous defaults
- Static replication of basket credit derivatives with first to default swaps
  - Brasch [2006]: “A note on efficient pricing and risk calculation of credit basket products”
- Super-replication: Walker [2005]
- Mean-variance hedging (local minimization): Elouerkhaoui [2006]

Risk within CDO tranches

- Risks as seen from different models
  - Aggregate loss or collective models
    - ➢Hedge with the index (iTraxx or CDX) only
  - Name per name or individual models
    - ➢Hedging using the set of underlying CDS
      - **Important issue for the hedging of equity tranches**
- Hedging in different models
  - Structural models
  - Copula and contagion models
  - Practitioner's approach
  - Multivariate Poisson models
  - Aggregate loss models
  - Econometric approach
  - Intensity models

Models for multivariate credit risk analysis

- **Structural models:**
  - **defaultable bonds seen as equity barrier options**
- Multiname credit derivatives can be perfectly hedged in a Black-Cox framework
  - Defaults are predictable
  - **Not very realistic:**
    - perfect correlation between equity returns and credit spreads
    - Term-structure of credit spreads
  - **Huge numerical issues**
  - **Not so far from copula models:**
    - Hull, Pedrescu & White

Models for multivariate credit risk analysis

- Contagion models or Copula models
  - Credit spreads are deterministic between two default dates
- Multivariate Poisson models
  - Allow for multiple defaults
- Aggregate loss models
  - Direct specification of loss dynamics
  - CDO tranches only involve European options on aggregate loss
  - Aggregate loss : Marked Point Process
- Intensity models
  - Cox or doubly stochastic Poisson processes, conditionally independent defaults
  - Defaults are not informative
  - No jumps in credit spreads at default times

Models for multivariate credit risk analysis

- Contagion models (interacting intensities)
  - Jumps of credit spreads of survival names at default times
  - Jarrow & Yu, Yu, Frey & Backhaus
- Copula models
  - Starting point : copula of default times
  - Copula specification states the dependence between default times
  - Marginal default time distributions are self-calibrated onto credit spread curves
  - Intensities in copula models
  - Related to partial derivatives of the copula
  - May be difficult to compute
  - Default intensities are deterministic between two default times
  - Jump at default times
  - Contagion effects in copula models

Hedging in different modeling framework

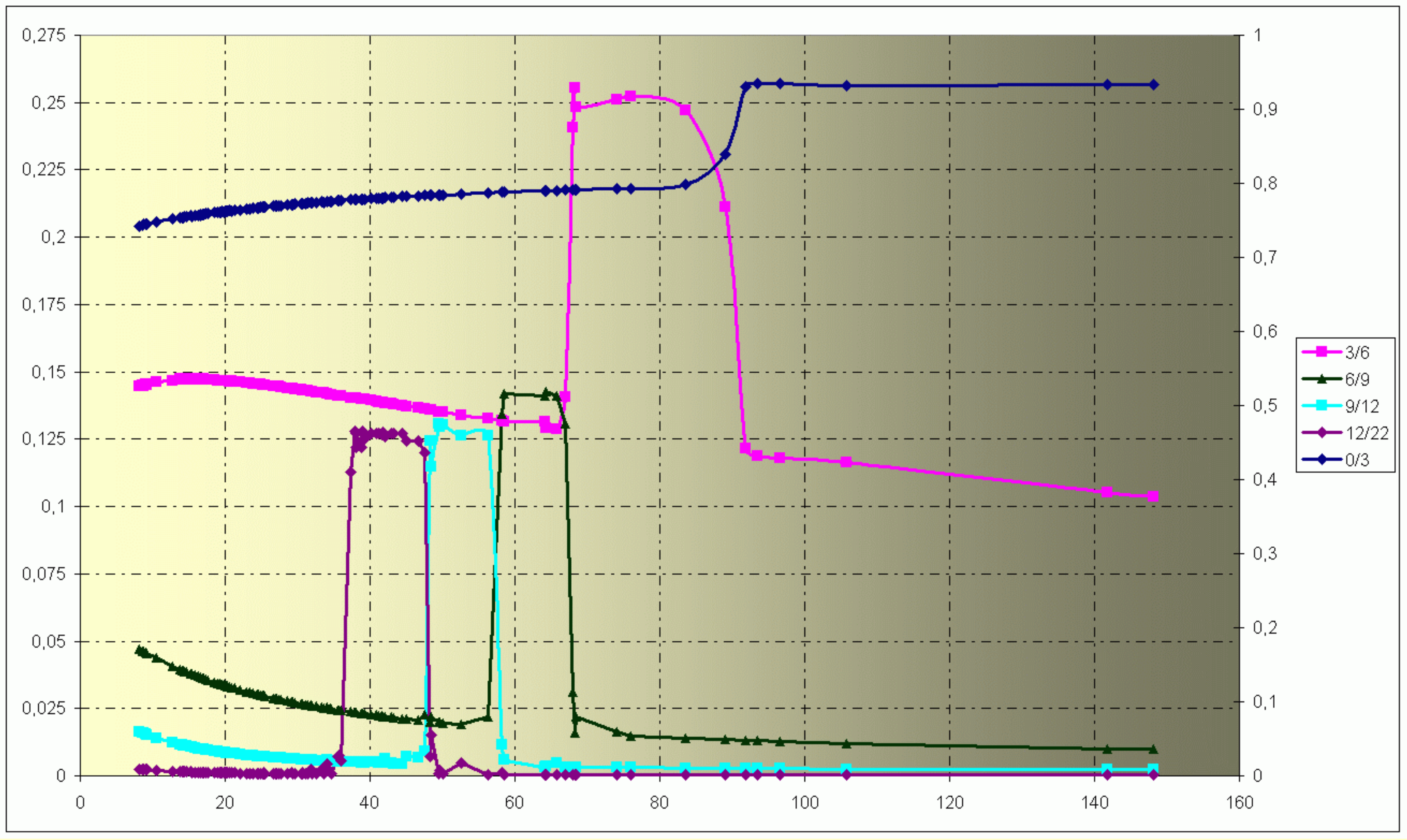
- Copula and contagion models: theory
  - Default intensities are only related to past defaults
  - In other words, credit spread risk derives from default risk
- Smooth copula precludes simultaneous defaults
  - In previous models, perfect hedge of multiname credit derivatives with single name CDS
  - Complete markets
  - Representation theorems for multivariate point processes
  - Only default risk, no “true” credit spread risk
  - Work in progress
    - ➢Bielecki, Jeanblanc & Rutkowski

Models for multivariate credit risk analysis

- Copula models: practice very different from theory
- Practical implementation of hedging strategies
- <u>Focus on credit spread risk only</u>
- Price of a CDO tranche depends upon marginal credit curves and the copula
- Compute CDS hedge ratio by bumping the marginal credit curves and compute the CDO price increment
- Local sensitivity analysis
  - Model dependent
  - No guarantee that local hedging leads to a correct global hedge
  - Does gamma effects offset theta effects?

Models for multivariate credit risk analysis

- **Credit deltas within a stochastic correlation model**
  - **Burstchell et al**



Models for multivariate credit risk analysis

- Copula models: gamma effects
- Homogeneous portfolio
  - Gamma matrix of a CDO tranche (wrt credit spreads)

$$\begin{pmatrix} I & B & B & B & B \\ B & I & B & B & B \\ B & B & I & B & B \\ B & B & B & I & B \\ B & B & B & B & I \end{pmatrix}$$

  - $(s_1, \ldots, s_n)$ change in credit spreads
    - ➢Assume credit delta hedging with CDS
    - ➢First order change in PV are equal to zero

Models for multivariate credit risk analysis

- Copula models: gamma effects
  - Assume $s_2 = \cdots = s_n = 0$
    - ➢Change in PV $\frac{I}{2} s_1^2$ idiosyncratic gamma effect
  - Assume $s_1 = \cdots = s_n = s$
    - ➢Change in PV $\frac{n}{2}\left(I + (n-1)B\right) s^2$ parallel gamma
- Homogeneous portfolio
  - Credit spread covariance matrix

$$\sigma^2 \Delta t \begin{pmatrix} 1 & \rho & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho & \rho \\ \rho & \rho & 1 & \rho & \rho \\ \rho & \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & \rho & 1 \end{pmatrix}$$

$$\boxed{\text{expected gamma P\&L} = n \frac{\sigma^2 \Delta t}{2} \left((1-\rho) I + \rho (I + (n-1)B) \right),}$$

  - $(n-1)B\rho$ high spread correlation sensitivity

Models for multivariate credit risk analysis

- Hedging CDO tranches in the base correlation approach
  - Tranchelets on standard indices
  - Bespoke portfolios
- Correlation depends upon the expected loss of the tranche
- Change in credit spreads changes the marginal credit curves and the implied correlation parameter
  - Sticky deltas
- Still main focus upon credit spread hedging
  - Still dispersion risk (idiosyncratic gamma) and parallel spread risk

Models for multivariate credit risk analysis

- **Hedging CDO tranches with liquid tranches**
  - Case of tranchelets on iTraxx or CDX
  - Not the same hedging instruments
- **Entropic calibration**
  - Perfect copula type approach
  - Start from some specification of conditional default probabilities
  - g_0 a priori density function of conditional default probabilities
  - Look for some a posteriori density function of cdp:

$$\min_g \int g(p) \ln \frac{g(p)}{g_0(p)} dp$$

  - consistency constraints with liquid tranches prices $\int_0^1 (p - k_i)^+ g(p) dp = \pi_i$

$$g(p) = g_0(p) \exp\left(\lambda + \sum_{i=0}^{I} \lambda_i (p - k_i)^+ \right)$$

  - Hedge ratios: compute partial derivatives of tranchelets wrt π_i

Models for multivariate credit risk analysis

- Multivariate Poisson models
  - Shock models
  - Default indicators are driven by a multivariate Poisson model
    - Lindskog & McNeil, Elouerkhaoui, Duffie & Singleton
  - Common and idiosyncratic shocks
    - Common shocks can be fatal or non fatal
    - A name can survive a non fatal shock
  - Armageddon risk
    - possibly large values for senior tranches
  - Intensities are deterministic between two shocks
    - Not really any credit spread risk

Models for multivariate credit risk analysis

- Multivariate Poisson models
  - Possibility of simultaneous defaults
    - Name 1 and 2 may default altogether
    - Name 1 and 3 may default altogether
    - Name 2 and 3 may default altogether
    - Name 1, 2, and 3 may default altogether
  - This drives the dependence
  - **High degree of default risk incompleteness**
    - 2^n states of the world
    - n hedging instruments (single name CDS)

Models for multivariate credit risk analysis

- Aggregate loss models
  - Increasing Market Point Process
  - Aggregate loss intensity = sum of name default intensities
  - Magnitude of jumps = 1 – recovery of defaulted name
- Markovian models
  - SPA, Schönbucher
  - Markov chain (or more general) processes for the aggregate loss
- Non Markovian
  - Giesecke & Goldberg
  - Self-exciting processes, Hawkes, ACD type
  - Loss intensity only depends upon past losses
  - Top-down approach ?

Hedging in different modeling framework

- Hedging in aggregate loss models
  - No notion of idiosyncratic gamma
  - Individual credit spreads are perfectly correlated
  - Jumps in aggregate loss process (default risk)
  - Change in loss intensity: parallel Gamma
- Hedging on a name per name basis
- Or based upon the index: same hedge ratios for all names
  - Hedging equity tranche with an aggregate loss model can become problematic
  - High sensitivity to heterogeneity between credit spreads
  - Hedge ratios for riskier names are likely to be higher
  - Does not take into account idiosyncratic gamma

Models for multivariate credit risk analysis

- Econometric approach to credit spread hedging
- Hedging liquid tranches with the index
  - iTraxx or CDX
  - Look for historical data on tranche premiums and index credit spread
  - Try to relate through some regression analysis changes in tranche premiums to changes in spreads
  - Check the hedging performance of different models
    - ➢Houdain & Guegan
    - ➢Similar ideas in equity derivatives markets
    - ➢Baskhi, Cao & Chen

Models for multivariate credit risk analysis

- Intensity models
  - Default arrivals are no longer predictable
  - Model conditional local probabilities of default $\lambda(t)\,dt$
  - τ : default date, $\lambda(t)$ *risk intensity* or *hazard rate*

$$\lambda_i(t)dt = P\left[\tau_i \in [t, t+dt[\,\middle|\, \tau_i > t\right]$$

  - *Marginal default intensity*
- Multivariate case: no simultaneous defaults
  - Model starts from specifying default intensities
- Multivariate Cox processes
  - Credit spreads do not jump at default times
  - Duffie Singleton, Lando, …

Hedging First to default swaps: introduction

- Consider a basket of M defaultable bonds
  - multiple counterparties
- First to default swaps
  - protection against the first default
- Hedging and valuation of basket default swaps
  - involves the joint (multivariate) modeling of default arrivals of issuers in the basket of bonds.
  - Modeling accurately the dependence between default times is a critical issue.

Hedging First to default swaps: introduction

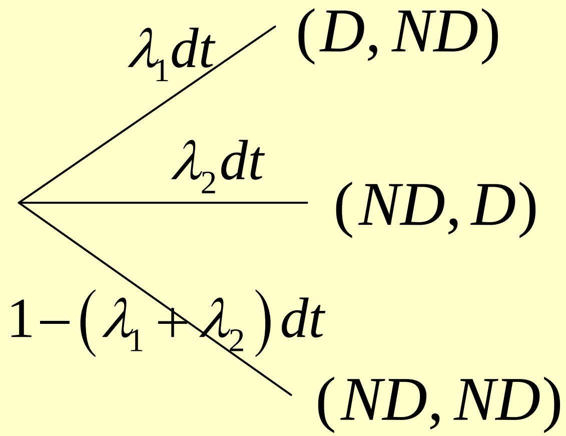
- Hedging Default Risk in Basket Default Swaps
- Example: first to default swap from a basket of two risky bonds.
  - If the first default time occurs before maturity,
  - The seller of the first to default swap pays the non recovered fraction of the defaulted bond.
- Assume that the two bonds cannot default simultaneously
  - We moreover assume that default on one bond has *no effect* on the credit spread of the remaining bond.
- How can the seller be protected *at default time* ?
  - The only way to be protected at default time is to hold two default swaps with the *same nominal* than the *nominal* of the bonds.
  - The *maturity* of underlying default swaps does not matter.

Hedging First to default swaps: introduction

- Some notations :
  - τ_1, τ_2 default times of counterparties 1 and 2,
  - $\mathcal{H}_t$ available information at time *t*,
  - P historical probability,
  - λ_1 , λ_2 : (historical) risk neutral intensities:
    - $P\left[\tau_i \in [t, t+dt[\,\middle|\, H_t\right] = \lambda_i dt, \; i = 1,2$
- Assumption : « Local » independence between default events
  - Probability of 1 and 2 defaulting altogether:
    - $P\left[\tau_1 \in [t, t+dt[, \tau_2 \in [t, t+dt[\,\middle|\, H_t\right] = \lambda_1 dt \times \lambda_2 dt \text{ in } (dt)^2$
  - Local independence: simultaneous joint defaults can be neglected

Hedging First to default swaps: introduction

- Building up a tree:
  - Four possible states: (D,D), (D,ND), (ND,D), (ND,ND)
  - Under no simultaneous defaults assumption $p_{(D,D)}=0$
  - Only three possible states: (D,ND), (ND,D), (ND,ND)
  - Identifying (historical) tree probabilities:



$$\begin{cases} p_{(D,D)} = 0 \Rightarrow p_{(D,ND)} = p_{(D,D)} + p_{(D,ND)} = p_{(D,.)} = \lambda_1 dt \\ p_{(D,D)} = 0 \Rightarrow p_{(ND,D)} = p_{(D,D)} + p_{(ND,D)} = p_{(.,D)} = \lambda_2 dt \\ p_{(ND,ND)} = 1 - p_{(D,.)} - p_{(.,D)} \end{cases}$$

Hedging First to default swaps: introduction

- Cash flows of (digital) CDS on counterparty 1:
  - $\lambda_1 \phi_1 dt$ CDS premium, ϕ_1 default risk premium

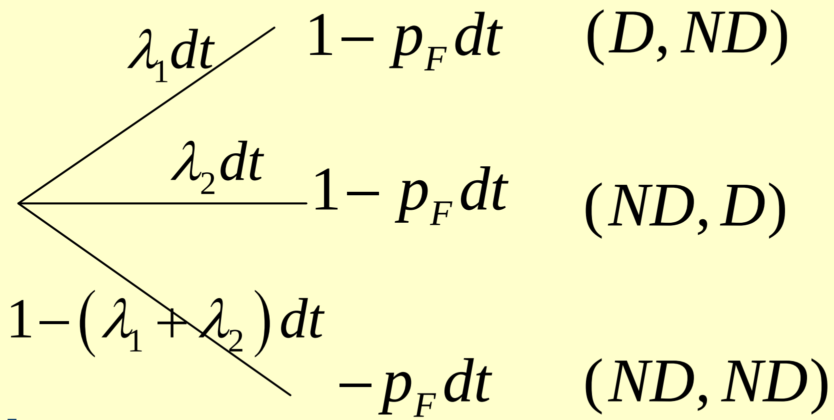
$\lambda_1 dt$	$1-\lambda_1\phi_1 dt$	(D, ND)
$\lambda_2 dt$	$-\lambda_1\phi_1 dt$	(ND, D)
$1-(\lambda_1+\lambda_2)dt$	$-\lambda_1\phi_1 dt$	(ND, ND)

- Cash flows of (digital) CDS on counterparty 1:

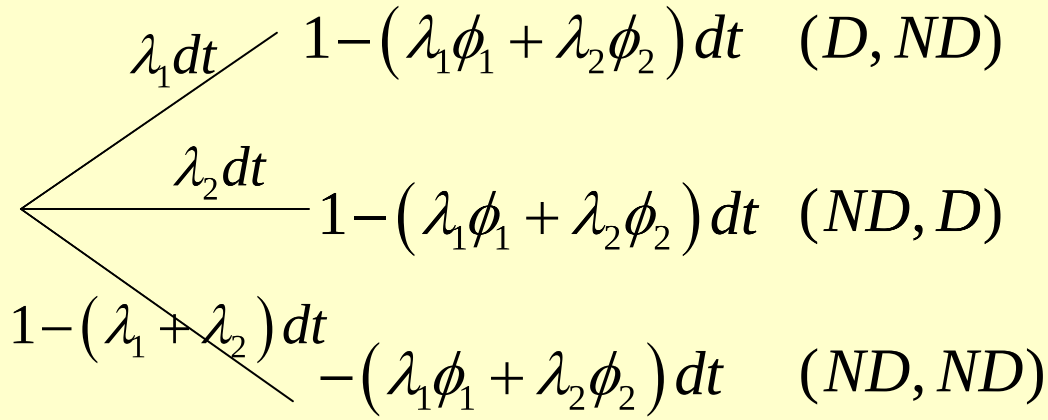
$\lambda_1 dt$	$-\lambda_2\phi_2 dt$	(D, ND)
$\lambda_2 dt$	$1-\lambda_2\phi_2 dt$	(ND, D)
$1-(\lambda_1+\lambda_2)dt$	$-\lambda_2\phi_2 dt$	(ND, ND)

Hedging First to default swaps: introduction

- Cash flows of (digital) first to default swap (with premium p_F):



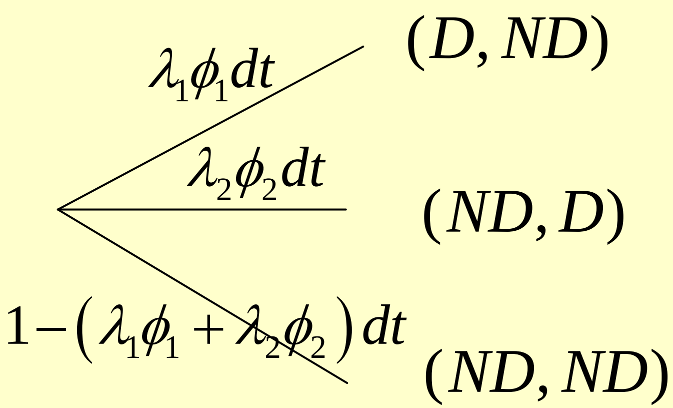
- Cash flows of holding CDS 1 + CDS 2:



- Absence of arbitrage opportunities imply:
  - $p_F = \lambda_1 \phi_1 + \lambda_2 \phi_2$
  - Perfect hedge of first to default swap by holding 1 CDS 1 + 1 CDS 2

Hedging First to default swaps: introduction

- Three possible states: (D,ND), (ND,D), (ND,ND)
- Three tradable assets: CDS1, CDS2, risk-free asset
  - ➢The market is still « complete »
- Risk-neutral probabilities
  - Used for computing prices
  - Consistent pricing of traded instruments
  - Uniquely determined from CDS premiums
  - $p_{(D,D)}=0$, $p_{(D,ND)}=\lambda_1\,\phi_1 dt$, $p_{(ND,D)}=\lambda_2\,\phi_2 dt$, $p_{(ND,ND)}=1-(\lambda_1\,\phi_1+\lambda_2\,\phi_2)\,dt$



Hedging First to default swaps: introduction

- *hedge ratios* for first to default swaps
- Consider a first to default swap associated with a basket of two defaultable loans.
  - Hedging portfolios based on standard underlying default swaps
  - Hedge ratios if:
    - simultaneous default events
    - *Jumps* of credit spreads at default times
- Simultaneous default events:
  - If counterparties default *altogether*, holding the *complete* set of default swaps is a conservative (and thus expensive) hedge.
  - In the *extreme* case where default *always* occur altogether, we only need a single default swap on the loan with largest nominal.
  - In other cases, holding a *fraction* of underlying default swaps does not hedge default risk (if *only one* counterparty defaults).

Hedging First to default swaps: introduction

- Default hedge ratios for first to default swaps and contagion
- What occurs if there is a *jump in the credit spread* of the second counterparty after default of the first ?
  - default of first counterparty means *bad news* for the second.
  - Contagion effects
- If hedging with short-term default swaps, no capital gain at default.
  - Since PV of short-term default swaps is not *sensitive* to credit spreads.
- This is not the case if hedging with long term default swaps.
  - If credit spreads jump, PV of long-term default swaps jumps.
- Then, the amount of hedging default swaps can be reduced.
  - This reduction is *model-dependent*.

Hedging <u>First to default swaps</u>: introduction

- Default hedge ratios for first to default swaps and stochastic credit spreads
- If one uses short maturity CDS to hedge the FTD?
  - Sell protection on FTD
  - Buy protection on underlying CDS
  - Short maturity CDS: no contagion
  - But, roll-over the hedge until first to default time
  - Negative exposure to an increase in CDS spreads
- If one uses long maturity CDS to hedge the FTD
  - unknown cost of unwinding the remain CDS
  - Credit spreads might have risen or decreased

Hedging credit spread risk for large portfolios

- When dealing with the risk management of CDOs, traders concentrate upon credit spread and correlation risk
- What about default risk ?
  - For large indices, default of one name has only a small effect on the aggregate loss
- Model framework
  - Given probability Q such that:
    - ➢Defaultable bond prices are martingales
    - ➢Default times follow a multivariate Cox process
    - ➢Q equivalent to historical probability P
    - ➢Bounded risk premiums

Hedging credit spread risk for large portfolios

- Notations and model framework
  - $\tau_1, \ldots, \tau_n$ default times
  - $N_i(t) = 1_{\{\tau_i \le t\}}, i = 1, \ldots, n$ default indicators
  - $H_t = \bigvee_{i=1,\ldots,n} \sigma\left(N_i(s), s \le t\right)$ natural filtration of default times
  - F_t background (credit spread filtration)
  - $G_t = H_t \vee F_t$ enlarged filtration, P historical measure
  - $l_i(t,T), i = 1, \ldots, n$ time t price of an asset paying $N_i(T)$ at time T

Assumption 1 *There exists a probability Q equivalent to P such that:*

1. *for $i = 1, \ldots, n$, the price processes of defaultable claims $l^i(., T)$ are $(Q, \mathcal{G})$ martingales:*

$$l^i(t,T) = E^Q[N_i(T) \mid \mathcal{G}_t], \tag{2.1}$$

for $0 \leq t \leq T$.

2. *the default times follow a multivariate Cox process:*

$$\tau_i = \inf\left\{t \in \mathbb{R}^+, U_i \geq \exp\left(-\int_0^t \lambda_{i,u} du\right)\right\}, \quad i = 1, \ldots, n \tag{2.2}$$

where $\lambda_1, \ldots, \lambda_n$ are strictly positive, $\mathcal{F}$ - progressively measurable processes, $U_1, \ldots, U_n$ are independent random variables uniformly distributed on $[0,1]$ under Q and $\mathcal{F}$ and $\sigma(U_1, \ldots, U_n)$ are independent under Q.

3. $E^Q\left[\left(\frac{dP}{dQ}\right)^2\right] < \infty^5$.

Hedging credit spread risk for large portfolios

- Remarks
  - For notational simplicity default-free rates are equal to zero
  - Existence of n hedging defaultable bonds
    - Could be *n* CDS as well
  - Existence of (non unique) martingale measure *Q* from perfect arbitrage free markets
  - Equivalence between:
    - *Q*-multivariate Cox process
    - Or “conditionally independent defaults”
    - Or “no contagion effects”
    - Or “defaults are non informative”
    - Or (*H*) hypothesis holds

Hedging credit spread risk for large portfolios

Lemma 2.1 *Every* $(Q, \mathcal{F})$ *square integrable martingale is also a* $(Q, \mathcal{G})$ *square integrable martingale.*

- No contagion effects
  - credit spreads drive defaults but defaults do not drive credit spreads
  - For a large portfolio, default risk is perfectly diversified
  - Only remains credit spread risks

Remark 2.2 While $(\tau_1, \dots, \tau_n)$ is a multivariate Cox process under Q, it may not be a Cox process under P. For instance, we may have some contagion effects under P (see Kusuoka [18]).

Remark 2.3 The joint survival function is such that $S(t_1, \dots, t_n) = E^Q \left[\prod_{i=1}^n \exp\left(-\Lambda_{i,t_i}\right) \right]$ for $t_1, \dots, t_n \in \mathbb{R}^+$. Since it is continuous, we must have $Q(\tau_i = \tau_j) = 0$ for $i \neq j$ which precludes simultaneous defaults.

- No simultaneous defaults
  - Otherwise market would be incomplete

Hedging credit spread risk for large portfolios

- **Purpose: hedging of stylized CDOs, i.e. options on the aggregate loss**

We will further consider payoffs of the type $(l_n(T) - K)^+ = \left(\frac{1}{n}\sum_{i=1}^{n} N_i(T) - K\right)^+$, for some $K \in [0,1]$ corresponding to so-called *zero-coupon CDOs*. Though zero-coupon CDOs are actually traded in the market, the most commonly traded CDOs involve more complex loss payments (see Laurent and Gregory [21]). To illustrate the risk management approach, we think it is more suitable to deal with simple payoffs.

- **Practical hedge is extremely tricky**
  - **Need to hedge both default and credit spread risk**
  - **Recall that traders focus mainly on credit spread risk**
  - **Since default risk is already partly diversified at the index level**
  - **Forget about default? back to the *F* filtration**

Hedging credit spread risk for large portfolios

- Construction of the hedging strategy
- Step 1: consider some pseudo defaultable bonds
  - i.e. project defaultable bond prices on the *F* filtration
  - shadow bonds similar to well diversified portfolios
    - Björk & Naslund, de Donno
  - shadow market only involves credit spread risk
- Step 2: approximate the CDO tranche payoff
  - Replace CDO payoff by its smoothed projection on the *F* filtration

Hedging credit spread risk for large portfolios

- Step 3: compute perfect hedge ratios
  - With respect to pseudo defaultable bonds 1, … , *n*
  - Assume that Shadow market is complete
    - **This can be relaxed (see paper)**
  - Numerical issues are left aside
    - High dimensionality
    - Markovian
    - Use of semi-analytical techniques
    - Not detailed in the paper
- Step 4: apply the hedging strategy to the **true** defaultable bonds

Hedging credit spread risk for large portfolios

- Main result
  - Bound on the hedging error following the previous hedging strategy
  - **When hedging an actual CDO tranche with actual defaultable bonds**
  - Hedging error decreases with the number of names
    - Default risk diversification
- Provides a hedging technique for CDO tranches
  - Known theoretical properties
  - Good theoretical properties rely on no simultaneous defaults, no contagion effects assumptions

Hedging credit spread risk for large portfolios

- Technical background
  - Projection of default indicators on the information generated by credit spreads
  - Smooth projection of the aggregate loss
  - No default risk in the market with incomplete information

Definition 3.1 *We denote by $p^i(.)$, the* **default-free running loss process** *associated with name $i \in \{0, \ldots, n\}$, which is such that for $0 \leq t \leq T$:*

$$p^i(t) \triangleq E^Q[N_i(t) \mid \mathcal{F}_t] = Q(\tau_i \leq t \mid \mathcal{F}_t) = 1 - \exp(-\Lambda_{i,t}) . \quad (3.1)$$

Definition 3.2 *The* **default free T forward loss process** *associated with name $i \in \{0, \ldots, n\}$, denoted by $p^i(., T)$ is such that for $0 \leq t \leq T$:*

$$p^i(t, T) \triangleq E^Q\left[p^i(T) \mid \mathcal{F}_t\right] = E^Q\left[N_i(T) \mid \mathcal{F}_t\right] = Q(\tau_i \leq T \mid \mathcal{F}_t). \quad (3.2)$$

Hedging credit spread risk for large portfolios

Definition 3.5 default-free aggregate running loss process *The default free aggregate running loss at time t is such that for $0 \leq t \leq T$:*

$$p_n(t) \triangleq \frac{1}{n} \sum_{i=1}^{n} p^i(t). \tag{3.7}$$

Assumption 2 *There exists some bounded $\mathcal{F}$ - predictable processes $\theta_1(.), \ldots, \theta_n(.)$ such that:*

$$(p_n(T) - K)^+ = E^Q\left[(p_n(T) - K)^+\right] + \frac{1}{n} \sum_{i=1}^{n} \int_0^T \theta_i(t) dp^i(t, T) + z_n, \tag{4.2}$$

where z_n is $\mathcal{F}_T$-measurable, of Q-mean zero and Q-strongly orthogonal to $p^1(., T), \ldots, p^n(., T)$.

Remark 4.3 The previous equation is simply the $(Q, \mathcal{F})$ Galtchouk - Kunita - Watanabe decomposition of $(p_n(T) - K)^+$. $\theta_1(.), \ldots, \theta_n(.)$ correspond to the optimal $(Q, \mathcal{F})$ mean-variance hedging strategy based upon the abstract forward price processes $p^1(., T)$, $\ldots, p^n(., T)$.

Hedging credit spread risk for large portfolios

Remark 4.5 The key point in Assumption (2) is the boundedness of the θ_i's. Let us remark that the individual credit deltas are equal to $\frac{\theta_i(t)}{n}$. For simplicity, we will thereafter assume that $0 \leq \theta_i(.) \leq 1$ for $i = 1, \ldots, n$. This boundedness assumption is related to the propagation of convexity property. We refer to Bergenthum and Rüschendorf [1], Ekström and Tysk [11] and the references therein for some discussion in a multivariate jump diffusion setting.

Proposition 1 *Under Assumptions (1) and (2), the hedging error ε_n defined as:*

$$\varepsilon_n = (l_n(T) - K)^+ - E^Q\left[(l_n(T) - K)^+\right] - \frac{1}{n}\sum_{i=1}^{n}\int_0^T \theta_i(t) dl^i(t,T), \tag{4.4}$$

is such that $E^P[\| \varepsilon_n \|]$ is bounded by:

$$\frac{1}{\sqrt{2n}}\left(1 + \left(E^Q\left[\left(\frac{dP}{dQ}\right)^2\right]\right)^{1/2}\right) + \frac{1}{n}\left(E^Q\left[\left(\frac{dP}{dQ}\right)^2\right]\right)^{1/2}\left(\sum_{i=1}^n \left(Q(\tau_i \leq T) + E^Q\left[B_i\right]_T\right)\right)^{1/2}$$
$$+E^P[\| z_n \|]. \tag{4.5}$$

Hedging credit spread risk for large portfolios

Remark 4.6 The terms $E^Q[[B_i]_T]$ are related to the riskiness associated with credit spreads. The smaller the "volatility" associated with the credit spreads, the better the approximation hedge will be. Provided that the $E^Q[[B_i]_T]$ are uniformly bounded, that the risk premium term $E^Q\left[\left(\frac{dP}{dQ}\right)^2\right]$ also remains bounded and that the credit spread market is complete, the previous proposition states that the $\mathcal{L}^1(P)$ norm of the hedging error tends to zero at the speed $n^{-1/2}$ as n tends to infinity.

Hedging credit spread risk for large portfolios

- Standard valuation approach in derivatives markets
  - ➢Complete markets
  - ➢Price = cost of the hedging/replicating portfolio
- Mixing of dynamic hedging strategies
  - for credit spread risk
- And diversification/insurance techniques
  - For default risk
- Thought provocative
  - To construct a practical hedging strategy, do not forget default risk
  - Equity tranche [0,3%]
  - iTraxx or CDX first losses cannot be considered as smooth