Risk Management aspects of CDOs

CDOs after the crisis: Valuation and risk management reviewed

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Presentation related to the paper
***Hedging default risks of CDOs in Markovian contagion models* (2008)**
Available on www.defaultrisk.com

Preliminary or obituary?

- On human grounds, shrinkage rather than enlargement of the job market
- Thanks to the crisis, our knowledge of the flaws of the various competing models has dramatically improved…
  - We know that we don't know and why
  - No new paradigm has yet emerged (if ever)
  - Paradoxically, academic research is making good progress
  - … but at its own pace
- Model to be presented is low tech, unrealistic, nothing new
- But deserves to be known (this is pure speculation)
- Provides an academic view on practical issues
  - Does not intend to give the insights of a trader or a risk manager

Overview

- CDO Business context
  - Decline of the one factor Gaussian copula model for risk management purposes
  - Recent correlation crisis
  - Unsatisfactory credit deltas for CDO tranches
- Tree approach to hedging defaults
  - From theoretical ideas
  - To practical implementation of hedging strategies
  - Robustness of the approach?
- Mathematical framework
- Empirical results

CDO Business context

- We are within a financial turmoil
  - Lots of restructuring and risk management of trading books
  - Collapse of highly leveraged products (CPDO)
  - February and March 2008 crisis on iTraxx and CDX markets
    - Surge in credit spreads
    - Extremely high correlations
    - Trading of [60-100%] tranches
    - Emergence of recovery rate risk
  - What is really a default event ?
    - How to cope with Fed or Treasury activism?
  - Questions about the pricing of bespoke tranches
    - Unreliability of projection techniques

CDO Business context

Morgan Stanley

MORGAN STANLEY RESEARCH

March 10, 2008
Structured Credit Analytics

CDX and iTraxx – Correlation Analysis and Delta Neutral Return

CDX Series 9

Index	Tranche	Bid	Ask	Mid 1 Week Change	Mid 1 Month Change	Index Level	Delta	Detach Corr.[1]	Corr. Skew	1 Week Change	Corr. Skew Z Score	Corr. Skew Rel. Value	CDX vs. iTRAXX Corr Skew	Value vs. iTraxx	Delta Neutral Returns 1 Week	Delta Neutral Returns 1 Month	Delta Neutral Returns 1 Year
5yr	Index	185.0	185.0	25.0	69.0												
5yr[2]	0-3%	67.5%	68.5%	10.4%	13.0%	185 bp	2.7	38%		-7.4%	1.19	RICH	(1.10)	CHEAP	-6.6%	0.6%	50.0%
5yr	3-7%	731	738	138	219		3.0	64%	25%	2.7%	1.07	RICH	0.97	FAIR	-2.1%	3.2%	9.0%
5yr	7-10%	405	415	75	141		1.8	73%	9%	0.5%	(0.31)	FAIR	0.21	FAIR	-1.2%	1.4%	-4.7%
5yr	10-15%	207	214	12	73		1.3	87%	14%	2.1%	1.15	RICH	0.96	FAIR	0.6%	1.2%	-1.8%
5yr	15-30%	126	130	13	62		1.0	108%	21%	1.2%	(0.01)	FAIR	0.18	FAIR	0.4%	-0.3%	-1.3%
5yr	30-100%	78	80	10	45		0.7								0.3%	-0.6%	-1.6%
7yr	Index	178.0	178.0	20.0	62.0												
7yr[2]	0-3%	71.0%	72.0%	8.0%	11.3%	178 bp	2.0	39%		-4.2%	1.38	RICH			-5.0%	1.8%	42.9%
7yr	3-7%	785	795	130	193		3.2	62%	23%	1.6%	1.00	FAIR			-2.6%	5.7%	18.2%
7yr	7-10%	452	460	79	106		2.5	71%	8%	0.1%	(0.66)	FAIR			-1.8%	5.4%	-2.8%
7yr	10-15%	256	265	21	73		1.6	83%	12%	1.6%	(0.09)	FAIR			0.5%	3.0%	-2.0%
7yr	15-30%	139	144	14	65		1.1	104%	21%	0.3%	(0.73)	FAIR			0.3%	-0.1%	-1.7%
7yr	30-100%	81	83	12	48		0.7								0.1%	-0.9%	-2.1%
10yr	Index	174.0	174.0	18.0	57.0												
10yr[2]	0-3%	74.0%	74.8%	7.9%	9.9%	174 bp	1.5	39%		-5.5%	1.33	RICH	1.31	RICH	-5.2%	1.5%	31.6%
10yr	3-7%	910	920	110	204		3.4	57%	18%	3.3%	3.13	RICH	3.16	RICH	-1.6%	7.5%	37.4%
10yr	7-10%	523	533	67	98		2.5	64%	7%	0.4%	(0.64)	FAIR	(0.64)	FAIR	-0.9%	8.5%	1.8%
10yr	10-15%	300	308	22	95		1.8	76%	12%	1.1%	(0.60)	FAIR	(0.60)	FAIR	1.0%	2.6%	3.0%
10yr	15-30%	150	155	14	63		1.2	99%	23%	0.3%	(1.99)	CHEAP	(2.01)	CHEAP	0.5%	0.0%	-1.6%
10yr	30-100%	79	82	10	45		0.7								0.2%	-1.2%	-2.9%
HY	Index	718.8	718.8	23.6	83.1												
HY[3]	0-10%	92.1%	92.6%	1.6%	2.5%	719 bp	0.7	44%		-2.4%	1.05	RICH			-1.1%	-0.4%	10.2%
HY	10-15%	75.3%	75.8%	2.5%	6.1%		1.8	49%	5%	1.2%	(0.12)	FAIR			-1.4%	-1.4%	8.0%
HY	15-25%	1,290	1,305	81	222		2.3	72%	23%	1.4%	2.73	RICH			-1.0%	-1.5%	1.6%
HY	25-35%	680	695	48	145		1.5	92%	19%	0.2%	(0.21)	FAIR			-0.6%	-1.6%	-6.3%
HY	35-100%	202	208	(2)	32		0.6								0.5%	0.0%	-2.1%
LCDX	Index	440.0	440.0	7.8	29.6												
LCDX[3]	0-5%	86.6%	87.5%	3.9%	-0.2%	440 bp	1.6	78%		-6.2%					-3.5%	3.0%	
LCDX[3]	5-8%	66.9%	67.6%	4.3%	1.0%		2.7								-3.6%	3.6%	
LCDX	8-12%	1140.0	1152.0	30.0	35.0		3.5								0.0%	3.0%	
LCDX	12-15%	740.0	748.0	-7.0	107.0		2.4								0.9%	-1.4%	
LCDX	15-100%	206.0	210.0	-13.5	20.0		0.6								0.7%	-0.8%	

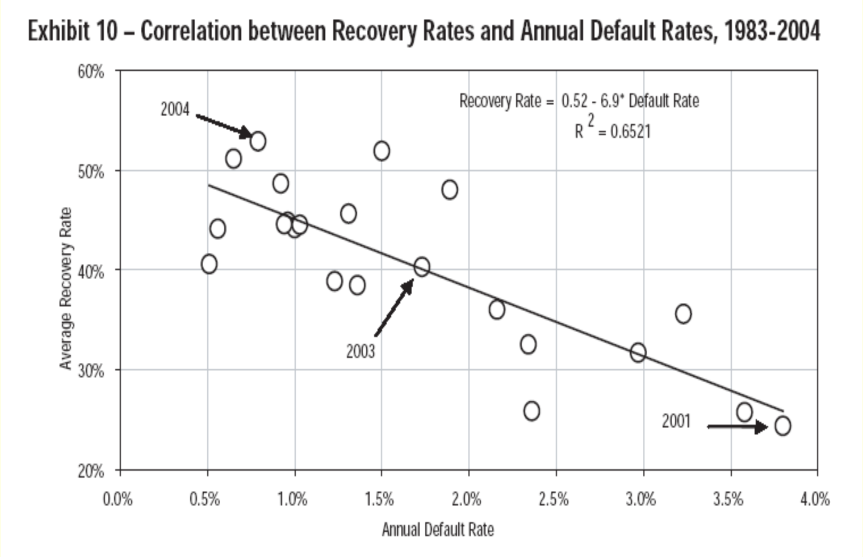
[1]Correlation of tranche with 0% attachment and the same detachment point as the benchmark tranche, implied from market prices of benchmark tranches
[2]Points upfront plus 500 bp running
[3]Points upfront plus 0 bp running
Source: Morgan Stanley

CDO Business context

- Extremely high spreads for senior and super senior tranches
  - Issues with the right-end of the loss distribution
  - Prices in March were not consistent with a fixed recovery rate of 40%
    - See Burtschell et al. [2008], updated version
    - See anecdotal evidence from previous slide
  - In mid-September 2008, no implied correlation could be found for the [30-100%] tranche on the CDX IG even with a 0% recovery
    - Constraints related to the increase and concavity of expected losses on base tranches may not be fulfilled
    - Seemingly, inconsistencies between prices and arbitrage opportunities

CDO Business context

- **Recovery rates**
  - **Market agreement of a fixed recovery rate of 40% is inadequate**



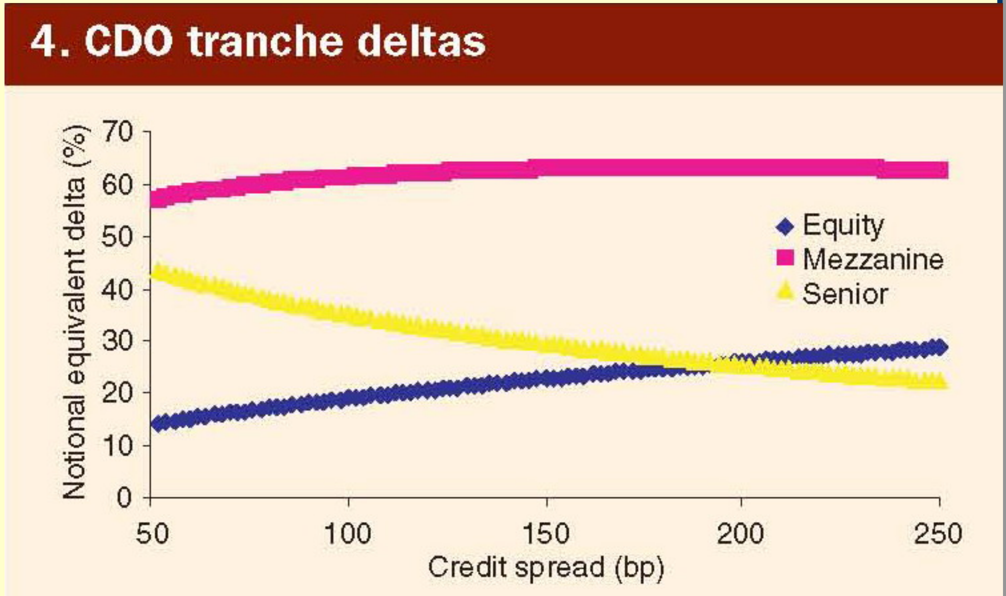
  - **Currently a major issue in the CDO market**
    - Krekel (2008), Amraoui and Hitier (2008)
  - **Use of state dependent stochastic recovery rates will dramatically change the credit deltas**
  - **Management of recovery rate risk?**

CDO Business context

- Some basic issues are back:
  - What is really a default event?
    - Restructuring, take over
    - Recent example of Wachovia
  - and can we define recovery rates properly and consistently across various products (CDS, CLNs, CDOs) and time horizons ?
    - Recent example of Washington Mutual
    - Application of ISDA Protocol to Lehman Brothers
- Jarrow et al. (2008): “distressed debt prices and recovery rate estimation”
  - Raise serious doubts about recovery rate estimation
  - Question the notions of “economic” and “reported” default dates

CDO Business context

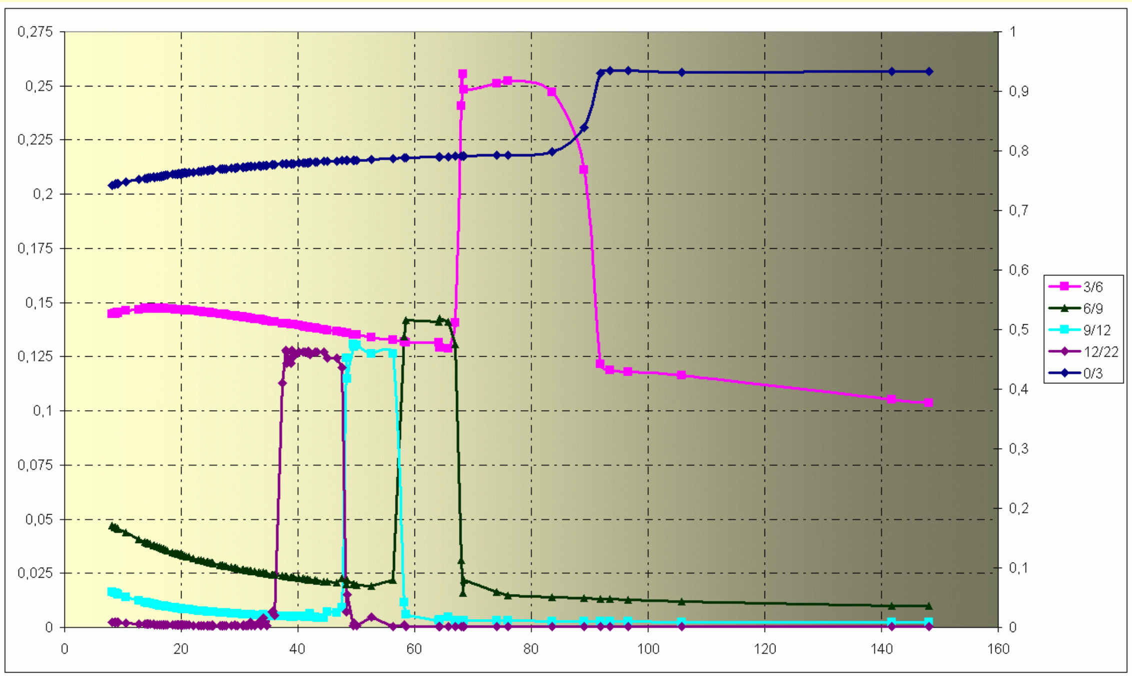
- CDS hedge ratios are computed by bumping the marginal credit curves
  - In 1F Gaussian copula framework
  - Focus on credit spread risk
  - individual name effects
  - Bottom-up approach
  - Smooth effects
  - Pre-crisis…



- At first sight, poor theoretical properties
  - Does not lead to a replication of CDO tranche payoffs
  - Not a hedge against defaults…
  - Unclear issues with respect to the management of correlation risks
  - To be discussed further (break-even correlation)

CDO Business context

- Decline of the one factor Gaussian copula model
- Credit deltas in "high correlation states"
  - Close to comonotonic default dates (current market situation)
  - Deltas are equal to zero or one depending on the level of spreads
    - ➢ Individual effects are too pronounced
    - ➢ Unrealistic i-gammas
    - ➢ Morgan & Mortensen



CDO Business context

- The decline of the one factor Gaussian copula model + base correlation
  - This is rather a practical than a theoretical issue
- Negative tranche deltas frequently occur
  - Which is rather unlikely for out of the money call spreads
    - Though this could actually arise in an arbitrage-free model
    - Schloegl, Mortensen and Morgan (2008)
  - Especially with steep base correlations curves
    - In the base correlation approach, the deltas of base tranches are computed under different correlations
  - And with thin tranchelets
    - Often due to “numerical” and interpolation issues

CDO Business context

- No clear agreement about the computation of credit deltas in the 1F Gaussian copula model
  - Sticky correlation, sticky delta?
  - Computation with respect to credit default swap index, individual CDS?
  - Volatility in the difference between CDS index spread and the average spread of the names in the index
- Weird effects when pricing and risk managing bespoke tranches
  - Price dispersion due to “projection” techniques
  - Negative deltas effects magnified
  - Sensitivity to names out of the considered basket

CDO business context

- Recent advances such as the notion of “break-even” correlation shed new light on the Gaussian copula model
- Think of a structural model with correlated Brownian motions:
  - Default leg of a CDS is like a barrier option
  - Defaults are predictable: no need to account for default risk
  - Correlation between credit spreads equals “correlation” between default events
  - Perfect replication of a CDO tranche can be achieved with the underlying CDS
    - **To cope with credit spread risk**
- One factor Gaussian copula model close to structural model
  - Hull, Predescu and White (2005), Cousin and Laurent (2008)

CDO business context

- Credit deltas in Gaussian copula models may be viewed as approximations of replication deltas in the previous structural model
  - Provided that flat correlation is equal to the correlation of credit spreads
- Drawbacks in the previous approach:
  - Jumps in asset values
    - ➢ **Associated with tail dependence in credit spreads**
    - ➢ **And fat tails in loss distributions**
  - Creates incompleteness
  - Spreads and stocks may move in the same way
    - ➢ **Due to state financial support**
    - ➢ **Collapse of the standard structural model**
    - ➢ **And debt-equity arbitrage…**

Tree approach to hedging defaults

- Complete markets
  - As many risks as hedging instruments
- Perfect replication of payoffs by dynamically trading a small number of « underlying assets »
  - Local volatility type framework
  - Obviously, a stylized view on risk management: model risk
- That is further investigated in the presentation
  - Dynamic trading of CDS to replicate CDO tranche payoff
  - Based on Laurent, Cousin and Fermanian (2008)

Tree approach to hedging defaults

- What are we trying to achieve?
- Show that under some (stringent) assumptions the market for CDO tranches is complete
  - CDO tranches can be perfectly replicated by dynamically trading CDS
  - Exhibit the building of the unique risk-neutral measure
- Display the analogue of the local volatility model of Dupire or Derman & Kani for credit portfolio derivatives
  - One to one correspondence between CDO tranche quotes and model dynamics (continuous time Markov chain for losses)
- Show the practical implementation of the model with market data
  - Deltas correspond to “sticky implied tree”

Tree approach to hedging defaults

- Main theoretical features of the complete market model
  - No simultaneous defaults
    - **Unlike multivariate Poisson models**
  - Credit spreads are driven by defaults
  - Jumps in credit spreads at default times
    - ➢ Contagion model (Jarrow & Yu)
      - **Enron failure was informative**
      - **Not consistent with the "conditional independence" assumption**
    - ➢ Credit spreads are deterministic between two defaults
  - Bottom-up approach
    - ➢ Aggregate loss intensity is derived from individual loss intensities
  - Correlation dynamics is also driven by defaults
    - ➢ Defaults lead to an increase in dependence

Tree approach to hedging defaults

- Changes in the dependence structure between default times
  - In the Gaussian copula world, change in the correlation parameters in the copula
  - The present value of the default leg of an equity tranche decreases when correlation increases
- Dependence parameters and credit spreads may be highly correlated

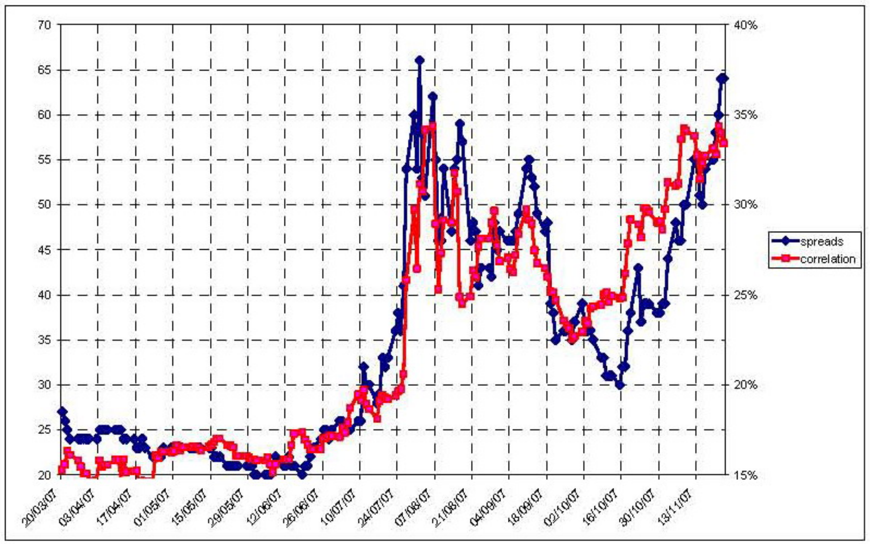
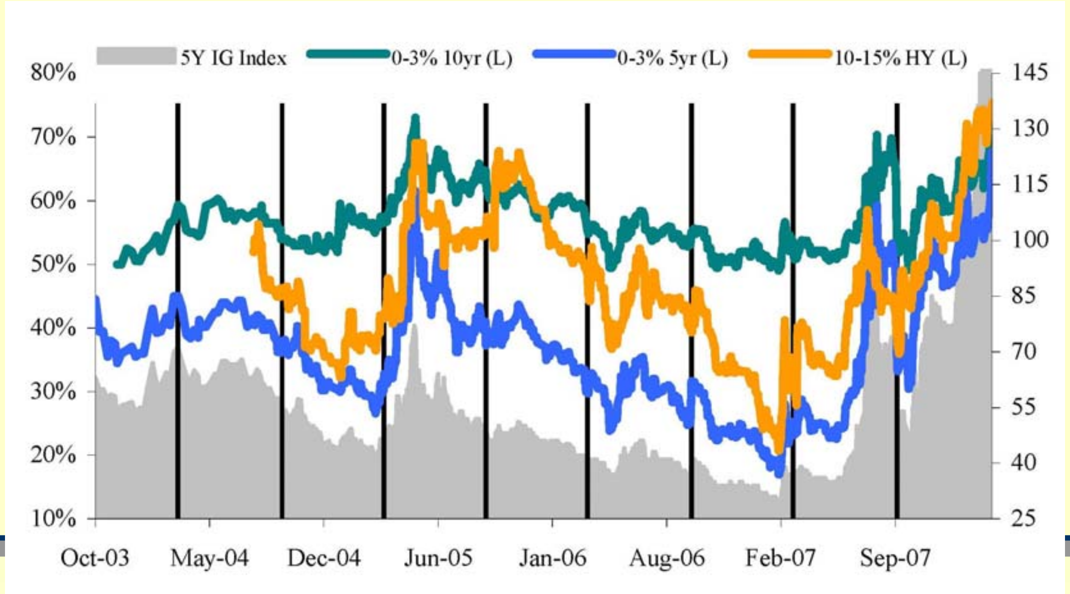


Figure 9. Credit spreads on the five years iTraxx index (Series 7) in bps on the left axis. Implied correlation on the equity tranche on the right axis

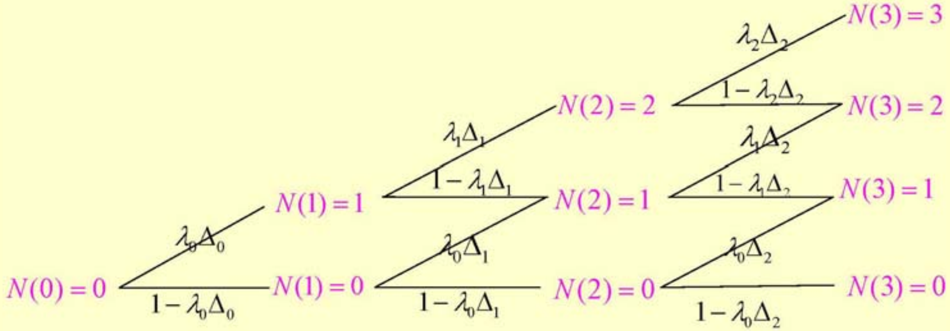
Tree approach to hedging defaults

- Without additional assumptions the model is intractable
  - Homogeneous portfolio
    - Only need of the CDS index
    - No individual name effect, no i-Gamma
    - Top-down approach
      - **Only need of the aggregate loss dynamics**
- Parallel shifts in credit spreads
  - On March 10, 2008, the 5Y CDX IG index spread quoted at 194 bp pa
  - starting from 30 bp pa on February 2007



Tree approach to hedging defaults

- Without additional assumptions the model is intractable
  - Markovian dynamics
    - ➢Pricing and hedging CDO tranches within a binomial tree
  - Perfect calibration the loss dynamics from CDO tranche quotes
    - ➢Thanks to forward induction in the tree
    - ➢On the right, loss intensities wrt number of defaults



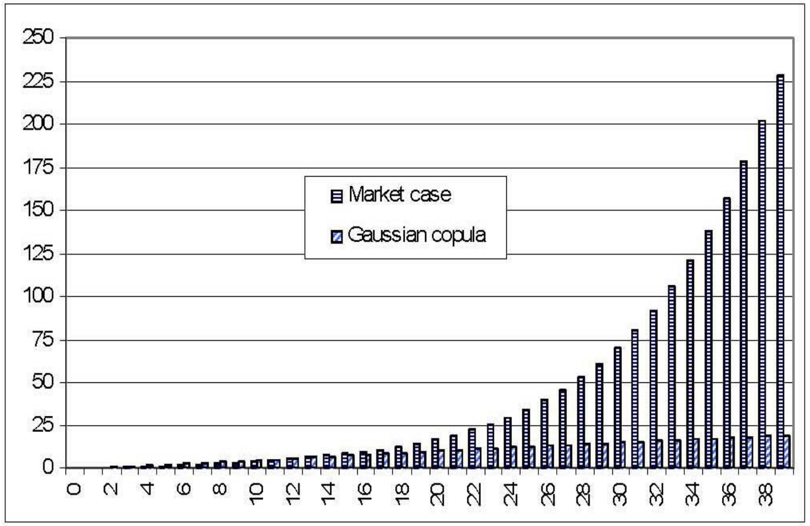


Figure 6. Loss intensities for the Gaussian copula and market case examples. Number of defaults on the $x-$axis.

Tree approach to hedging defaults

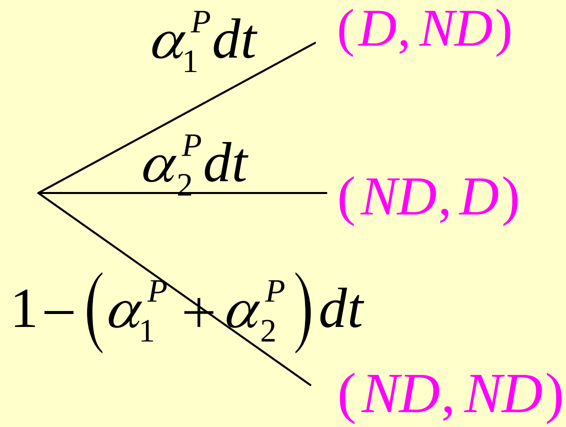
- We will start with two names only
- Firstly in a static framework
  - Look for a First to Default Swap
  - Discuss historical and risk-neutral probabilities
- Further extending the model to a dynamic framework
  - Computation of prices and hedging strategies along the tree
  - Pricing and hedging of tranchelets
- Multiname case: homogeneous Markovian model
  - Computation of risk-neutral tree for the loss
  - Computation of dynamic deltas
- Technical details can be found in the paper:
  - "hedging default risks of CDOs in Markovian contagion models"

Tree approach to hedging defaults

- Some notations :
  - τ_1, τ_2 default times of counterparties 1 and 2,
  - $\mathcal{H}_t$ available information at time *t*,
  - P historical probability,
  - α_1^P, α_2^P : (historical) default intensities:
    - $P\left[\tau_i \in \left[t, t+dt\right[\middle| H_t\right] = \alpha_i^P dt,\ i = 1,2$
- Assumption of « local » independence between default events
  - Probability of 1 and 2 defaulting altogether:
    - $P\left[\tau_1 \in \left[t, t+dt\right[, \tau_2 \in \left[t, t+dt\right[\middle| H_t\right] = \alpha_1^P dt \times \alpha_2^P dt \text{ in } \left(dt\right)^2$
  - Local independence: simultaneous joint defaults can be neglected

Tree approach to hedging defaults

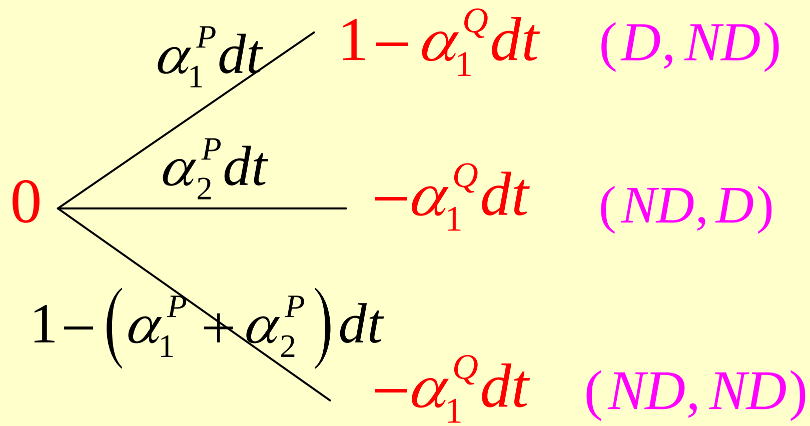
- Building up a tree:
  - Four possible states: (D,D), (D,ND), (ND,D), (ND,ND)
  - Under no simultaneous defaults assumption $p_{(D,D)}=0$
  - Only three possible states: (D,ND), (ND,D), (ND,ND)
  - Identifying (historical) tree probabilities:



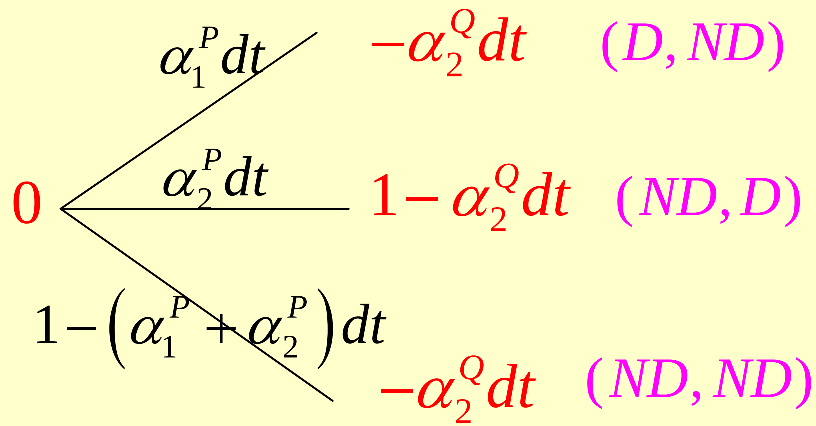
$$\begin{cases} p_{(D,D)}=0 \Rightarrow p_{(D,ND)} = p_{(D,D)} + p_{(D,ND)} = p_{(D,.)} = \alpha_1^P dt \\ p_{(D,D)}=0 \Rightarrow p_{(ND,D)} = p_{(D,D)} + p_{(ND,D)} = p_{(.,D)} = \alpha_2^P dt \\ p_{(ND,ND)} = 1 - p_{(D,.)} - p_{(.,D)} \end{cases}$$

Tree approach to hedging defaults

- Stylized cash flows of short term digital CDS on counterparty 1:
  - $\alpha_1^Q dt$ CDS 1 premium

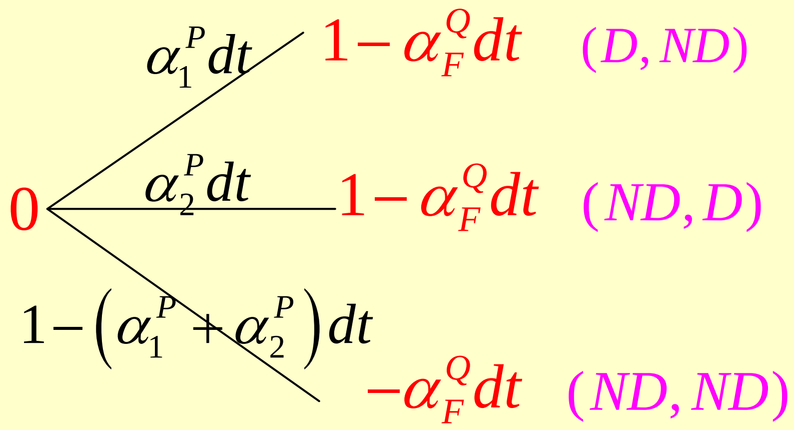


- Stylized cash flows of short term digital CDS on counterparty 2:

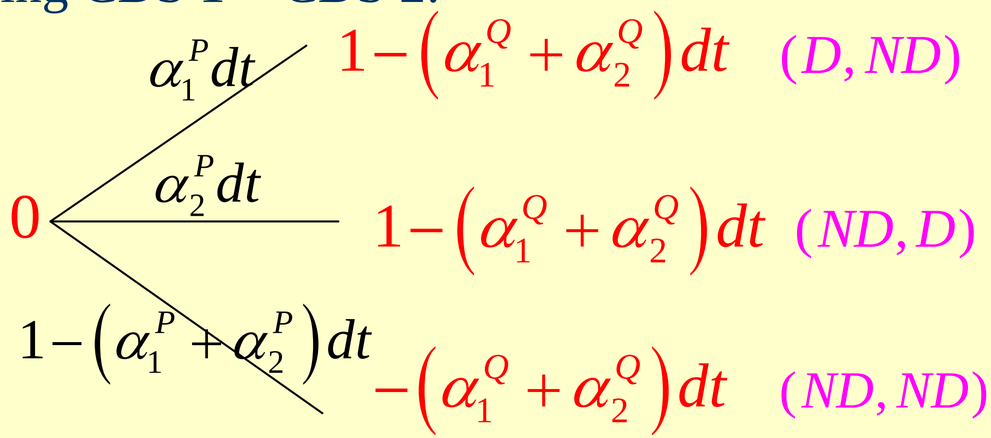


Tree approach to hedging defaults

- Cash flows of short term digital first to default swap with premium $\alpha_F^Q dt$:



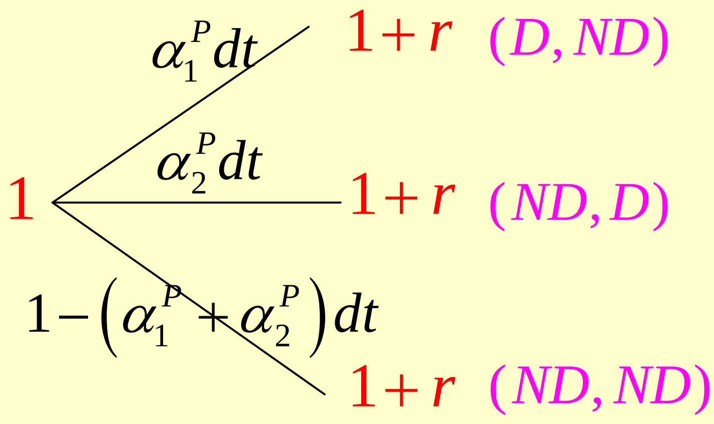
- Cash flows of holding CDS 1 + CDS 2:



- Perfect hedge of first to default swap by holding 1 CDS 1 + 1 CDS 2
  - Delta with respect to CDS 1 = 1, delta with respect to CDS 2 = 1

Tree approach to hedging defaults

- Absence of arbitrage opportunities imply:
  - $\alpha_F^Q = \alpha_1^Q + \alpha_2^Q$
- Arbitrage free first to default swap premium
  - Does not depend on historical probabilities α_1^P, α_2^P
- Three possible states: (*D*,*ND*), (*ND*,*D*), (*ND*,*ND*)
- Three tradable assets: CDS1, CDS2, risk-free asset



- For simplicity, let us assume $r = 0$

Tree approach to hedging defaults

- Three state contingent claims
  - Example: claim contingent on state (D, ND)
  - Can be replicated by holding
  - 1 CDS 1 + $\alpha_1^Q dt$ risk-free asset

$$? \quad \begin{cases} \alpha_1^P dt & 1 \quad (D, ND) \\ \alpha_2^P dt & 0 \quad (ND, D) \\ 1-\left(\alpha_1^P + \alpha_2^P\right)dt & 0 \quad (ND, ND) \end{cases}$$

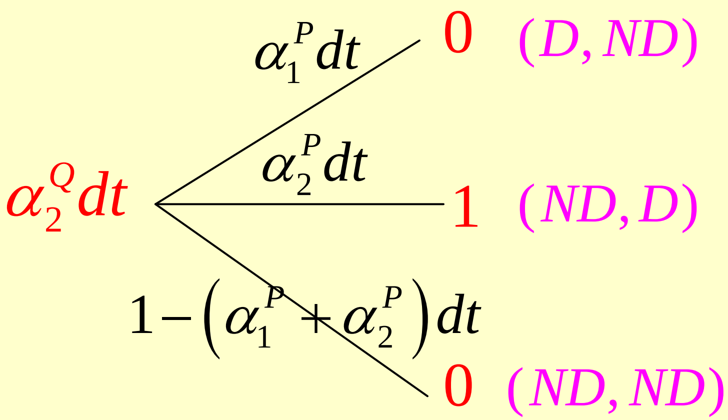
$$\alpha_1^Q dt \quad \begin{cases} \alpha_1^P dt & \alpha_1^Q dt \quad (D, ND) \\ \alpha_2^P dt & \alpha_1^Q dt \quad (ND, D) \\ 1-\left(\alpha_1^P + \alpha_2^P\right)dt & \alpha_1^Q dt \quad (ND, ND) \end{cases} \quad + \quad 0 \quad \begin{cases} \alpha_1^P dt & 1-\alpha_1^Q dt \quad (D, ND) \\ \alpha_2^P dt & -\alpha_1^Q dt \quad (ND, D) \\ 1-\left(\alpha_1^P + \alpha_2^P\right)dt & -\alpha_1^Q dt \quad (ND, ND) \end{cases}$$

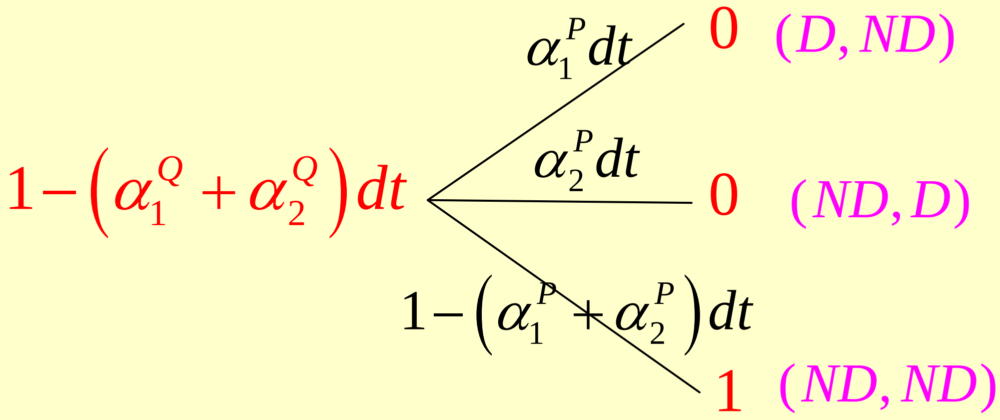
  - Replication price = $\alpha_1^Q dt$

$$\alpha_1^Q dt \quad \begin{cases} \alpha_1^P dt & 1 \quad (D, ND) \\ \alpha_2^P dt & 0 \quad (ND, D) \\ 1-\left(\alpha_1^P + \alpha_2^P\right)dt & 0 \quad (ND, ND) \end{cases}$$

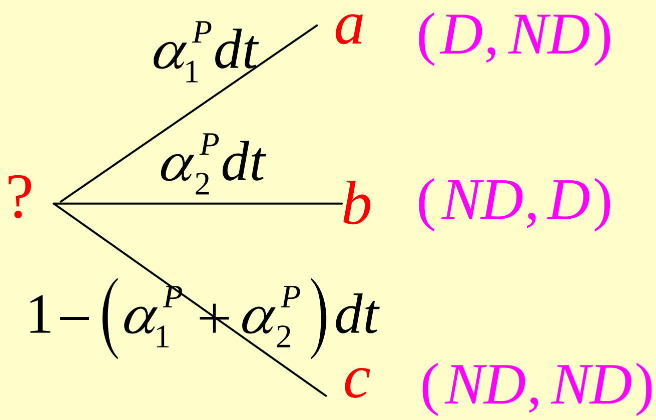
Tree approach to hedging defaults

- Similarly, the replication prices of the (ND, D) and (ND, ND) claims





- Replication price of:

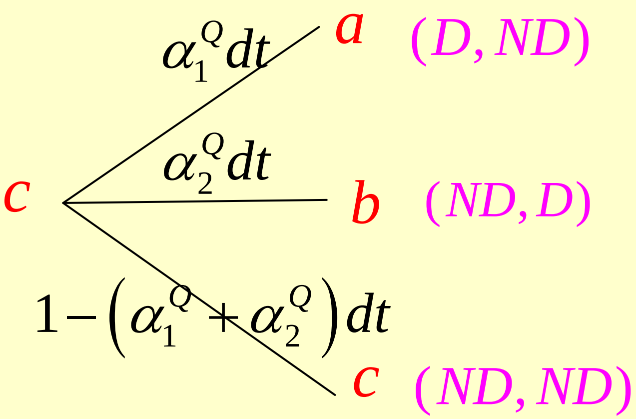


- Replication price = $\alpha_1^Q dt \times a + \alpha_2^Q dt \times b + \left(1-(\alpha_1^Q+\alpha_2^Q)dt\right)c$

Tree approach to hedging defaults

- Replication price obtained by computing the expected payoff
  - Along a risk-neutral tree

$$\alpha_1^Q dt \times a + \alpha_2^Q dt \times b + \left(1 - (\alpha_1^Q + \alpha_2^Q) dt\right) c$$



- Risk-neutral probabilities
  - Used for computing replication prices
  - Uniquely determined from short term CDS premiums
  - No need of historical default probabilities

Tree approach to hedging defaults

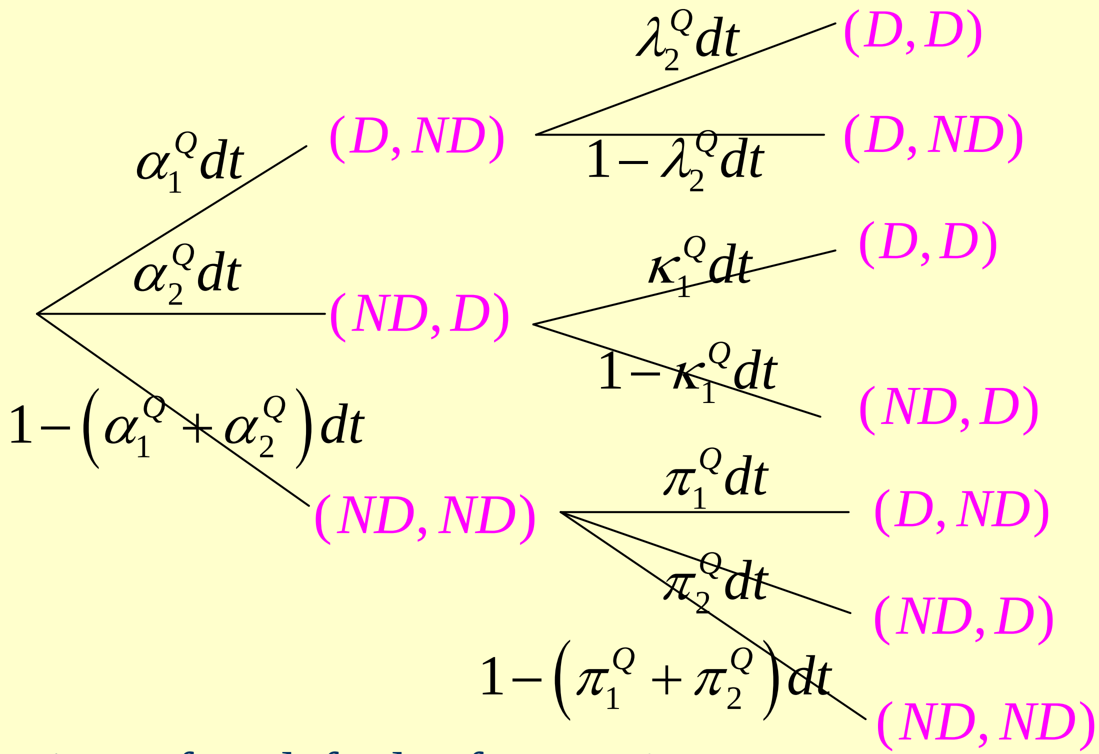
- Computation of deltas
  - Delta with respect to CDS 1: δ_1
  - Delta with respect to CDS 2: δ_2
  - Delta with respect to risk-free asset: p
    - p also equal to up-front premium

$$\begin{cases} a = p + \delta_1 \times \overbrace{\left(1 - \alpha_1^Q dt\right)}^{\text{payoff CDS 1}} + \delta_2 \times \overbrace{\left(-\alpha_2^Q dt\right)}^{\text{payoff CDS 2}} \\ b = p + \delta_1 \times \left(-\alpha_1^Q dt\right) + \delta_2 \times \left(1 - \alpha_2^Q dt\right) \\ c = p + \delta_1 \times \underbrace{\left(-\alpha_1^Q dt\right)}_{\text{payoff CDS 1}} + \delta_2 \times \underbrace{\left(-\alpha_2^Q dt\right)}_{\text{payoff CDS 2}} \end{cases}$$

  - As for the replication price, deltas only depend upon CDS premiums

Tree approach to hedging defaults

- Dynamic case:



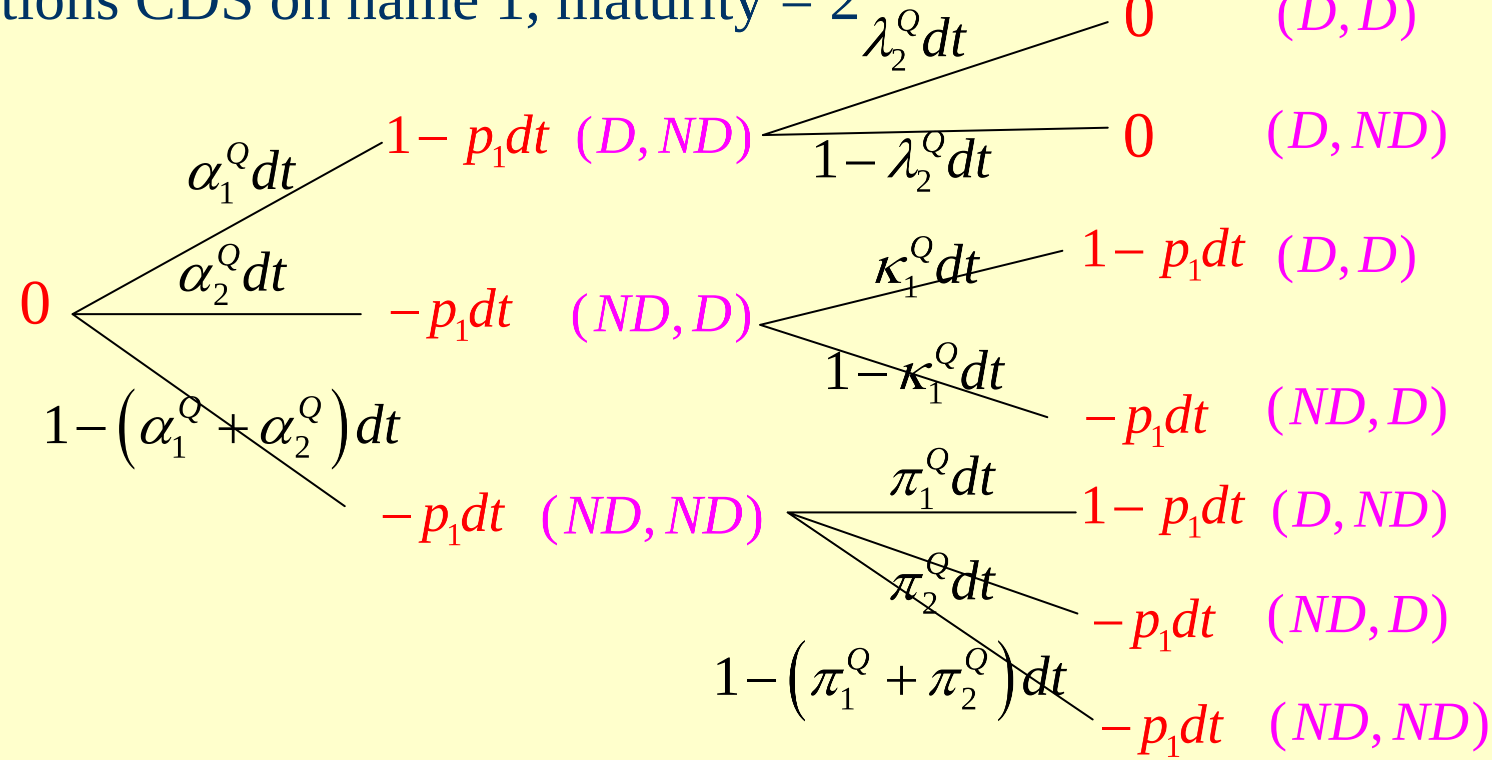
  - $\lambda_2^Q dt$ CDS 2 premium after default of name 1
  - $\kappa_1^Q dt$ CDS 1 premium after default of name 2
  - $\pi_1^Q dt$ CDS 1 premium if no name defaults at period 1
  - $\pi_2^Q dt$ CDS 2 premium if no name defaults at period 1
- Change in CDS premiums due to contagion effects
  - Usually, $\pi_1^Q < \alpha_1^Q < \kappa_1^Q$ and $\pi_2^Q < \alpha_2^Q < \lambda_2^Q$

Tree approach to hedging defaults

- Computation of prices and hedging strategies by backward induction
  - use of the dynamic risk-neutral tree
  - Start from period 2, compute price at period 1 for the three possible nodes
  - + hedge ratios in short term CDS 1,2 at period 1
  - Compute price and hedge ratio in short term CDS 1,2 at time 0
- Example: term structure of credit spreads
  - computation of CDS 1 premium, maturity = 2
  - $p_1 dt$ will denote the periodic premium
  - Cash-flow along the nodes of the tree

Tree approach to hedging defaults

- Computations CDS on name 1, maturity = 2

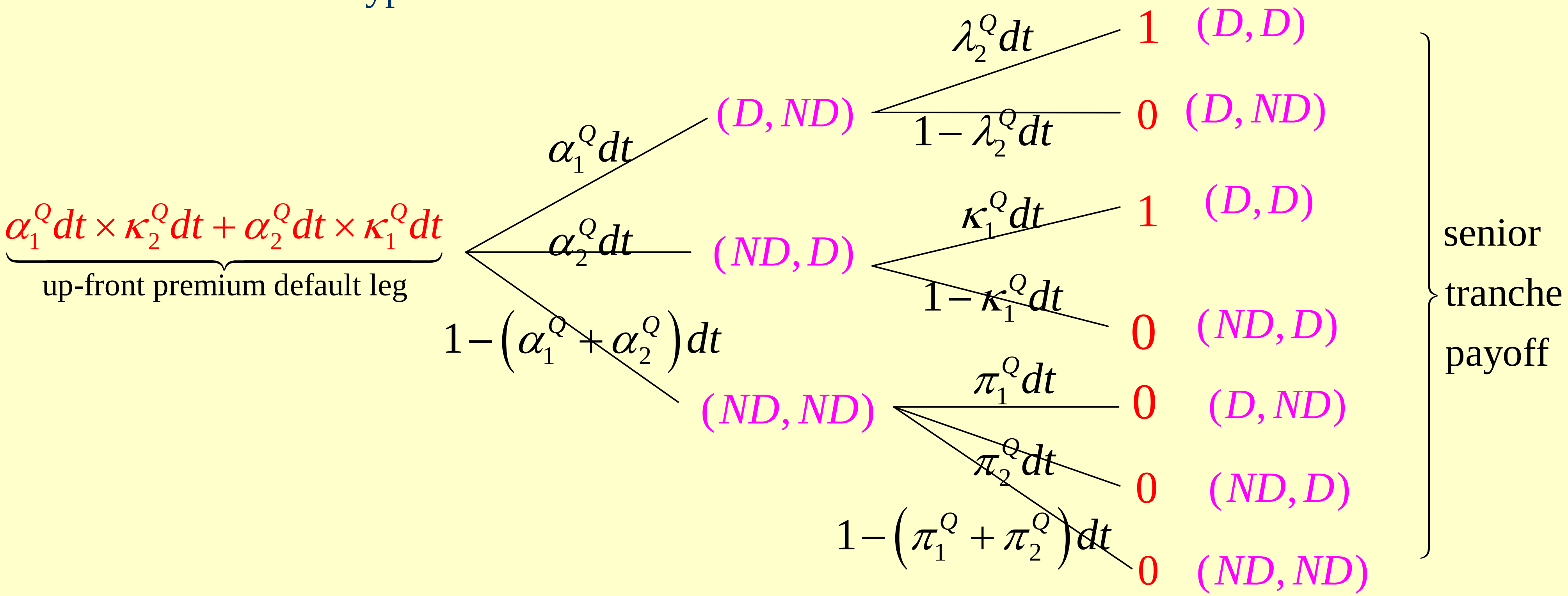


- Premium of CDS on name 1, maturity = 2, time = 0, $p_1 dt$ solves for:

$$0 = \left(1-p_1\right)\alpha_1^Q + \left(-p_1 + \left(1-p_1\right)\kappa_1^Q - p_1\left(1-\kappa_1^Q\right)\right)\alpha_2^Q$$
$$+\left(-p_1 + \left(1-p_1\right)\pi_1^Q - p_1\pi_2^Q - p_1\left(1-\pi_1^Q-\pi_2^Q\right)\right)\left(1-\alpha_1^Q-\alpha_2^Q\right)$$

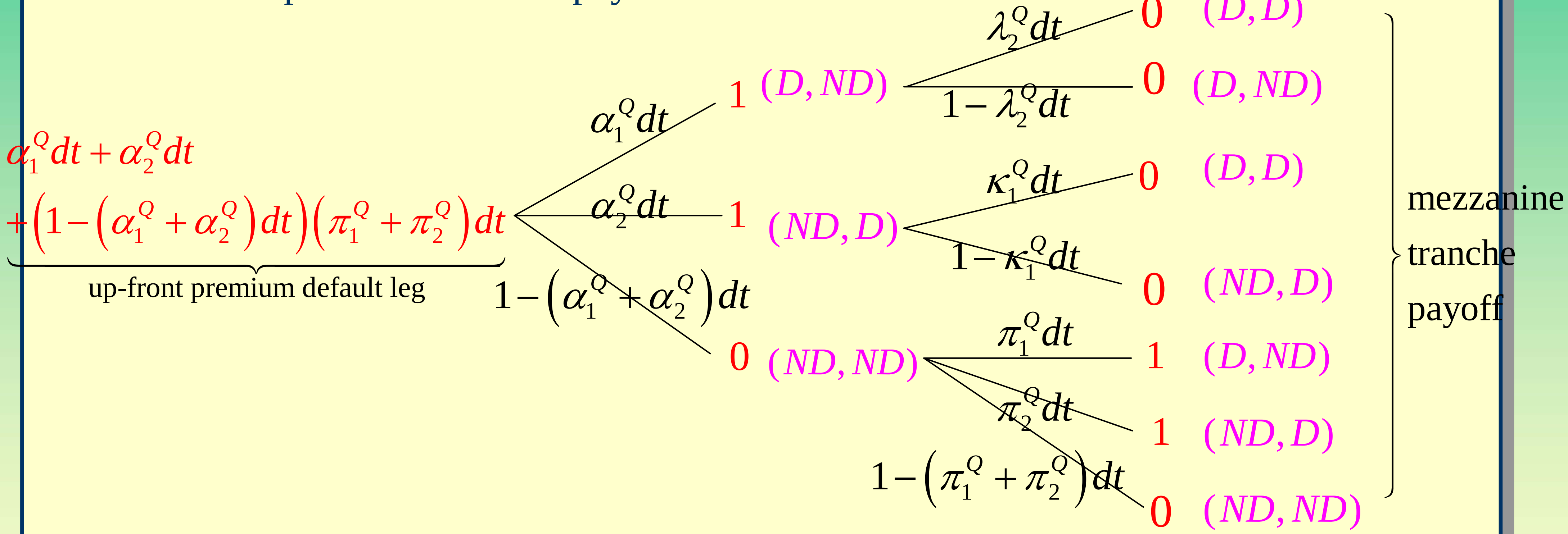
Tree approach to hedging defaults

- **Stylized example: default leg of a senior tranche**
  - Zero-recovery, maturity 2
  - Aggregate loss at time 2 can be equal to 0,1,2
    - Equity type tranche contingent on no defaults
    - Mezzanine type tranche : one default
    - Senior type tranche : two defaults



Tree approach to hedging defaults

- **Stylized example: default leg of a mezzanine tranche**
  - Time pattern of default payments



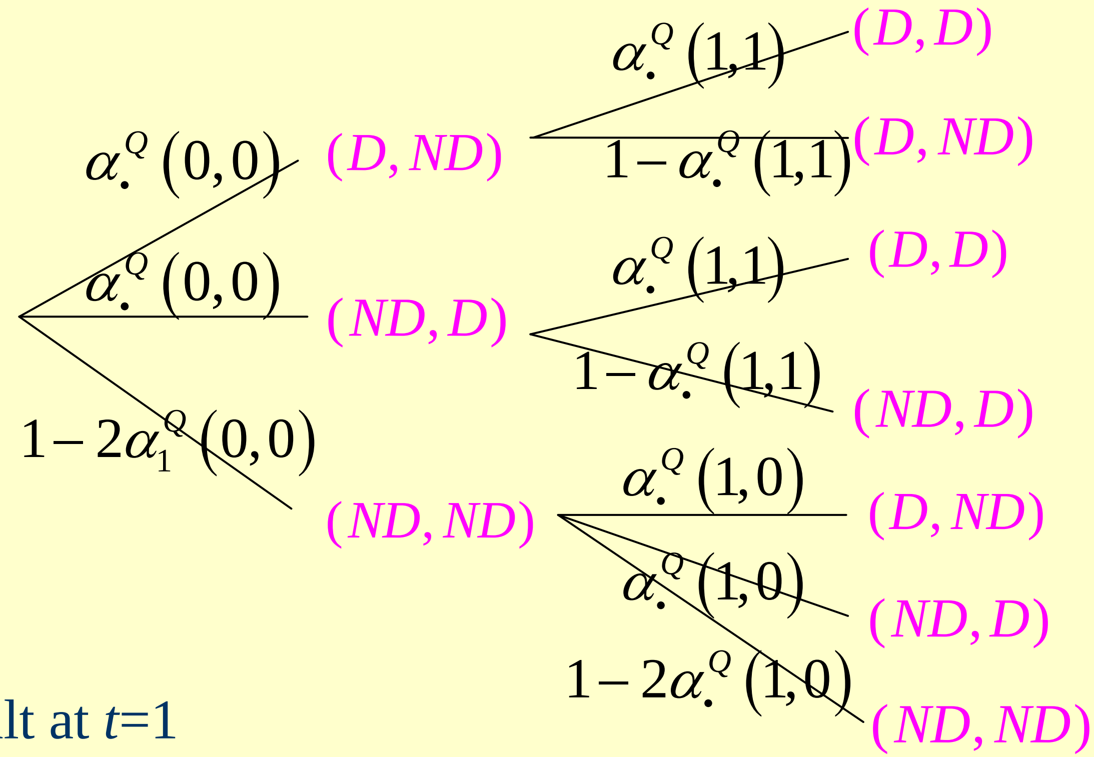
  - Possibility of taking into account discounting effects
  - The timing of premium payments
  - Computation of dynamic deltas with respect to short or actual CDS on names 1,2

Tree approach to hedging defaults

- In theory, one could also derive dynamic hedging strategies for standardized CDO tranches
  - Numerical issues: large dimensional, non recombining trees
  - Homogeneous Markovian assumption is very convenient
    - ➢CDS premiums at a given time *t* only depend upon the current number of defaults $N(t)$
  - CDS premium at time 0 (no defaults) $\alpha_1^Q dt = \alpha_2^Q dt = \alpha_\bullet^Q \left(t = 0, N(0) = 0\right)$
  - CDS premium at time 1 (one default) $\lambda_2^Q dt = \kappa_1^Q dt = \alpha_\bullet^Q \left(t = 1, N(t) = 1\right)$
  - CDS premium at time 1 (no defaults) $\pi_1^Q dt = \pi_2^Q dt = \alpha_\bullet^Q \left(t = 1, N(t) = 0\right)$

Tree approach to hedging defaults

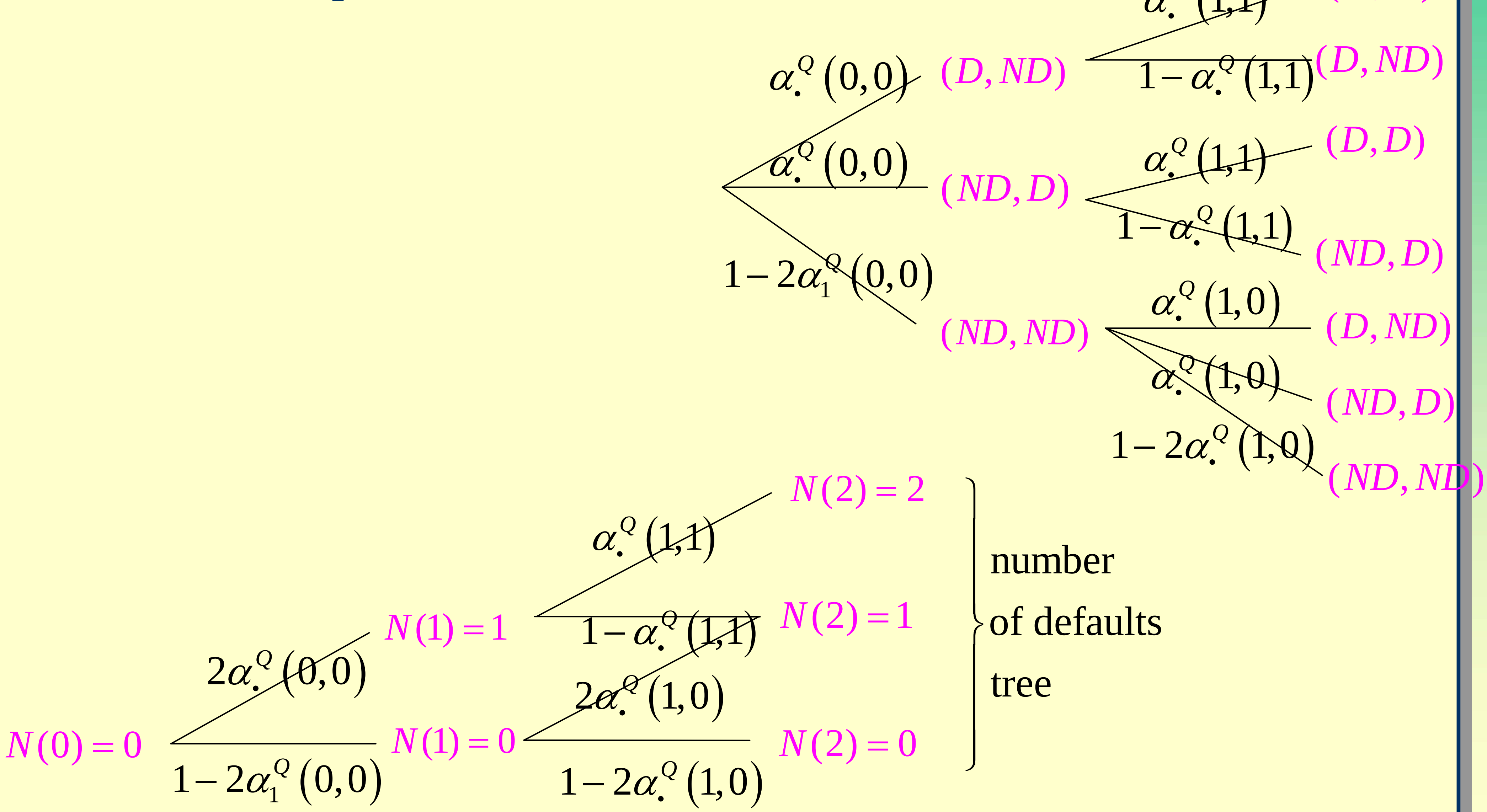
- Tree in the homogeneous case



  - If we have $N(1)=1$, one default at t=1
  - The probability to have $N(2)=1$, one default at t=2…
  - Is $1-\alpha_{\bullet}^{Q}(1,1)$ and does not depend on the defaulted name at t=1
  - $N(t)$ is a **Markov process**
  - **Dynamics of the number of defaults can be expressed through a binomial tree**

Tree approach to hedging defaults

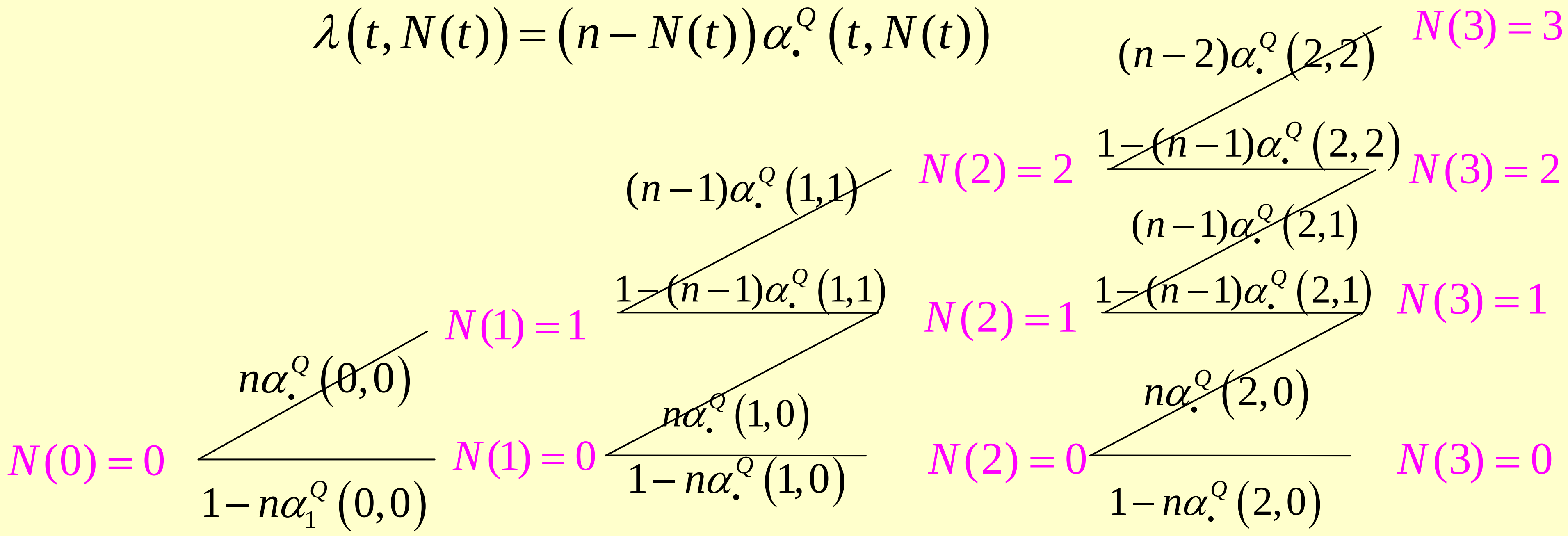
- **From name per name to number of defaults tree**



Tree approach to hedging defaults

- Easy extension to *n* names
  - Predefault name intensity at time *t* for $N(t)$ defaults: $\alpha_{\bullet}^{Q}\left(t,N(t)\right)$
  - Number of defaults intensity : sum of surviving name intensities:

$$\lambda\left(t,N(t)\right)=\left(n-N(t)\right)\alpha_{\bullet}^{Q}\left(t,N(t)\right)$$



  - $\alpha_{\bullet}^{Q}(0,0),\alpha_{\bullet}^{Q}(1,0),\alpha_{\bullet}^{Q}(1,1),\alpha_{\bullet}^{Q}(2,0),\alpha_{\bullet}^{Q}(2,1),\ldots$ can be easily calibrated
  - on marginal distributions of $N(t)$by forward induction.

Mathematical Framework

- n obligors
- Default times: $\tau_1, \ldots, \tau_n$
  - (Ω, A, P) Probability space
- Default indicator processes: $N_i(t) = 1_{\{\tau_i \le t\}}, i = 1, \ldots, n$
- $H_{i,t} = \sigma\left(N_i(s), s \le t\right), i = 1, \ldots n;\ H_t = \bigvee_{i=1}^{n} H_{i,t}$
  - Natural filtration of default times
  - Ordered default times: $\tau^1, \ldots, \tau^n$
  - No simultaneous defaults: $\tau^1 < \ldots < \tau^n, P - a.s.$
- $\alpha_1^P, \ldots, \alpha_n^P \quad (P, H_t)$ intensities
  - $t \to N_i(t) - \int_0^t \alpha_i^P(s)ds \quad (P, H_t)$ martingales

Mathematical Framework

- Instantaneous digital CDS
  - Traded at t
  - Stylized cash-flow at t+dt:

$$\underbrace{dN_i(t)}_{\substack{\text{default}\\ \text{payment}}} - \underbrace{\alpha_i(t)dt}_{\substack{\text{premium}\\ \text{payment}}}$$

- Default free interest rate: r
- Payoffs of self-financed strategies:

$$\underbrace{V_0}_{\substack{\text{initial}\\ \text{investment}}} e^{rT} + \sum_{i=1}^{n} \int_0^T \underbrace{\delta_i(s)}_{\substack{\text{holdings}\\ \text{in CDS } i}} e^{r(T-s)} \left(dN_i(s) - \alpha_i(s)ds\right)$$

  - $\delta_1(\bullet), \ldots, \delta_n(\bullet)$ $H_t -$ predictable processes

Mathematical Framework

- Absence of arbitrage opportunities: $\left\{ \underbrace{\alpha_i(t)}_{\substack{\text{CDS}\\\text{premium}}} > 0 \right\} \overset{P-a.s.}{=} \left\{ \underbrace{\alpha_i^P(t)}_{\substack{(P-H_t)\\\text{intensity}}} > 0 \right\}$
- As a consequence: $\exists! Q \sim P$,
  - such that $\alpha_1, \ldots, \alpha_n$ are the (Q, H_t) intensities of default times
- $M : H_T -$ measurable, Q –integrable payoff
- Integral representation theorem of point processes (Brémaud)

$$M = E^Q[M] + \sum_{i=1}^{n} \int_0^T \underbrace{\theta_i(s)}_{H_s-\text{predictable}} \left(\underbrace{dN_i(s) - \alpha_i(s)ds}_{\substack{\text{CDS}\\\text{cash-flow}}} \right)$$

Mathematical Framework

- Integral representation theorem implies completeness of the credit market
  - Perfect replication of claims which depend only upon the default history
    - ➢With CDS on underlying names and default-free asset
    - ➢CDO tranches
  - Q: unique martingale measure
  - Replication price of M at time t: $V_t = E^Q\left[M e^{-r(T-t)} \middle| H_t \right]$
  - Note that the holdings of CDS only depend upon default history
    - ➢Credit spread risk is not taken into account

Mathematical Framework

- Need of additional assumptions to effectively compute dynamic hedging strategies:

$$\begin{cases} \alpha_i(t) = \alpha\left(t, N(t)\right), \; i = 1,\dots,n \\ N(t) = \sum_{i=1}^{n} N_i(t), \text{ number of defaults at time } t \end{cases}$$

  - CDS spreads only depend upon the current credit status
    - ➢Markov property
  - CDS spreads only depend on the number of defaults
    - ➢Mean-field
  - All names have the same short-term credit spread
    - ➢Homogeneity

Mathematical Framework

- $N(t) = \sum_{i=1}^{n} 1_{\{\tau_i \le t\}}$ number of default process
- is a continuous time Q- Markov chain
  - Pure death process
  - Generator of the Chain $\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & 0 & & & 0 \\ 0 & & \bullet & \bullet & & & 0 \\ 0 & & & & \bullet & & 0 \\ 0 & & & & & \bullet & 0 \\ 0 & & & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
  - $\lambda\left(t, N(t)\right)$ is the intensity of the pure jump process $N(t)$
    - is also the aggregate loss intensity

$$\lambda(t, N(t)) = \underbrace{\left(n - N(t)\right)}_{\substack{\text{number of} \\ \text{non-defaulted} \\ \text{names}}} \times \underbrace{\alpha\left(t, N(t)\right)}_{\substack{\text{individual} \\ \text{pre-default} \\ \text{intensity}}}$$

Mathematical Framework

- Replication price for a CDO tranche $V_t = V_{CDO}\left(t, N(t)\right)$
- Only depends on the number of defaults
  - And of the individual characteristics of the tranche
    - Seniority, maturity, features of premium payments
- Thanks to the "homogeneity" between names:
  - All hedge ratios with respect to individual CDS are equal
  - Only hedge with the CDS index + risk-free asset
- Replicating hedge ratio:

$$\delta\left(t, N(t)\right) = \frac{V_{CDO}\left(t, N(t)+1\right) - V_{CDO}\left(t, N(t)\right)}{V_{CDS\ \text{Index}}\left(t, N(t)+1\right) - V_{CDS\ \text{Index}}\left(t, N(t)\right)}$$

Empirical results

- Calibration of loss intensities
  - From marginal distributions of aggregate losses
  - Or onto CDO tranche quotes
  - Use of forward Kolmogorov equations
    - For the Markov chain
  - Easy to solve for a pure death process
- Loss intensities with respect to the number of defaults
  - For simplicity, assumption of time homogeneous intensities
  - Increase in intensities: contagion effects
  - Compare flat and steep base correlation structures

3%	6%	9%	12%	22%
18%	28%	36%	42%	58%

Table 8. Base correlations with respect to attachment points.

Number of names: 125
Default-free rate: 4%
5Y credit spreads: 20 bps
Recovery rate: 40%

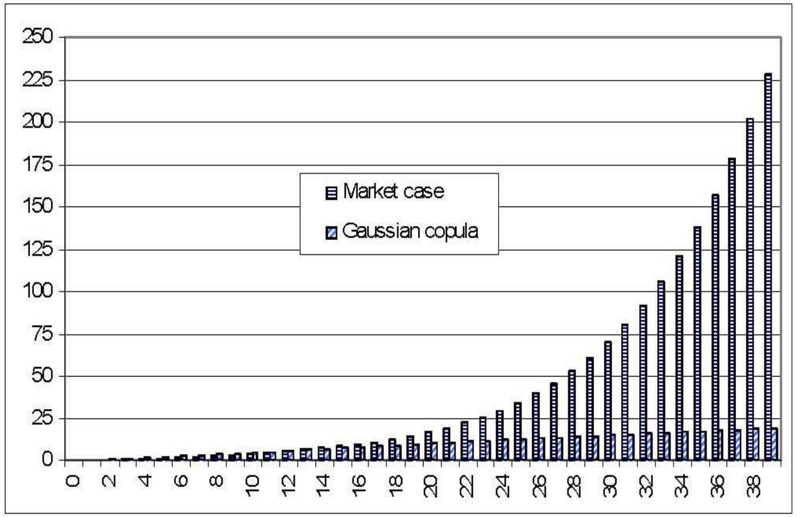


Figure 6. Loss intensities for the Gaussian copula and market case examples. Number of defaults on the x – axis.

Empirical results

- Dynamics of the credit default swap index in the Markov chain

Nb Defaults	Weeks			
	0	14	56	84
0	20	19	17	16
1	0	31	23	20
2	0	95	57	43
3	0	269	150	98
4	0	592	361	228
5	0	1022	723	490
6	0	1466	1193	905
7	0	1870	1680	1420
8	0	2243	2126	1945
9	0	2623	2534	2423
10	0	3035	2939	2859

Table 9. Dynamics of credit default swap index spread $s_{IS}(i,k)$ in basis points per annum.

  - The first default leads to a jump from 19 bps to 31 bps
  - The second default is associated with a jump from 31 bps to 95 bps
  - Explosive behavior associated with upward base correlation curve

Empirical results

- What about the credit deltas?
  - In a homogeneous framework, deltas with respect to CDS are all the same
  - Perfect dynamic replication of a CDO tranche with a credit default swap index and the default-free asset
  - Credit delta with respect to the credit default swap index
  - = change in PV of the tranche / change in PV of the CDS index

Nb Defaults	OutStanding Nominal	Weeks			
		0	14	56	84
0	3.00%	0.541	0.617	0.823	0.910
1	2.52%	0	0.279	0.510	0.690
2	2.04%	0	0.072	0.166	0.304
3	1.56%	0	0.016	0.034	0.072
4	1.08%	0	0.004	0.006	0.012
5	0.60%	0	0.002	0.002	0.002
6	0.12%	0	0.001	0.000	0.000
7	0.00%	0	0	0	0

Table 11. Delta of the default leg of the $[0,3\%]$ equity tranche with respect to the credit default swap index ($\delta_d(i,k)$).

Empirical results

- Dynamics of credit deltas:

Nb Defaults	OutStanding Nominal	Weeks			
		0	14	56	84
0	**3.00%**	0.541	0.617	0.823	0.910
1	**2.52%**	0	0.279	0.510	0.690
2	**2.04%**	0	0.072	0.166	0.304
3	**1.56%**	0	0.016	0.034	0.072
4	**1.08%**	0	0.004	0.006	0.012
5	**0.60%**	0	0.002	0.002	0.002
6	**0.12%**	0	0.001	0.000	0.000
7	**0.00%**	0	0	0	0

Table 11. Delta of the default leg of the $[0,3\%]$ equity tranche with respect to the credit default swap index ($\delta_d(i,k)$).

  - Deltas are between 0 and 1
  - Gradually decrease with the number of defaults
    - ➢Concave payoff, negative gammas
  - When the number of defaults is > 6, the tranche is exhausted
  - Credit deltas increase with time
    - ➢Consistent with a decrease in time value

Empirical results

- Market and theoretical deltas at inception
  - Market deltas computed under the Gaussian copula model
    - Base correlation is unchanged when shifting spreads
    - “Sticky strike” rule
    - Standard way of computing CDS index hedges in trading desks

	[0-3%]	[3-6%]	[6-9%]	[9-12%]	[12-22%]
market deltas	27	4.5	1.25	0.6	0.25
model deltas	21.5	4.63	1.63	0.9	NA

- Smaller equity tranche deltas for in the Markov chain model
    - How can we explain this?

Empirical results

- Smaller equity tranche deltas in the Markov chain model
  - Default is associated with an increase in dependence
    - ➢Contagion effects

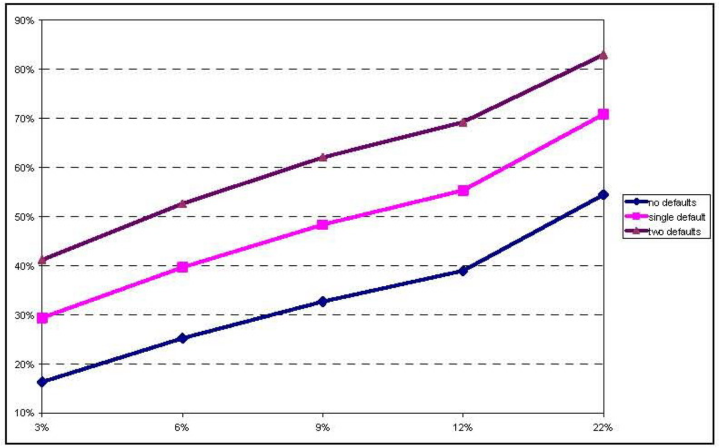


Figure 8. Dynamics of the base correlation curve with respect to the number of defaults. Detachment points on the x–axis. Base correlations on the y–axis.

  - Increasing correlation leads to a decrease in the PV of the equity tranche
    - ➢Sticky implied tree deltas
  - Recent market shifts go in favour of the contagion model

Empirical results

- The current crisis is associated with joint upward shifts in credit spreads
  - Systemic risk
- And an increase in base correlations

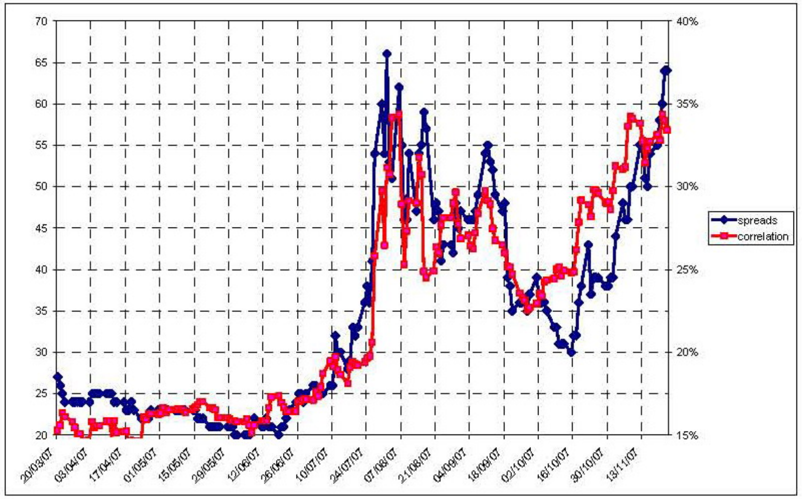


Figure 9. Credit spreads on the five years iTraxx index (Series 7) in bps on the left axis. Implied correlation on the equity tranche on the right axis

- Sticky implied tree deltas are well suited in regimes of fear
  - Derman: “regimes of volatility” (1999)

Empirical results

- We have experienced three defaults on the CDX NA IG so far
- Good material to look further for contagion effects
  - Financials are subject to systemic and thus contagion risk
  - However, the model does not account for feedback effects to central banks looking for stabilizing the financial system
  - The model translates a base correlation skew into contagion effects
  - Does not provide more than market expectations in CDO tranches
  - For example, large spreads in senior and super senior tranches are a sign of fear of a global market turndown

Empirical results

- Comparing with results provided by:
  - Arnsdorf and Halperin "*BSLP: Markovian Bivariate Spread-Loss Model for Portfolio Credit Derivatives*" Working Paper, JP Morgan (2007), Figure 7

	[0-3%]	[3-6%]	[6-9%]	[9-12%]	[12-22%]
market deltas	26.5	4.5	1.25	0.65	0.25
model deltas	21.9	4.81	1.64	0.79	0.38

  - Computed in March 2007 on the iTraxx tranches
  - Two dimensional Markov chain, shift in credit spreads

	[0-3%]	[3-6%]	[6-9%]	[9-12%]	[12-22%]
market deltas	27	4.5	1.25	0.6	0.25
model deltas	21.5	4.63	1.63	0.9	0.6

  - Note that our results, related to default deltas, are quite similar
    - ➢ Equity tranche deltas are smaller in contagion models than Gaussian copula credit deltas

Empirical results

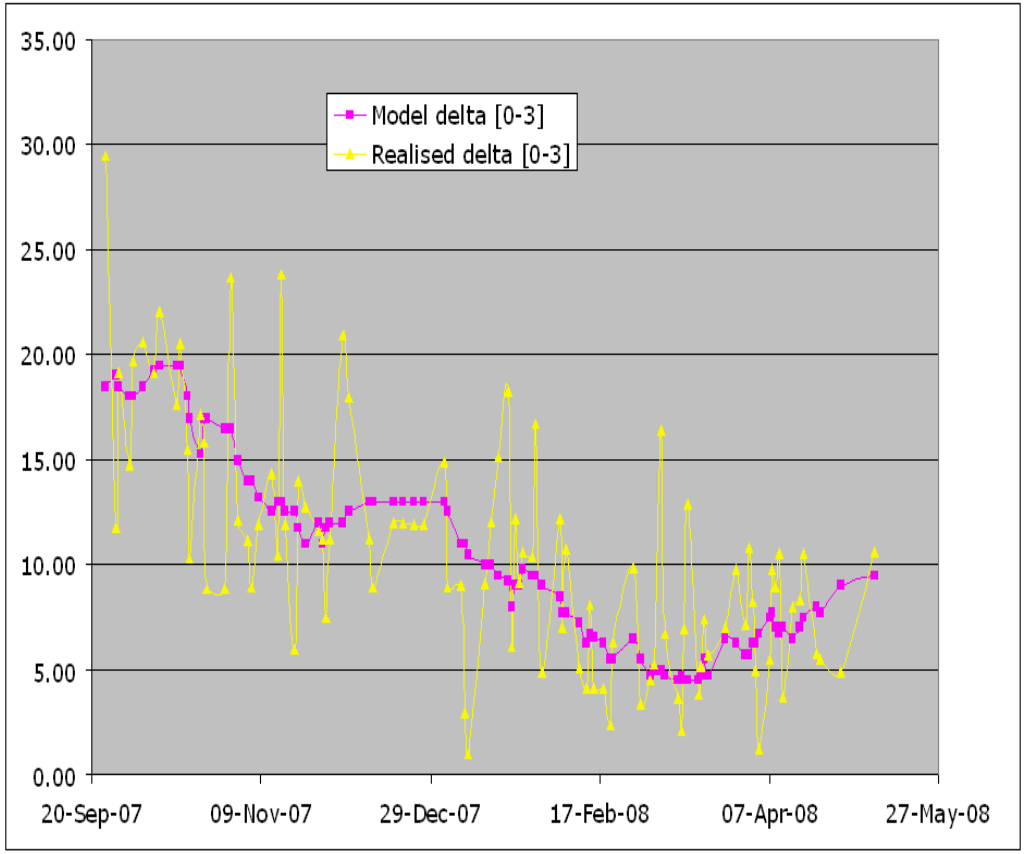
- Cont and Kan: "*Dynamic hedging of portfolio credit derivatives*" (2008)
- **Spread deltas**
  - Gaussian copula model
  - Local intensity corresponds to our contagion model
  - BSLP corresponds to Arnsdorf and Halperin (2007)
  - GPL: generalized Poisson loss model of Brigo *et al*. (2006)
- This shows some kind of robustness
- Picture becomes more complicated when considering other hedging criteria…

Tranche	Gauss	Local	BSLP	GPL
0 - 3	24.48	24.52	24.79	24.48
3 - 6	5.54	5.45	5.30	5.54
6 - 9	1.79	1.80	1.80	1.79
9 - 12	0.87	0.85	0.88	0.87
12 - 22	0.35	0.35	0.32	0.35
22 - 100	0.08	0.08	0.09	0.08

Spread deltas computed for 5Y Europe iTraxx on 20 September 2006

Empirical results

- Back-test study on iTraxx Series 8 equity tranche
- Comparison of realized spread deltas on the equity tranche and model (implied tree) deltas
- Good hedging performance compared with the Gaussian copula model
  - **During the credit crisis**
  - **Discrepancy with results of Cont and Kan (2008)?**



Source: S. Amraoui BNP Paribas

Empirical results

- Cont and Kan (2008) show rather poor performance of "jump to default" deltas
  - Even in the recent crisis period
- However, unsurprisingly, the credit deltas ("jump to default") seem to be rather sensitive to the calibration of contagion parameters on quoted CDO tranches
- Right pictures represent aggregate loss intensities
  - Huge contagion effects for the first six defaults in Cont *et al*. (2008)
  - Much smaller contagion effects for the first defaults in Laurent *et al*. (2007)

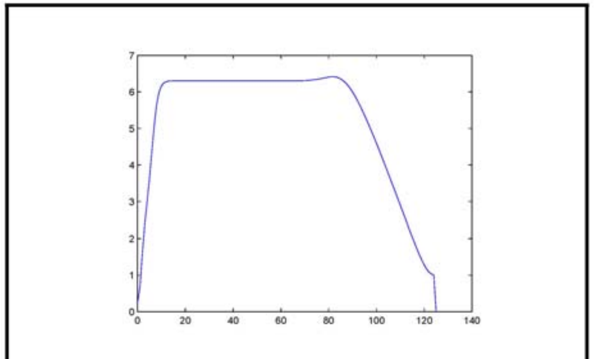


Figure 2: Dependence of default intensity on number of defaults for $t = 1$ year: ITRAXX Europe Series 6, March 15 2007..

Cont, Minca and Savescu (2008)

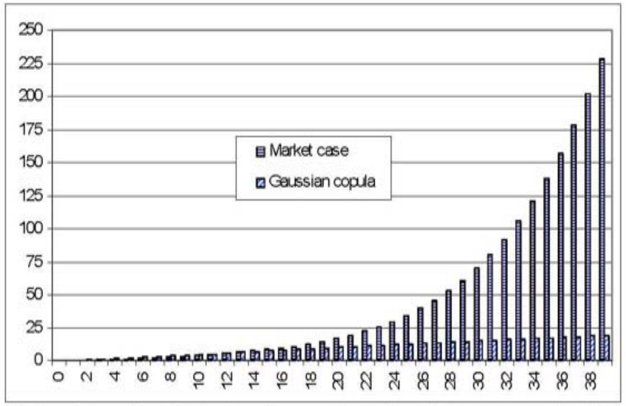


Figure 6. Loss intensities for the Gaussian copula and market case examples. Number of defaults on the x – axis.

Laurent, Cousin and Fermanian (2007)

Empirical results

- Frey and Backhaus: "*Dynamic hedging of synthetic CDO tranches with spread risk and default contagion*" (2007)

Tranche	[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
Spread	26 %	84 bp	24 bp	14 bp	11 bp
Tranche Correlation	17.30 %	3.22 %	9.93 %	15.81 %	27.46 %
Gauss Cop. Δ	$\boxed{0.61}$	0.23	0.06	0.03	0.07

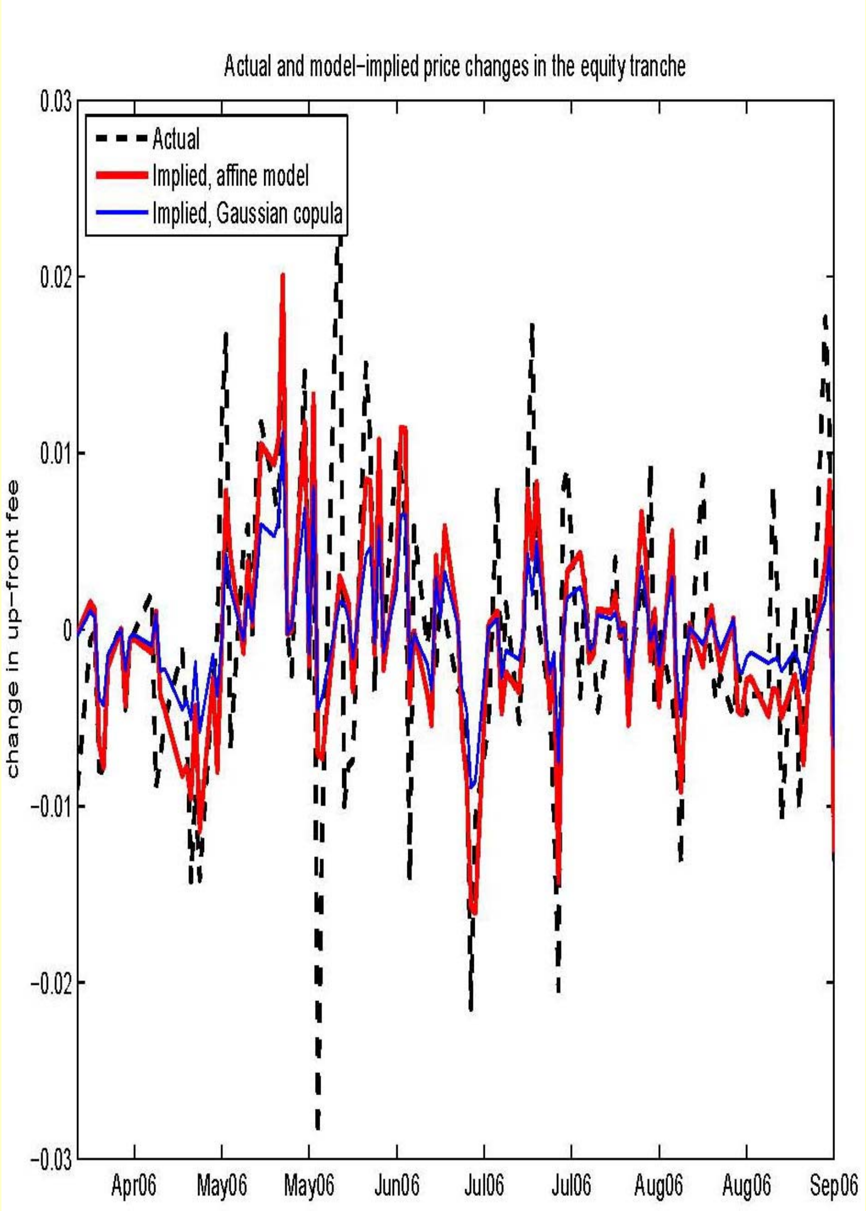
VOD: Value on default

	VOD in the Markov model	VOD in the Copula model
$[0, 3]$	$\boxed{0.344}$	1.002
$[3, 6]$	0.138	0.171
$[6, 9]$	0.058	0.023
$[9, 12]$	0.039	0.008
$[12, 22]$	0.107	0.010

Much smaller deltas in the contagion model than in Gaussian copula model

Empirical results

- Laurent: “A note on the risk management of CDO” (2007)
  - provides a theoretical framework for hedging credit spread risk only while default risk is diversified at the portfolio level
  - no default contagion, correlation between defaults are related to “correlation” between credit spreads
- Feldhütter: “An empirical investigation of an intensity-based model for pricing CDO tranches” (2008)
  - comparison of hedging performance of a Duffie and Garleanu (2001) reduced-form model and one factor Gaussian copula
  - Use of information at time *t*+1 to compute hedge ratios at time *t*
  - Higher deltas for the equity tranche in the affine model compared with the 1F Gaussian copula (market deltas)



Empirical results

- Consistent results with the affine model of Eckner (2007) based on December 2005 CDX data

Tranches	[0-3%]	[3-7%]	[7-10%]	[10-15%]	[15-30%]
market deltas	18.5	5.5	1.5	0.8	0.4
AJD deltas	21.7	6.0	1.1	0.4	0.1
contagion model deltas	17.9	6.3	2.5	1.3	0.8

  - Market deltas, “intensity” model credit deltas in Eckner (2007) and contagion model deltas
  - Goes into the opposite direction when comparing with the contagion model
- Note that Feldhütter (2008) and Eckner (2007) are pre-crisis
- And are according to a “sticky delta rule” (Derman) which is reflects irrational exuberance or greed
  - And might be appropriate for the pre-crisis period

Conclusion

- Main theoretical features of the complete market model
  - No simultaneous defaults
    - **Unlike multivariate Poisson models**
  - Credit spreads are driven by defaults
    - ➢ Contagion model
      - **Jumps in credit spreads at default times**
    - ➢ Credit spreads are deterministic between two defaults
  - Bottom-up approach
    - ➢ Aggregate loss intensity is derived from individual loss intensities
  - Correlation dynamics is also driven by defaults
    - ➢ Defaults lead to an increase in dependence

Conclusion

- What did we learn from the previous approaches?
  - Thanks to stringent assumptions:
    - credit spreads driven by defaults
    - homogeneity
    - Markov property
  - It is possible to compute a dynamic hedging strategy
    - Based on the CDS index
  - That fully replicates the CDO tranche payoffs
    - Model matches market quotes of liquid tranches
    - Very simple implementation
    - Credit deltas are easy to understand
  - Improve the computation of default hedges
    - Since it takes into account credit contagion
    - Provide some meaningful results in the current credit crisis

Additional selected references

- Arnsdorf, M., and Halperin, I., 2007, *BSLP: Markovian bivariate spread-loss model for portfolio credit derivatives*, working paper, JP Morgan.
- Bielecki, T.R., Jeanblanc, M., and Rutkowski, M., 2007, *Hedging of basket credit derivatives in default swap market*, Journal of Credit Risk, 3(1), 91–132.
- Bielecki, T.R., Crépey, S., Jeanblanc, M. and Rutkowski, M., 2007, *Valuation of basket credit derivatives in the credit migrations environment*, Handbook of Financial Engineering, J. Birge and V. Linetskyeds., Elsevier, 2007.
- Derman, E., 1999, *Regimes of volatility*, Quantitative Strategies Research Notes, Goldman Sachs.
- Eckner, A., 2007, *Computational techniques for basic affine models of portfolio credit risk*, working paper, Stanford University.
- Frey, R., and Backhaus, J., 2007a, *Pricing and hedging of portfolio credit derivatives with interacting default intensities*, working paper, University of Leipzig.
- Frey, R., and Backhaus, J., 2007b, *Dynamic hedging of synthetic CDO tranches with spread and contagion risk*, working paper, University of Leipzig.
- Herbertsson, A., 2007, *Pricing synthetic CDO tranches in a model with default contagion using the matrix-analytic approach*, working paper, Göteborg University.
- Schönbucher, P.J., 2006, *Portfolio losses and the term-structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives*, working paper, ETH Zürich.