



# *Copula and Other Models of Measuring Correlation and Dependence*

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*Advanced Correlation Modelling and Analysis  
Incisive Media Training Course  
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*A comparative analysis of CDO pricing models  
Beyond the Gaussian copula: stochastic and local correlation  
Available on [www.defaultrisk.com](http://www.defaultrisk.com)*



## *Beyond the Gaussian copula*

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- *One factor Gaussian copula*
  - Factors models, semi-analytical computations
  - Ordering of risks, Base correlation
  - Gaussian extensions, correlation sensitivities
  - Stochastic recovery rates
- *Model dependence/Choice of copula*
  - Student  $t$ , double  $t$ , Clayton, Marshall-Olkin, Stochastic correlation
  - Calibration methodology, empirical results
  - Distribution of conditional default probabilities
- *Beyond the Gaussian copula*
  - Marginal compound correlation
  - Stochastic correlation and state dependent correlation
  - Local correlation



## *Semi explicit pricing, conditional default probabilities*

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- Factor approaches to joint default times distributions:
  - *V: low dimensional factor*
  - *Conditionally on V, default times are independent.*
  - *Conditional default and survival probabilities:*

$$p_t^{i|V} = Q(\tau_i \leq t | V), \quad q_t^{i|V} = Q(\tau_i > t | V).$$

- Why factor models ?
  - *Tackle with large dimensions (i-Traxx, CDX)*
- Need of tractable dependence between defaults:
  - *Parsimonious modelling*
  - *Semi-explicit computations for CDO tranches*
  - *Large portfolio approximations*



## *Semi explicit pricing, conditional default probabilities*

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- Semi-explicit pricing for CDO tranches

- Laurent & Gregory [2003]

- *Default payments are based on the accumulated losses on the pool of credits:*

$$L(t) = \sum_{i=1}^n LGD_i 1_{\{\tau_i \leq t\}}, \quad LGD_i = N_i(1 - \delta_i)$$

- *Tranche premiums only involve call options on the accumulated losses*

$$E \left[ (L(t) - K)^+ \right]$$

- *This is equivalent to knowing the distribution of  $L(t)$*



## *Semi explicit pricing, conditional default probabilities*

- Characteristic function:  $\varphi_{L(t)}(u) = E \left[ e^{iuL(t)} \right]$ 
  - *By conditioning upon  $V$  and using conditional independence:*

$$\varphi_{L(t)}(u) = E \left[ \prod_{1 \leq j \leq n} \left( 1 - p_t^{j|V} + p_t^{j|V} \varphi_{1-\delta_j}(uN_j) \right) \right]$$

- *Distribution of  $L(t)$  can be obtained by FFT*
    - Or other recursion technique
- Only need of conditional default probabilities  $p_t^{i|V}$
- $p_t^{i|V}$  losses on a large homogeneous portfolio
  - *Approximation techniques for pricing CDOs*



## *Semi explicit pricing, conditional default probabilities*

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- One factor Gaussian copula:

- $V, \bar{V}_i, i = 1, \dots, n$  independent Gaussian,

$$V_i = \rho_i V + \sqrt{1 - \rho_i^2} \bar{V}_i$$

- *Default times:*  $\tau_i = F_i^{-1}(\Phi(V_i))$
- $F_i$  marginal distribution function of default times
- *Conditional default probabilities:*

$$p_t^{i|V} = \Phi \left( \frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \rho_i^2}} \right)$$



## *One factor Gaussian copula*

- CDO margins (bps pa)
  - *With respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads 100 bp*
  - *5 years maturity*

	<b>equity</b>	<b>mezzanine</b>	<b>senior</b>
<b>0%</b>	<b>5341</b>	<b>560</b>	<b>0.03</b>
<b>10%</b>	<b>3779</b>	<b>632</b>	<b>4.6</b>
<b>30%</b>	<b>2298</b>	<b>612</b>	<b>20</b>
<b>50%</b>	<b>1491</b>	<b>539</b>	<b>36</b>
<b>70%</b>	<b>937</b>	<b>443</b>	<b>52</b>
<b>100%</b>	<b>167</b>	<b>167</b>	<b>91</b>



## *One factor Gaussian copula*

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- Equity tranche premiums are decreasing wrt  $\rho$ 
  - *General result*
  - *Equity tranche premium is always decreasing with correlation parameter*
    - See Burtschell et al [2005] for more details about stochastic orders
  - *Guarantees uniqueness of « base correlation »*
  - *Monotonicity properties extend to Student  $t$ , Clayton and Marshall-Olkin copulas*





## *One factor Gaussian copula: extreme cases*

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- $\rho = 100\%$ 
  - *Equity tranche premiums decrease with correlation*
  - *Does  $\rho = 100\%$  correspond to some lower bound?*
  - *$\rho = 100\%$  corresponds to « comonotonic » default dates:*
  - *$\rho = 100\%$  is a model free lower bound for the equity tranche premium*
- $\rho = 0\%$ 
  - *Does  $\rho = 0\%$  correspond to the higher bound on the equity tranche premium?*
  - *$\rho = 0\%$  corresponds to the independence case between default dates*
  - *The answer is no, negative dependence can occur*
  - *Base correlation does not always exist*







## One factor Gaussian copula and extensions

### ■ Gaussian extensions

- *Intra & intersector correlations*

$$V_i = \rho_{s(i)} W_{k(i)} + \sqrt{1 - \rho_{s(i)}^2} \bar{V}_i$$

- *i, name, s(i) sector*

- *$\rho$  systemic correlation*

$$W_{s(i)} = \rho W + \sqrt{1 - \rho^2} \bar{W}_{s(i)}$$

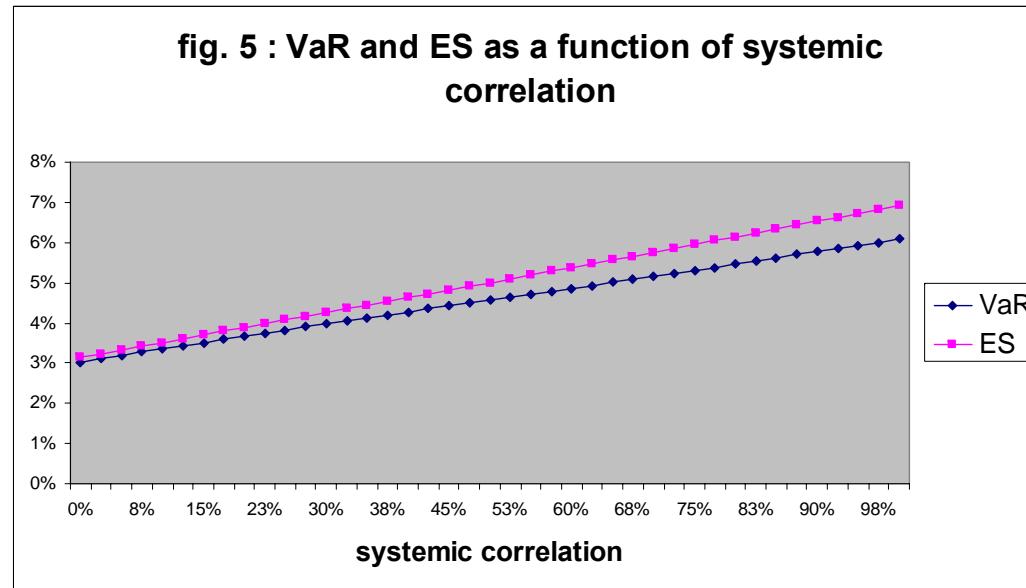
- *Accounting for sector diversification in risk assessment*

- *Risk measures based on unexpected losses,  $\alpha = 99.9\%$*

	$\zeta$ (VaR)	$\kappa$ (Expected Shortfall)
$\rho = 100\%$ (Basel II)	6,1%	6,9%
$\rho = 50\%$ (multifactor model)	4,6%	5,0%
Relative variation	-25%	-27%

# One factor Gaussian copula and extensions

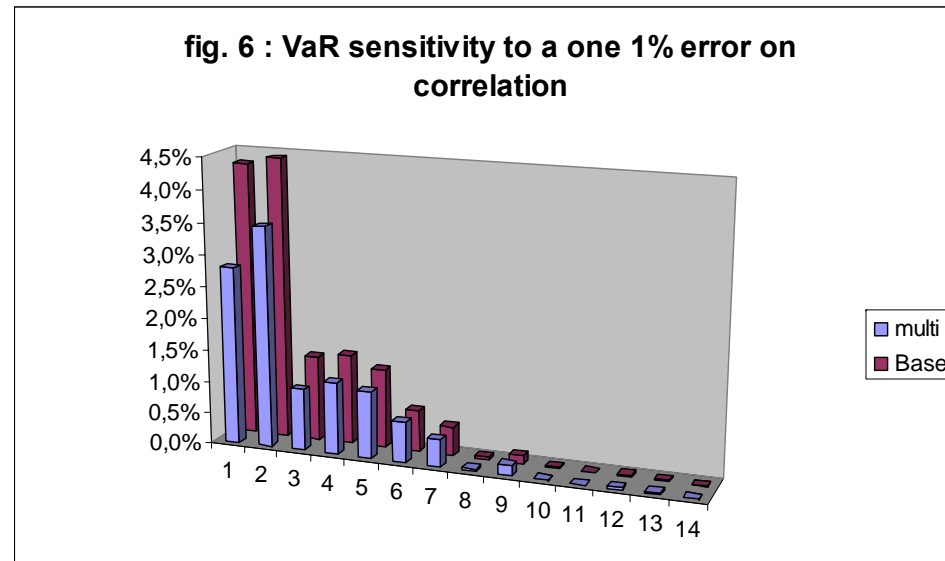
- VaR, Expected Shortfall and systemic correlation



- *Risk measures change almost linearly wrt to systemic correlation*
- *Basel II: no sector diversification*
- *Sector diversification lessens capital requirements*
  - See “**Aggregation and credit risk measurement in retail banking**”, Chabaane et al [2003]

# One factor Gaussian copula and extensions

- VaR and intrasector correlation



- *Elasticity of VaR wrt intrasector correlation parameters*

$$\frac{\rho_J}{\zeta} \times \frac{\partial \zeta}{\partial \rho_J}$$

- *Lines 1 and 2 correspond to subportfolios with highest credit quality*



## *One factor Gaussian copula and extensions*

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- Correlation between default dates and recovery rates

- *One factor Gaussian copula for default dates*  $\Psi_i = \sqrt{\rho}\Psi + \sqrt{1-\rho}\bar{\Psi}_i$

- *Losses Given Default also have a one factor structure:*

$$\xi_i = \sqrt{\beta}\xi + \sqrt{1-\beta}\bar{\xi}_i$$

- *Merton type LGD:*  $\max(0, 1 - e^{\mu + \sigma\xi_i})$

- *A two factor Gaussian model with factors  $\Psi, \xi$*

- *Correlation between defaults & recoveries and amongst recoveries*

- See **Credit Risk Assessment and Stochastic LGD's: an Investigation of Correlation Effects** in *Recovery Risk: The Next Challenge in Credit Risk Management*, Risk Books

## One factor Gaussian copula and extensions

- Correlation between default dates and recovery rates
  - *VaR and ES as a function of correlation parameters*

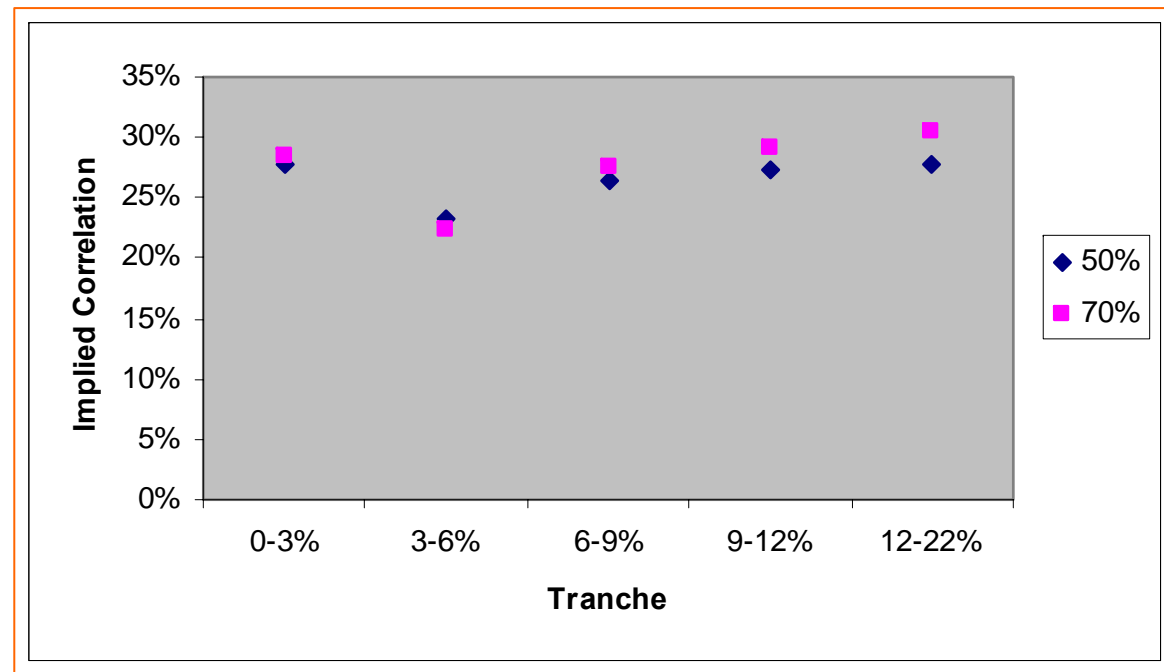
$\beta$ $\eta$	0%	20%	40%	60%	80%	100%
0%	158,9%	161,0%	164,2%	162,5%	159,3%	145,9%
	154,8%	160,2%	165,4%	164,7%	162,4%	152,1%
20%	157,5%	175,4%	182,6%	186,8%	186,0%	172,8%
	153,9%	175,6%	183,7%	188,6%	192,5%	179,8%
40%	160,2%	194,1%	207,9%	211,8%	212,6%	205,7%
	156,0%	196,6%	211,6%	218,7%	219,5%	217,2%
60%	158,2%	207,4%	227,0%	238,9%	240,8%	234,1%
	155,2%	210,3%	231,1%	243,0%	249,2%	243,4%
80%	159,6%	223,1%	244,1%	257,4%	264,5%	260,5%
	156,0%	229,4%	249,4%	265,1%	271,2%	273,4%
100%	158,1%	238,9%	262,7%	276,5%	283,3%	286,8%
	153,9%	246,4%	268,0%	287,3%	296,3%	296,6%

- *Taking into account correlation between default events and LGD leads to a substantial increase in VaR and Expected Shortfall*



## One factor Gaussian copula and extensions

- Correlation between default dates and recovery rates
  - *Correlation smile implied from the correlated recovery rates*
  - *Not as important as what is found in the market*





## *Model dependence / choice of copula*

- Stochastic correlation copula
  - $V, \bar{V}_i, i = 1, \dots, n$  independent Gaussian variables
  - $B_i = 1$  correlation  $\rho$ ,  $B_i = 0$  correlation  $\beta$

$$V_i = B_i \left( \rho V + \sqrt{1 - \rho^2} \bar{V}_i \right) + (1 - B_i) \left( \beta V + \sqrt{1 - \beta^2} \bar{V}_i \right)$$

$$\tau_i = F_i^{-1}(\Phi(V_i))$$

$$p_t^{i|V} = p \Phi \left( \frac{-\rho V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \rho^2}} \right) + (1 - p) \Phi \left( \frac{-\beta V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \beta^2}} \right)$$



## Model dependence / choice of copula

- Student  $t$  copula

- Embrechts, Lindskog & McNeil, Greenberg et al, Mashal et al, O'Kane & Schloegl, Gilkes & Jobst

$$\begin{cases} X_i = \rho V + \sqrt{1 - \rho^2} \bar{V}_i \\ V_i = \sqrt{W} \times X_i \\ \tau_i = F_i^{-1} (t_v (V_i)) \end{cases}$$

- $V, \bar{V}_i$  independent Gaussian variables
- $\frac{v}{W}$  follows a  $\chi_v^2$  distribution
- Conditional default probabilities (two factor model)

$$P_t^{i|V,W} = \Phi \left( \frac{-\rho V + W^{-1/2} t_v^{-1} (F_i(t))}{\sqrt{1 - \rho^2}} \right)$$



## *Model dependence / choice of copula*

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- *Clayton copula*

- *Schönbucher & Schubert, Rogge & Schönbucher, Friend & Rogge, Madan et al*

$$V_i = \psi \left( -\frac{\ln U_i}{V} \right) \quad \tau_i = F_i^{-1}(V_i) \quad \psi(s) = (1+s)^{-1/\theta}$$

- Marshall-Olkin construction of archimedean copulas
- *V: Gamma distribution with parameter  $\theta$*
- *$U_1, \dots, U_n$  independent uniform variables*
- *Conditional default probabilities (one factor model)*

$$p_t^{i|V} = \exp \left( V \left( 1 - F_i(t)^{-\theta} \right) \right)$$



## *Model dependence / choice of copula*

- Double  $t$  model (Hull & White)

$$V_i = \rho_i \left( \frac{\nu - 2}{\nu} \right)^{1/2} V + \sqrt{1 - \rho_i^2} \left( \frac{\bar{\nu} - 2}{\bar{\nu}} \right)^{1/2} \bar{V}_i$$

- $V, \bar{V}_i$  are independent Student  $t$  variables
  - with  $\nu$  and  $\bar{\nu}$  degrees of freedom

$$\tau_i = F_i^{-1} \left( H_i \left( V_i \right) \right)$$

- where  $H_i$  is the distribution function of  $V_i$

$$P_t^{i|V} = t_{\bar{\nu}} \left( \left( \frac{\bar{\nu}}{\bar{\nu} - 2} \right)^{1/2} \frac{H_i^{-1} \left( F_i(t) \right) - \rho_i \left( \frac{\nu - 2}{\nu} \right)^{1/2} V}{\sqrt{1 - \rho_i^2}} \right)$$



## *Model dependence / choice of copula*

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- Shock models (multivariate exponential copulas)
  - *Duffie & Singleton, Giesecke, Elouerkhaoui, Lindskog & McNeil, Wong*
- Modelling of default dates:  $V_i = \min(V, \bar{V}_i)$ 
  - $V, \bar{V}_i$  exponential with parameters  $\alpha, 1-\alpha$
  - Default dates  $\tau_i = S_i^{-1}(\exp(-\min(V, \bar{V}_i)))$ 
    - $S_i$  marginal survival function
  - *Conditionally on  $V$ ,  $\tau_i$  are independent.*
- Conditional default probabilities

$$q_t^{i|V} = 1_{V > -\ln S_i(t)} S_i(t)^{1-\alpha}$$



## *Model dependence / choice of copula*

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- Calibration procedure
  - *One parameter copulas*
  - *Fit Clayton, Student  $t$ , double  $t$ , Marshall Olkin parameters onto CDO equity tranches*
    - Computed under one factor Gaussian model
    - Or given market quotes on equity tranches
  - *Reprice mezzanine and senior CDO tranches*
    - Given the previous parameter



## *Model dependence / choice of copula*

- CDO margins (bps pa)
  - *With respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads 100 bp*
  - *5 years maturity*

	equity	mezzanine	senior
0%	5341	560	0.03
10%	3779	632	4.6
30%	2298	612	20
50%	1491	539	36
70%	937	443	52
100%	167	167	91





## *Model dependence / choice of copula*

$\rho$	0%	10%	30%	50%	70%	100%
$\theta$	0	0.05	0.18	0.36	0.66	$\infty$
$\rho_6^2$			14%	39%	63%	100%
$\rho_{12}^2$			22%	45%	67%	100%
$\rho$ $t(4)$ - $t(4)$	0%	12%	34%	55%	73%	100%
$\rho$ $t(5)$ - $t(4)$	0%	13%	36%	56%	74%	100%
$\rho$ $t(4)$ - $t(5)$	0%	12%	34%	54%	73%	100%
$\rho$ $t(3)$ - $t(4)$	0%	10%	32%	53%	75%	100%
$\rho$ $t(4)$ - $t(3)$	0%	11%	33%	54%	73%	100%
$\alpha$	0	28%	53%	69%	80%	100%

Table 5: correspondence between parameters



## *Model dependence / choice of copula*

$\rho$	0%	10%	30%	50%	70%	100%
Gaussian	560	633	612	539	443	167
Clayton	560	637	628	560	464	167
Student (6)			637	550	447	167
Student (12)			621	543	445	167
$t(4)-t(4)$	560	527	435	369	313	167
$t(5)-t(4)$	560	545	454	385	323	167
$t(4)-t(5)$	560	538	451	385	326	167
$t(3)-t(4)$	560	495	397	339	316	167
$t(4)-t(3)$	560	508	406	342	291	167
MO	560	284	144	125	134	167

Table 6: mezzanine tranche (bps pa)



## *Model dependence / choice of copula*

$\rho$	0%	10%	30%	50%	70%	100%
Gaussian	0.03	4.6	20	36	52	91
Clayton	0.03	4.0	18	33	50	91
Student (6)			17	34	51	91
Student (12)			19	35	52	91
$t(4)-t(4)$	0.03	11	30	45	60	91
$t(5)-t(4)$	0.03	10	29	45	59	91
$t(4)-t(5)$	0.03	10	29	44	59	91
$t(3)-t(4)$	0.03	12	32	47	71	91
$t(4)-t(3)$	0.03	12	32	47	61	91
MO	0.03	25	49	62	73	91

Table 7: senior tranche (bps pa)

Gaussian, Clayton and Student  $t$  CDO premiums are close

# Model dependence / choice of copula

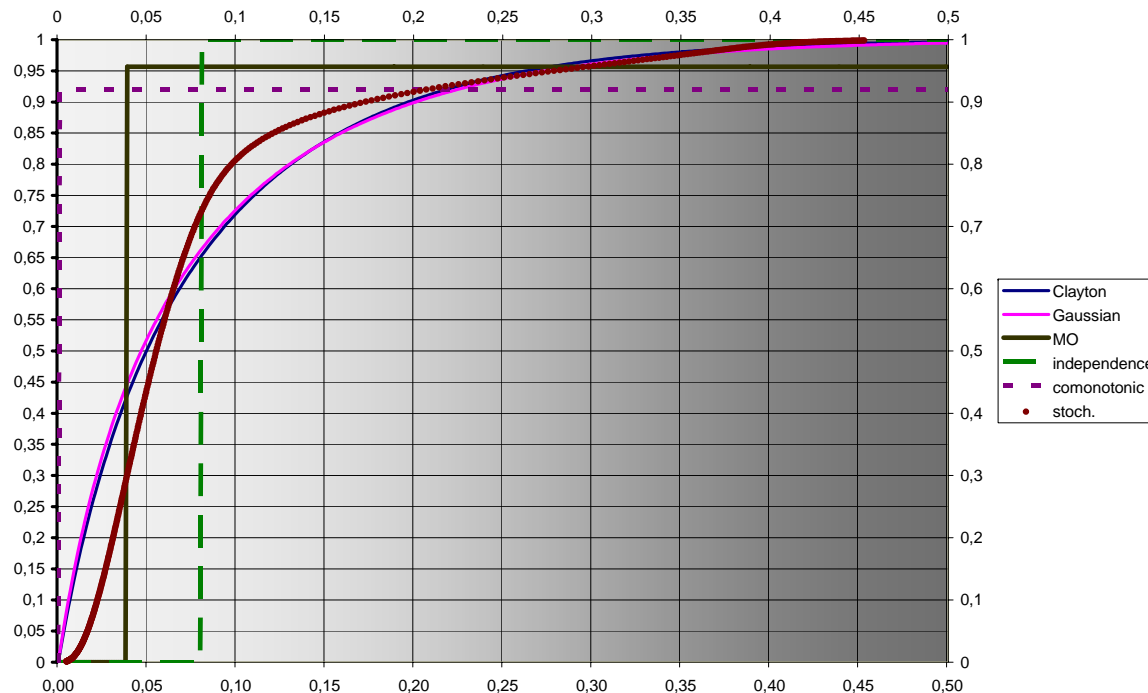
## Why Clayton and Gaussian copulas provide same SL premiums?

- Loss distributions depend on the **distribution** of conditional default probabilities

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

$$p_t^{i|V} = \Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1 - \rho^2}}\right)$$

- Distribution of conditional default probabilities are close for Gaussian and Clayton





## Matching the correlation skew

Tranches	Market	Gaussian	Clayton	Student (12)	$t(4)-t(4)$	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[3-6%]	101	163	163	164	82	122	14
[6-9%]	33	48	47	47	34	53	11
[9-12%]	16	17	16	15	22	29	11
[12-22%]	9	3	2	2	13	8	11

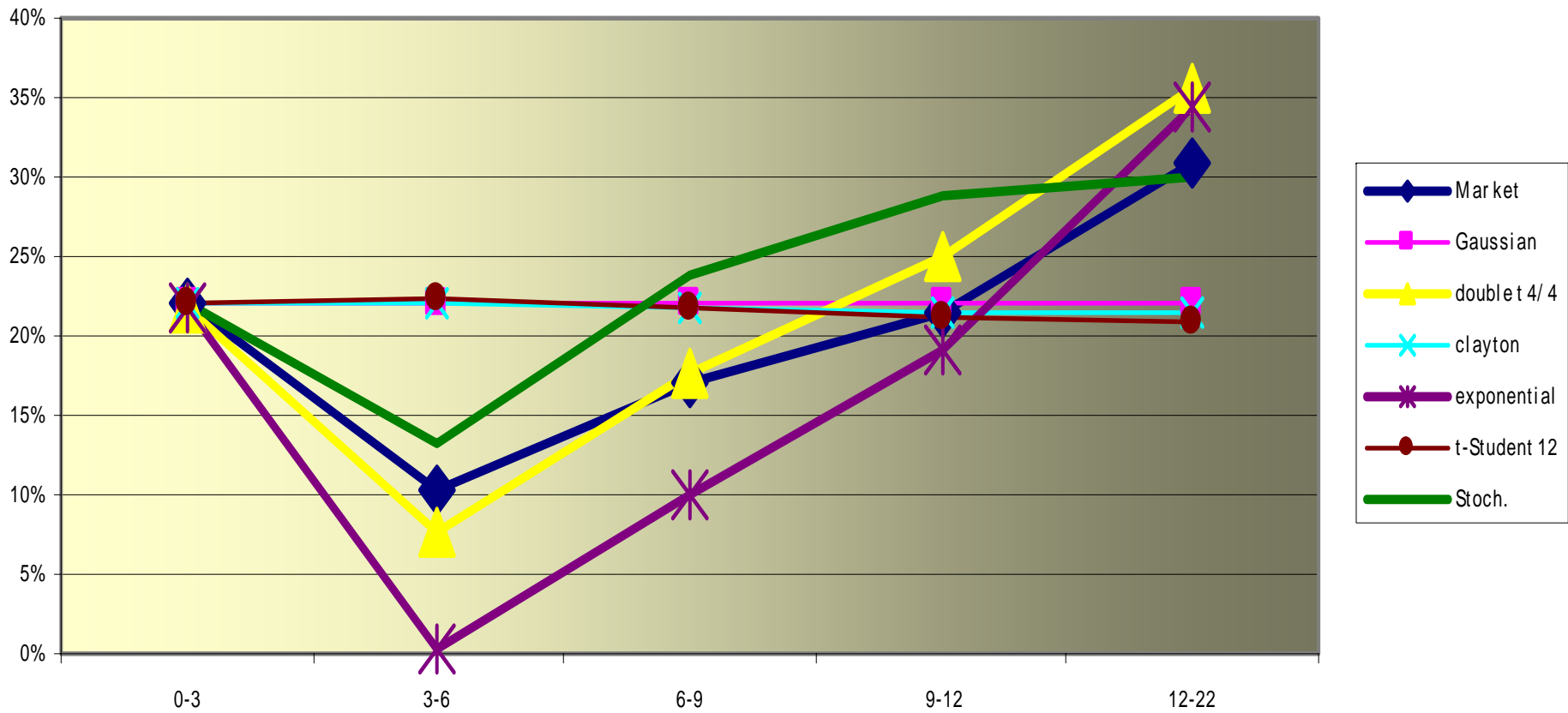
Table 17: CDO tranche premiums iTraxx (bps pa)

Tranches	Market	Gaussian	Clayton	Student (12)	$t(4)-t(4)$	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[0-6%]	466	503	504	504	456	479	418
[0-9%]	311	339	339	340	305	327	272
[0-12%]	233	253	253	254	230	248	203
[0-22%]	128	135	135	135	128	135	113

Table 18: “equity tranche” CDO tranche premiums iTraxx (bps pa)

# Matching the correlation skew

## implied compound correlation





## Beyond the Gaussian copula: stochastic and local correlation

- Stochastic correlation

- Latent variables  $V_i = \tilde{\rho}_i V + \sqrt{1 - \tilde{\rho}_i^2} \bar{V}_i, \quad i = 1, \dots, n$

$$\tilde{\rho}_i = (1 - B_s)(1 - B_i)\rho + B_s$$

$\tilde{\rho}_i$ , stochastic correlation,

$Q(B_s = 1) = q_s$ , systemic state,

$Q(B_i = 1) = q$ , idiosyncratic state

- Conditional default probabilities

$$p_t^{V, B_s=0} = (1 - q)\Phi\left(\frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1 - \rho^2}}\right) + qF(t), \quad F(t) \text{ default probability}$$

$$p_t^{V, B_s=1} = \mathbf{1}_{V \leq \Phi^{-1}(F(t))}, \quad \text{comonotonic}$$



## *Beyond the Gaussian copula: stochastic and local correlation*

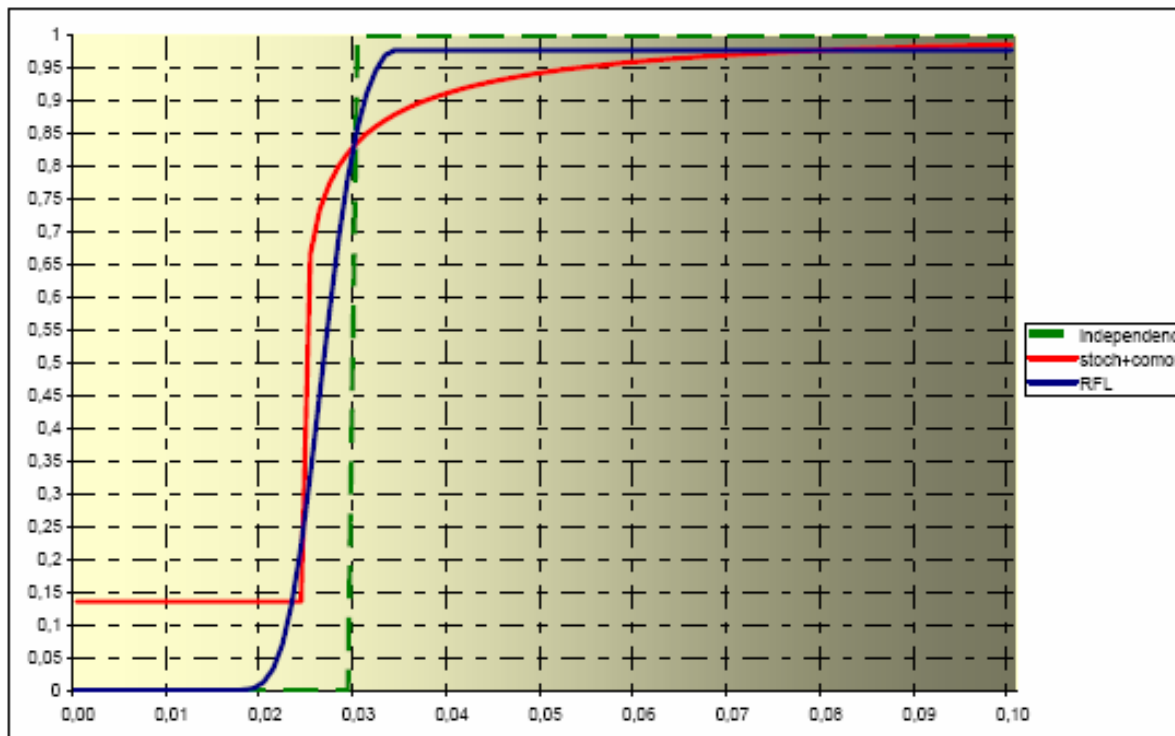
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- Stochastic correlation  $\tilde{\rho}_i = (1 - B_s)(1 - B_i)\rho + B_s$ 
  - *Semi-analytical techniques for pricing CDOs available*
  - *Large portfolio approximation can be derived*
  - *Allows for Monte Carlo*
  - $\nearrow \rho, \searrow q_s, \searrow q$  leads to increase senior tranche premiums
- State dependent correlation  $V_i = m_i(V)V + \sigma_i(V)\bar{V}_i, \quad i = 1, \dots, n$ 
  - *Local correlation*  $V_i = -\rho(V)V + \sqrt{1 - \rho^2(V)}\bar{V}_i$ 
    - Turc et al
  - *Random factor loadings*  $V_i = m + (l1_{V < e} + h1_{V \geq e})V + v\bar{V}_i$ 
    - Andersen & Sidenius



## Beyond the Gaussian copula: stochastic and local correlation

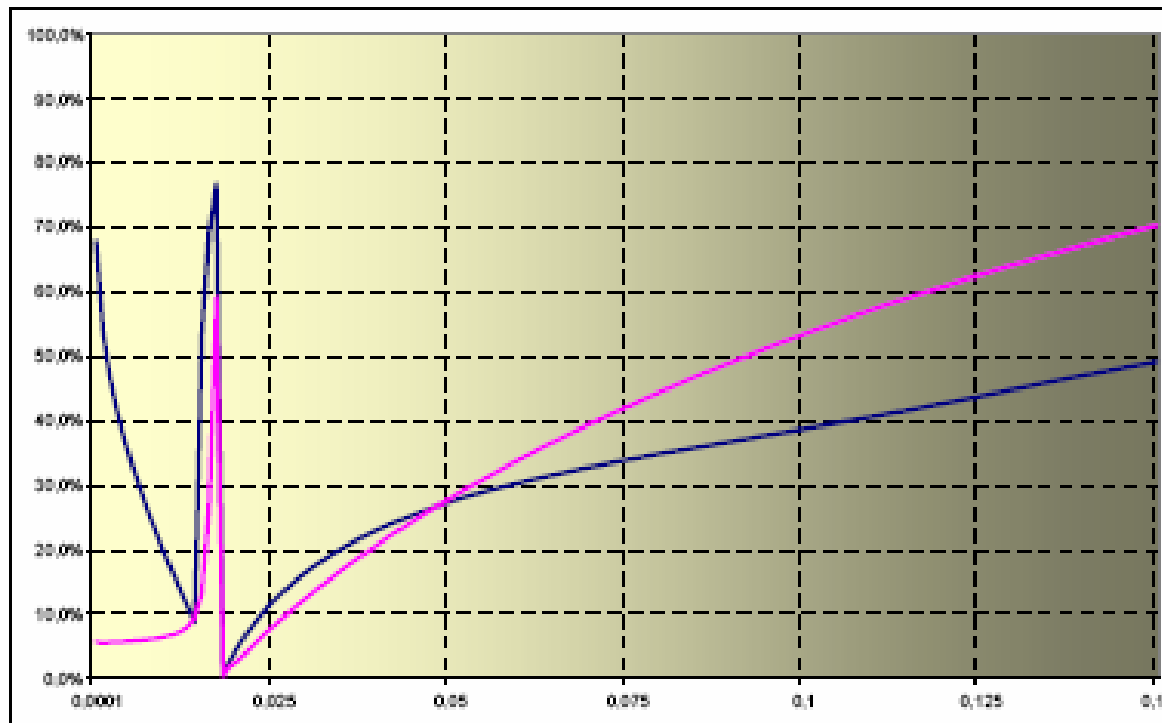
- Distribution functions of conditional default probabilities
  - *stochastic correlation vs RFL*



- *With respect to level of aggregate losses*
- *Also correspond to loss distributions on large portfolios*

## Beyond the Gaussian copula: stochastic and local correlation

- Marginal compound correlations:
  - *With respect to attachment – detachment point*
  - *Compound correlation of a  $[\alpha, \alpha]$  tranche*



- *Stochastic correlation vs RFL*

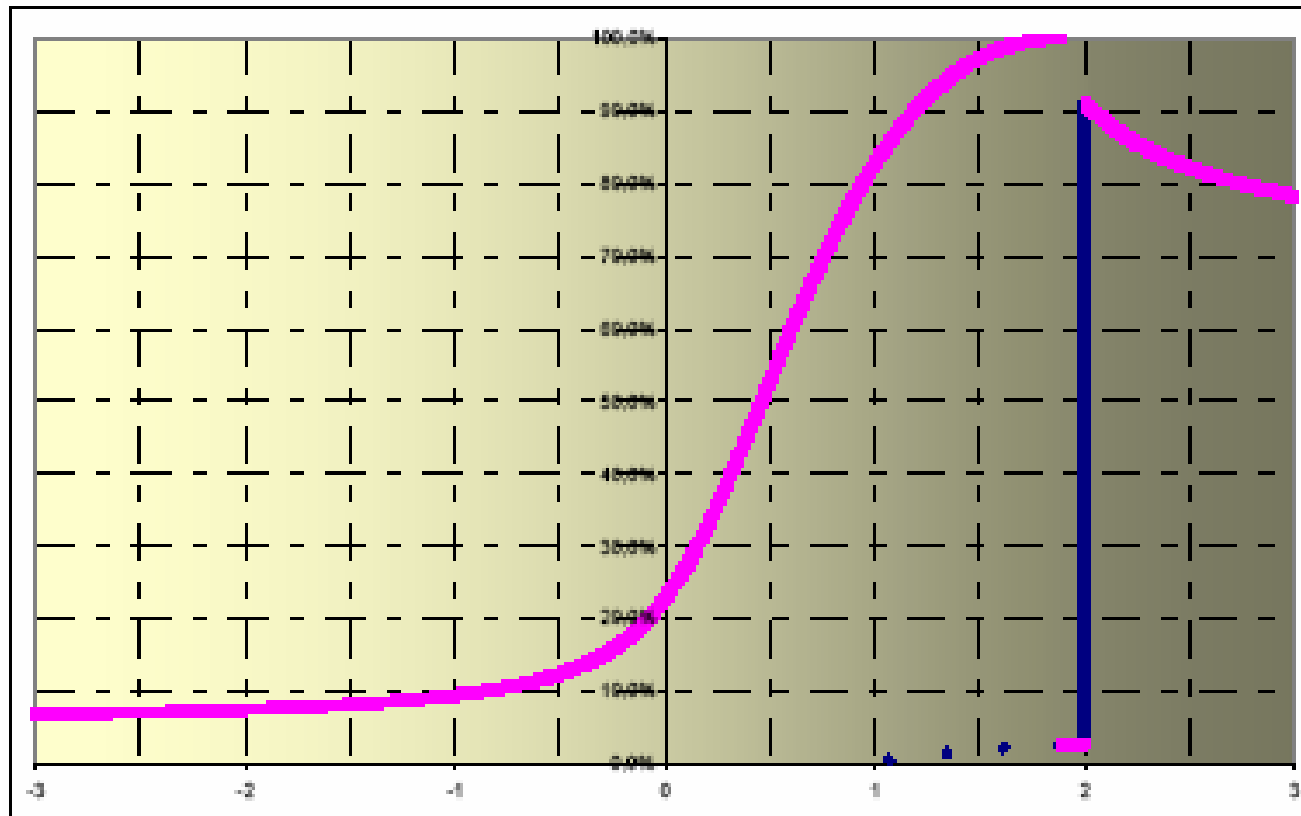
- Marginal compound correlation
  - *Can be obtained from the distribution function of conditional default probabilities*
  - *Need to solve a second order equation*
  - *There might be zero, one or two marginal compound correlations*
  - *Associated with the same conditional default probabilities*
  - *Always a zero marginal compound correlation at the expected loss*

- Local correlation

- *Can be obtained from the conditional default probability distribution*
- *Need to solve for a functional equation*
- *Fixed point algorithm*
- *Step one: solving for a second order equation similar to the one giving marginal compound correlation*
- *Local correlation at step one: rescaled marginal compound correlation*
- *Same issues of uniqueness and existence*

## Beyond the Gaussian copula: stochastic and local correlation

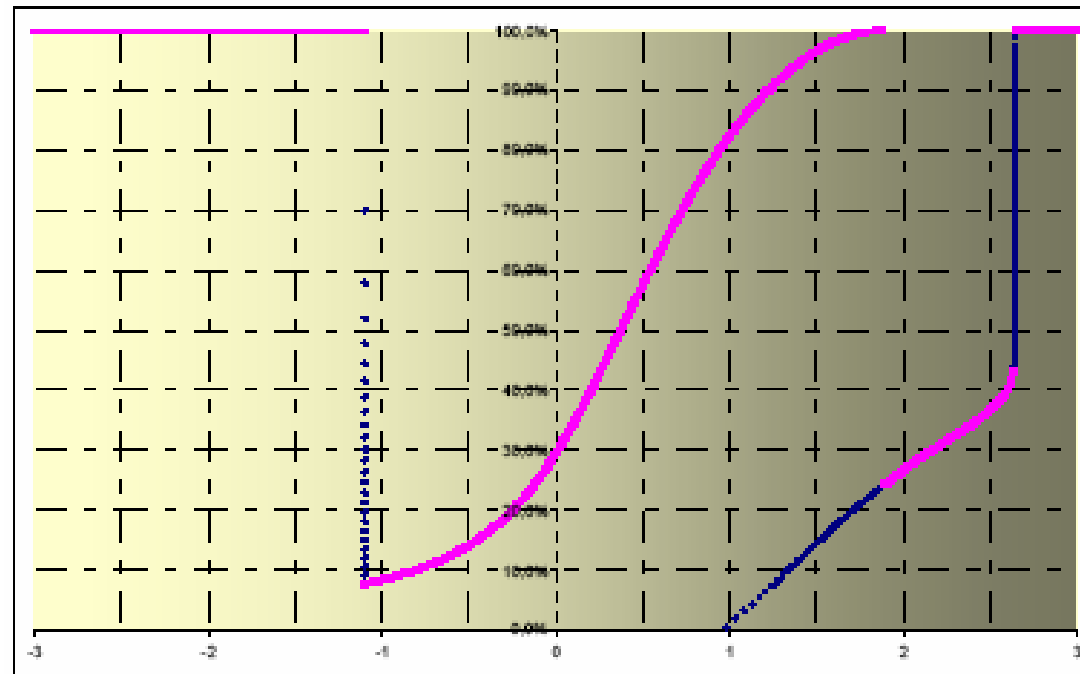
- Local correlation associated with RFL (as a function of the factor)



- Jump at threshold 2, low correlation level 5%, high correlation level 85%
- Possibly two local correlations

## Beyond the Gaussian copula: stochastic and local correlation

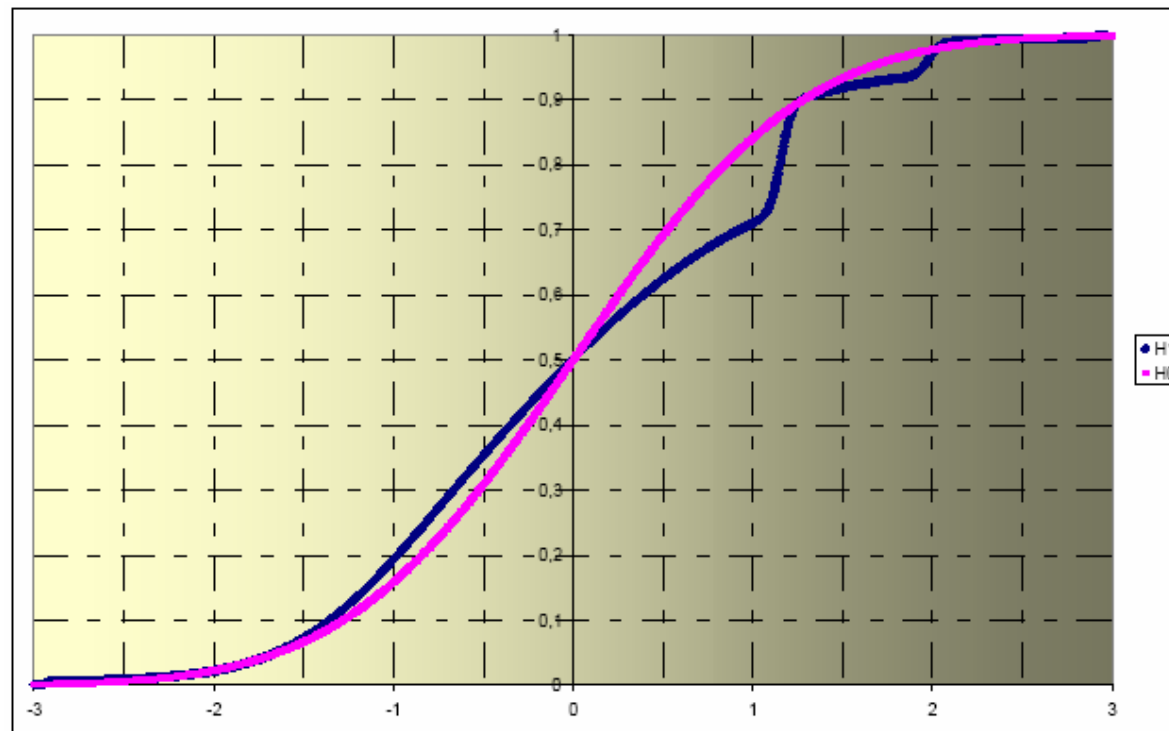
- Local correlation associated with stochastic correlation model
  - *With respect to factor  $V$*



- *Correlations of 1 for high-low values of  $V$  (comonotonic state)*
- *Possibly two local correlations leading to the same prices*
- *As for RFL, rather irregular pattern*

## Beyond the Gaussian copula: stochastic and local correlation

- Checking for the convergence of the fixed point algorithm



- *Good news: convergence at step one*



## Beyond the Gaussian copula: stochastic and local correlation

- Market fits: stochastic correlation model

Tranche	Market	Model	Market $\rho$	Model $\rho$
Index	36			
[0-3%]	24%	25%	16%	14%
[3-6%]	83	84	4%	4%
[6-9%]	27	27	12%	12%
[9-12%]	14	14	17%	17%
[12-22%]	9	9	28%	28%
[22-100%]	4	2	63%	56%

**Table 2.** Fit of model characterised by equation (2.8) to iTraxx market data on 31-August-2005. All values in bp pa unless otherwise stated. As usual the equity tranche is quoted as an up-front premium, in addition to the contractual 500 bp pa.  $q_s = 0.13$ ,  $q = 0.84$ ,  $\rho^2 = 73.5\%$ .

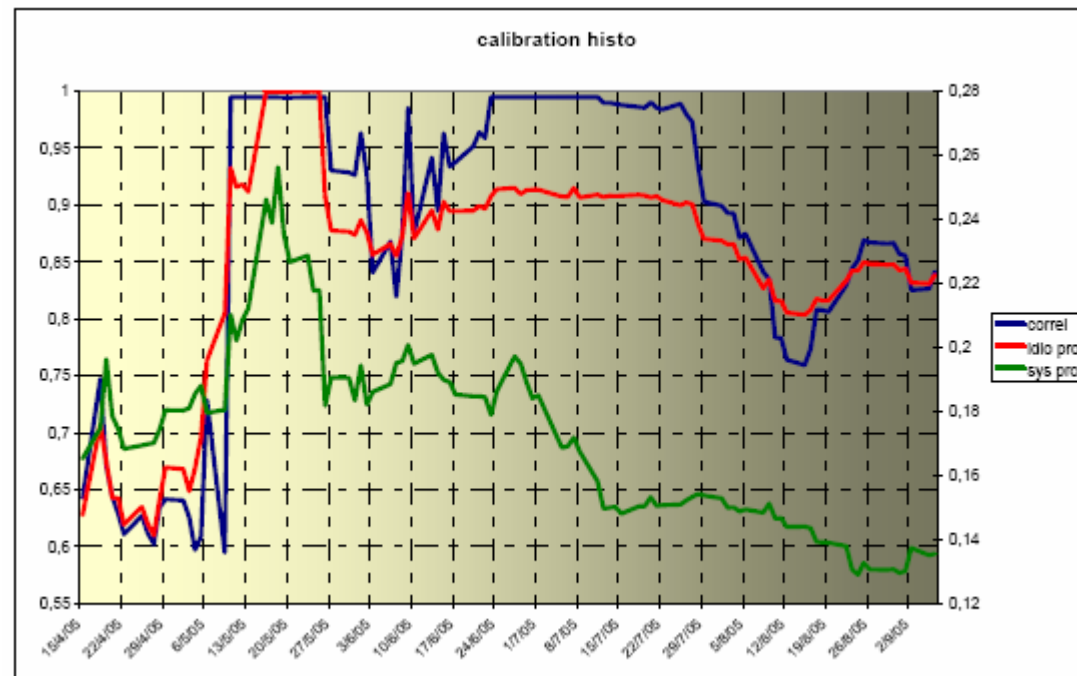
Tranche	Market	Model	Market $\rho$	Model $\rho$
Index	50			
[0-3%]	40%	38%	10%	13%
[3-7%]	126	139	2%	2%
[7-10%]	36	39	12%	13%
[10-15%]	20	17	20%	19%
[15-30%]	10	10	34%	34%
[30-100%]	2	3	59%	65%

**Table 3.** Fit of model characterised by equation (2.8) to CDX market data on 31-August-2005. All values in bp pa unless otherwise stated. As usual the equity tranche is quoted as an up-front premium, in addition to the contractual 500 bp pa.  $q_s = 0.15$ ,  $q = 0.84$ ,  $\rho^2 = 85.3\%$ .



## Beyond the Gaussian copula: stochastic and local correlation

- Calibration history (from 15 April 2005)
  - *Implied correlation, implied idiosyncratic and systemic probabilities*



- *Trouble in fitting during the crisis*
- *Since then, decrease in systemic probability*



## Conclusion

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- Analysis of dependence through Gaussian models
  - *CDO premiums, Risk measures*
  - *Stochastic orders, base correlations*
  - *Analytical techniques, large portfolio approximations*
- Matching the skew with second generation models
  - *RFL, double t*
  - *Conditional default probability distributions are the drivers*
  - *Technique can be extended to structural or intensity models*
- Beyond the Gaussian copula
  - *Stochastic, local & marginal compound correlation*
- Pricing bespoke portfolios, CDO squared with a consistent model