Comparison results for credit risk portfolios

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Contents

1 Comparison results
   • Motivation
   • De Finetti theorem and factor representation
   • Stochastic orders
   • Main results

2 Application to several popular CDO pricing models
   • Factor copula approaches
   • Structural model
   • Multivariate Poisson model
Motivation

- Specify the **dependence structure** of default indicators $D_1, \ldots, D_n$ which leads to:
  - an increase of the value of **call options** $E \left[(L_t - a)^+\right]$ for all strike level $a > 0$
  - an increase of **convex risk measures** on $L_t$ (TVaR, Wang risk measures)
- Comparison between homogeneous credit portfolios
  - $D_1, \ldots, D_n$ are assumed to be **exchangeable** Bernoulli random variables
  - De Finetti’s theorem leads to a **factor representation** of $D_1, \ldots, D_n$
- Application to several popular CDO pricing models
Homogeneity assumption: default indicators $D_1, \ldots, D_n$ forms an exchangeable Bernoulli random vector

**Definition (Exchangeability)**

A random vector $(D_1, \ldots, D_n)$ is exchangeable if its distribution function is invariant for every permutations of its coordinates: $\forall \sigma \in S_n$

$$(D_1, \ldots, D_n) \overset{d}{=} (D_{\sigma(1)}, \ldots, D_{\sigma(n)})$$

- Same marginals
Assume that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables.

Thanks to de Finetti’s theorem, there exists a random factor $	ilde{p}$ such that $D_1, \ldots, D_n$ are conditionally independent given $	ilde{p}$.

Denote by $F_{\tilde{p}}$ the distribution function of $	ilde{p}$, then:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum i d_i} (1 - p)^{n - \sum i d_i} F_{\tilde{p}}(dp)$$

Finite exchangeability only leads to a sign measure Jaynes (1986)

$	ilde{p}$ is characterized by:

$$\frac{1}{n} \sum_{i=1}^n D_i \xrightarrow{a.s.} \tilde{p} \quad \text{as} \quad n \to \infty$$

$	ilde{p}$ is exactly the loss of the infinitely granular portfolio (Bâle 2 terminology)
Stochastic orders

- The convex order compares the dispersion level of two random variables.
- **Convex order**: $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions $f$.
- **Stop-loss order**: $X \leq_{sl} Y$ if $E[(X - K)^+] \leq E[(Y - K)^+]$ for all $K \in \mathbb{R}$.
  - $X \leq_{sl} Y$ and $E[X] = E[Y] \iff X \leq_{cx} Y$.
- $X \leq_{cx} Y$ if $E[X] = E[Y]$ and $F_X$, the distribution function of $X$, and $F_Y$, the distribution function of $Y$, are such that:

![Graph showing the comparison of distribution functions $F_X$ and $F_Y$.]
Supermodular order

- The supermodular order captures the dependence level among coordinates of a random vector.
- \((X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n)\) if \(E[f(X_1, \ldots, X_n)] \leq E[f(Y_1, \ldots, Y_n)]\) for all supermodular function \(f\).

**Definition (Supermodular function)**

A function \(f : \mathbb{R}^n \to \mathbb{R}\) is supermodular if for all \(x \in \mathbb{R}^n, 1 \leq i < j \leq n\) and \(\varepsilon, \delta > 0\) holds:

\[
f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n) \\
\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)
\]

- Consequences of new defaults are always worse when other defaults have already occurred.
Review of literature

Müller (1997)

*Stop-loss order for portfolios of dependent risks*

\[(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \Rightarrow \sum_{i=1}^{n} M_i D_i \leq_{sl} \sum_{i=1}^{n} M_i D_i^*\]

Bäuerle and Müller (2005)

*Stochastic orders ans risk measures: Consistency and bounds*

\[X \leq_{sl} Y \Rightarrow \rho(X) \leq \rho(Y)\]

for all law-invariant, convex risk measures \(\rho\)

Lefèvre and Utev (1996)

*Comparing sums of exchangeable Bernoulli random variables*

\[\tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow \sum_{i=1}^{n} D_i \leq_{sl} \sum_{i=1}^{n} D_i^*\]
Main results

- Let us compare two credit portfolios with aggregate loss \( L_t = \sum_{i=1}^{n} M_i D_i \) and \( L^*_t = \sum_{i=1}^{n} M_i D^*_i \).
- Let \( D_1, \ldots, D_n \) be exchangeable Bernoulli random variables associated with the mixture probability \( \tilde{p} \).
- Let \( D^*_1, \ldots, D^*_n \) exchangeable Bernoulli random variables associated with the mixture probability \( \tilde{p}^* \).

**Theorem**

\[ \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D^*_1, \ldots, D^*_n) \]

- In particular, if \( \tilde{p} \leq_{cx} \tilde{p}^* \), then:
  - \( E[(L_t - a)^+] \leq E[(L^*_t - a)^+] \) for all \( a > 0 \).
  - \( \rho(L_t) \leq \rho(L^*_t) \) for all convex risk measures \( \rho \).
Main results

- Let $D_1, \ldots, D_n, \ldots$ be exchangeable Bernoulli random variables associated with the mixture probability $\tilde{p}$
- Let $D_1^*, \ldots, D_n^*, \ldots$ be exchangeable Bernoulli random variables associated with the mixture probability $\tilde{p}^*$

**Theorem (reverse implication)**

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^*.$$
Comparison results

Motivation
De Finetti theorem and factor representation
Stochastic orders
Main results

Application to several popular CDO pricing models

Factor copula approaches
Structural model
Multivariate Poisson model

Jean-Paul LAURENT and Areski COUSIN

Comparison results for credit risk portfolios
Ordering of CDO tranche premiums

Burtschell, Gregory, and Laurent (2008)

* A comparative analysis of CDO pricing models*

- Analysis of the dependence structure within some factor copula models such as:
  - Gaussian, Student $t$, Double $t$, Clayton, Marshall-Olkin copula
- An increase of the dependence parameter leads to:
  - a decrease of $[0\%, b]$ equity tranches premiums (which guaranties the uniqueness of the market base correlation)
  - an increase of $[a, 100\%]$ senior tranches premiums
Additive factor copula approaches

- The dependence structure of default times is described by some latent variables $V_1, \ldots, V_n$ such that:
  - $V_i = \rho V + \sqrt{1 - \rho^2} \tilde{V}_i, \ i = 1 \ldots n$
  - $V, \tilde{V}_i, \ i = 1 \ldots n$ independent
  - $\tau_i = G^{-1}(H_\rho(V_i)), \ i = 1 \ldots n$
    - $G$: distribution function of $\tau_i$
    - $H_\rho$: distribution function of $V_i$
  - $D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n$ are conditionally independent given $V$
  - $\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} E[D_i \mid V] = P(\tau_i \leq t \mid V) = \tilde{p}$
Additive factor copula approaches

Theorem

For any fixed time horizon $t$, denote by $D_i = 1\{\tau_i \leq t\}, \ i = 1\ldots n$ and $D_i^* = 1\{\tau_i^* \leq t\}, \ i = 1\ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- This framework includes popular factor copula models:
  - One factor Gaussian copula - the industry standard for the pricing of CDO tranches
  - Double t: Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2007)
  - Double Variance Gamma: Moosbrucker(2006)

- $V$ is a positive random variable with Laplace transform $\varphi^{-1}$
- $U_1, \ldots, U_n$ are independent Uniform random variables independent of $V$
- $V_i = \varphi^{-1} \left( -\frac{\ln U_i}{\sqrt{V}} \right), \ i = 1 \ldots n$ (Marshall and Olkin (1988))
  - $(V_1, \ldots, V_n)$ follows a $\varphi$-archimedean copula
  - $P(V_1 \leq v_1, \ldots, V_n \leq v_n) = \varphi^{-1}(\varphi(v_1) + \ldots + \varphi(v_n))$
- $\tau_i = G^{-1}(V_i)$
  - $G$: distribution function of $\tau_i$
- $D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n$ independent knowing $V$
- $\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} E[D_i \mid V] = P(\tau_i \leq t \mid V)$
Archimedean copula

Conditional default probability: \( \tilde{p} = \exp \{ -\varphi(G(t)V) \} \)

<table>
<thead>
<tr>
<th>Copula</th>
<th>Generator ( \varphi )</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>( t^{-\theta} - 1 )</td>
<td>( \theta \geq 0 )</td>
</tr>
<tr>
<td>Gumbel</td>
<td>( (-\ln(t))^\theta )</td>
<td>( \theta \geq 1 )</td>
</tr>
<tr>
<td>Franck</td>
<td>( -\ln \left( \frac{(1 - e^{-\theta t})}{(1 - e^{-\theta})} \right) )</td>
<td>( \theta \in \mathbb{R}^* )</td>
</tr>
</tbody>
</table>

Theorem

\( \theta \leq \theta^* \Rightarrow \tilde{p} \leq_{\text{cx}} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{\text{sm}} (D_1^*, \ldots, D_n^*) \)
Archimedean copula

- Clayton copula
- Mixture distributions are ordered with respect to the convex order

\[ P(\tau_i \leq t) = 0.08 \]
Hull, Predescu and White (2005)

Consider \( n \) firms

Let \( V_{i,t}, \ i = 1 \ldots n \) be their asset dynamics

\[
V_{i,t} = \rho V_t + \sqrt{1 - \rho^2} \bar{V}_{i,t}, \ i = 1 \ldots n
\]

\( V, \bar{V}_i, \ i = 1 \ldots n \) are independent standard Wiener processes

Default times as first passage times:

\[
\tau_i = \inf\{ t \in \mathbb{R}^+ | V_{i,t} \leq f(t) \}, \ i = 1 \ldots n, \ f : \mathbb{R} \to \mathbb{R} \ \text{continuous}
\]

\( D_i = 1\{\tau_i \leq T\}, \ i = 1 \ldots n \) are conditionally independent given \( \sigma(V_t, t \in [0, T]) \)
Theorem

For any fixed time horizon $T$, denote by $D_i = 1\{\tau_i \leq T\}$, $i = 1 \ldots n$ and $D_{i}^{*} = 1\{\tau_{i}^{*} \leq T\}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^{*}$, then:

$$\rho \leq \rho^{*} \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^{*}, \ldots, D_n^{*})$$
Structural model

Distributions of Conditionnal Default Probabilities

- $\sum_{i=1}^{n} D_i \overset{a.s.}{\rightarrow} \tilde{p}$
- $\sum_{i=1}^{n} D_i^* \overset{a.s.}{\rightarrow} \tilde{p}^*$

Empirically, mixture probabilities are ordered with respect to the convex order: $\tilde{p} \leq_{cx} \tilde{p}^*$
Multivariate Poisson model

- $\tilde{N}_t^i$ Poisson with parameter $\tilde{\lambda}$: idiosyncratic risk
- $N_t^i$ Poisson with parameter $\lambda$: systematic risk
- $(B_j^i)_{i,j}$ Bernoulli random variable with parameter $p$
- All sources of risk are independent
- $N_t^i = \tilde{N}_t^i + \sum_{j=1}^{N_t^i} B_j^i$, $i = 1 \ldots n$
- $\tau_i = \inf\{t > 0 | N_t^i > 0\}$, $i = 1 \ldots n$
Multivariate Poisson model

- Dependence structure of \((\tau_1, \ldots, \tau_n)\) is the Marshall-Olkin copula
- \(\tau_i \sim \text{Exp}(\bar{\lambda} + p\lambda)\)
- \(D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n\) are conditionally independent given \(N_t\)
- \(\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)\)
- Conditional default probability:

\[
\tilde{p} = 1 - (1 - p)^{N_t} \exp(-\bar{\lambda}t)
\]
Comparison of two multivariate Poisson models with parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\)

Supermodular order comparison requires equality of marginals:
\[\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda^*\]

3 comparison directions:
- \(p = p^*\): \(\bar{\lambda}\) v.s \(\lambda\)
- \(\lambda = \lambda^*\): \(\bar{\lambda}\) v.s \(p\)
- \(\bar{\lambda} = \bar{\lambda}^*\): \(\lambda\) v.s \(p\)
**Theorem ($p = p^*$)**

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*$, then:

$$\lambda \leq \lambda^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- **Computation of $E[(L_t - a)^+]$:**
  - 30 names
  - $M_i = 1, \ i = 1 \ldots n$

- When $\lambda$ increases, the aggregate loss increases with respect to stop-loss order.
Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p \lambda = \bar{\lambda}^* + p^* \lambda$, then:

$$p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \bar{p} \leq_{cx} \bar{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

Convex order for mixture probabilities
Multivariate Poisson model

**Theorem \((\lambda = \lambda^*)\)**

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda\), then:

\[ p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \bar{p} \leq cx \bar{p}^* \Rightarrow (D_1, \ldots, D_n) \leq sm (D_1^*, \ldots, D_n^*) \]

- Computation of \(E[(L_t - K)^+]\):
  - 30 names
  - \(M_i = 1, \; i = 1 \ldots n\)
- When \(p\) increases, the aggregate loss increases with respect to stop-loss order
Theorem ($\bar{\lambda} = \bar{\lambda}^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $p\lambda = p^*\lambda^*$, then:

$$p \leq p^*, \lambda \geq \lambda^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1, i = 1 \ldots n$
- When $p$ increases, the aggregate loss increases with respect to stop-loss order
Conclusion

- When considering an exchangeable vector of default indicators, the conditional independence assumption is not restrictive thanks to de Finetti’s theorem.

- The mixture probability (the factor) can be viewed as the loss of an infinitely granular portfolio.

- We completely characterize the supermodular order between exchangeable default indicator vectors in terms of the convex ordering of corresponding mixture probabilities.

- We show that the mixture probability is the key input to study the impact of dependence on CDO tranche premiums.

- Comparison analysis can be performed with the same method within a large number of popular CDO pricing models.