Comparison results for credit risk portfolios

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Contents

Motivation

- Specify the dependence structure of default indicators $D_1, \ldots, D_n$ which leads to:
  - an increase of the value of call options $E\left[(L_t - a)^+\right]$ for all strike level $a > 0$
  - an increase of convex risk measures on L_t (TVaR, Wang risk measures)
- Comparison between homogeneous credit portfolios
  - $D_1, \ldots, D_n$ are assumed to be exchangeable Bernoulli random variables
  - De Finetti's theorem leads to a factor representation of $D_1, \ldots, D_n$
- Application to several popular CDO pricing models

De Finetti theorem and factor representation

- Homogeneity assumption: default indicators $D_1, \ldots, D_n$ forms an exchangeable Bernoulli random vector

Definition (Exchangeability)

A random vector $(D_1, \ldots, D_n)$ is exchangeable if its distribution function is invariant for every permutations of its coordinates: $\forall \sigma \in S_n$

$$(D_1, \ldots, D_n) \stackrel{d}{=} (D_{\sigma(1)}, \ldots, D_{\sigma(n)})$$

- Same marginals

De Finetti theorem and factor representation

- Assume that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables
- Thanks to de Finetti's theorem, there exists a random factor $\tilde{p}$ such that
- $D_1, \ldots, D_n$ are conditionally independent given $\tilde{p}$
- Denote by $F_{\tilde{p}}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum_i d_i}(1-p)^{n-\sum_i d_i} F_{\tilde{p}}(dp)$$

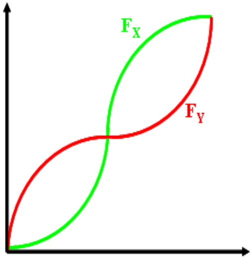
  - Finite exchangeability only leads to a sign measure Jaynes (1986)
- $\tilde{p}$ is characterized by:

$$\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p} \quad \text{as} \quad n \longrightarrow \infty$$

  - $\tilde{p}$ is exactly the loss of the infinitely granular portfolio (Bâle 2 terminology)

Stochastic orders

- The convex order compares the dispersion level of two random variables
- Convex order: $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions f
- Stop-loss order: $X \leq_{sl} Y$ if $E[(X-K)^+] \leq E[(Y-K)^+]$ for all $K \in \mathbb{R}$
  - $X \leq_{sl} Y$ and $E[X] = E[Y] \Leftrightarrow X \leq_{cx} Y$
- $X \leq_{cx} Y$ if $E[X] = E[Y]$ and F_X, the distribution function of X and F_Y, the distribution function of Y are such that:



Supermodular order

- The supermodular order captures the dependence level among coordinates of a random vector
- $(X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n)$ if $E[f(X_1, \ldots, X_n)] \leq E[f(Y_1, \ldots, Y_n)]$ for all supermodular function f

Definition (Supermodular function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **supermodular** if for all $x \in R^n$, $1 \leq i < j \leq n$ and $\varepsilon, \delta > 0$ holds

$$f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n)$$
$$\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$$

- Consequences of new defaults are always worse when other defaults have already occurred

Review of literature

Müller(1997)

Stop-loss order for portfolios of dependent risks

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \Rightarrow \sum_{i=1}^{n} M_i D_i \leq_{sl} \sum_{i=1}^{n} M_i D_i^*$$

Bäuerle and Müller(2005)

Stochastic orders ans risk measures: Consistency and bounds

$$X \leq_{sl} Y \Rightarrow \rho(X) \leq \rho(Y)$$

for all law-invariant, convex risk measures ρ

Lefèvre and Utev(1996)

Comparing sums of exchangeable Bernoulli random variables

$$\tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow \sum_{i=1}^{n} D_i \leq_{sl} \sum_{i=1}^{n} D_i^*$$

Main results

- Let us compare two credit portfolios with aggregate loss $L_t = \sum_{i=1}^{n} M_i D_i$ and $L_t^* = \sum_{i=1}^{n} M_i D_i^*$
- Let $D_1, \ldots, D_n$ be exchangeable Bernoulli random variables associated with the mixture probability $\tilde{p}$
- Let $D_1^*, \ldots, D_n^*$ exchangeable Bernoulli random variables associated with the mixture probability $\tilde{p}^*$

Theorem

$$\tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- In particular, if $\tilde{p} \leq_{cx} \tilde{p}^*$, then:
  - $E[(L_t - a)^+] \leq E[(L_t^* - a)^+]$ for all $a > 0$.
  - $\rho(L_t) \leq \rho(L_t^*)$ for all convex risk measures ρ

Main results

- Let $D_1, \ldots, D_n, \ldots$ be exchangeable Bernoulli random variables associated with the mixture probability $\tilde{p}$
- Let $D_1^*, \ldots, D_n^*, \ldots$ be exchangeable Bernoulli random variables associated with the mixture probability $\tilde{p}^*$

Theorem (reverse implication)

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^*.$$

Contents

Ordering of CDO tranche premiums

Burtschell, Gregory, and Laurent(2008)
A comparative analysis of CDO pricing models

- Analysis of the dependence structure within some factor copula models such as:
  - Gaussian, Student t, Double t, Clayton, Marshall-Olkin copula
- An increase of the dependence parameter leads to:
  - a decrease of $[0\%, b]$ equity tranches premiums (which guaranties the uniqueness of the market base correlation)
  - an increase of $[a, 100\%]$ senior tranches premiums

Additive factor copula approaches

- The dependence structure of default times is described by some latent variables $V_1, \ldots, V_n$ such that:
- $V_i = \rho V + \sqrt{1-\rho^2}\bar{V}_i, \ i = 1 \ldots n$
- $V, \bar{V}_i, \ i = 1 \ldots n$ independent
- $\tau_i = G^{-1}(H_\rho(V_i)), \ i = 1 \ldots n$
    - G: distribution function of τ_i
    - H_ρ: distribution function of V_i
- $D_i = 1_{\{\tau_i \le t\}}, \ i = 1 \ldots n$ are conditionally independent given V
- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid V] = P(\tau_i \le t \mid V) = \tilde{p}$

Additive factor copula approaches

Theorem

For any fixed time horizon t, denote by $D_i = 1_{\{\tau_i \leq t\}}$, $i = 1 \ldots n$ and $D_i^* = 1_{\{\tau_i^* \leq t\}}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) ρ and ρ^*, then:

$$\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- This framework includes popular factor copula models:
  - One factor Gaussian copula - the industry standard for the pricing of CDO tranches
  - Double t: Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2007)
  - Double Variance Gamma: Moosbrucker(2006)

Archimedean copula

Schönbucher and Schubert(2001), Gregory and Laurent(2003), Madan *et al.*(2004), Friend and Rogge(2005)

- V is a positive random variable with Laplace transform φ^{-1}
- $U_1, \ldots, U_n$ are independent Uniform random variables independent of V
- $V_i = \varphi^{-1}\left(-\frac{\ln U_i}{V}\right)$, $i = 1 \ldots n$ (Marshall and Olkin (1988))
  - $(V_1, \ldots, V_n)$ follows a φ-archimedean copula
  - $P(V_1 \leq v_1, \ldots, V_n \leq v_n) = \varphi^{-1}\left(\varphi(v_1) + \ldots + \varphi(v_n)\right)$
- $\tau_i = G^{-1}(V_i)$
  - G: distribution function of τ_i
- $D_i = 1_{\{\tau_i \leq t\}}$, $i = 1 \ldots n$ independent knowing V
- $\frac{1}{n}\sum_{i=1}^n D_i \xrightarrow{a.s} E[D_i \mid V] = P(\tau_i \leq t \mid V)$

Archimedean copula

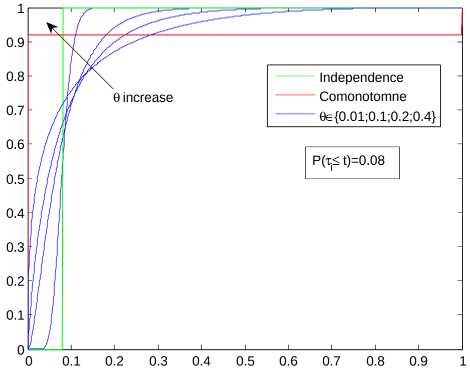
- Conditional default probability: $\tilde{p} = \exp\left\{-\varphi(G(t)V)\right\}$

Copula	Generator φ	Parameter
Clayton	$t^{-\theta} - 1$	$\theta \geq 0$
Gumbel	$(-\ln(t))^{\theta}$	$\theta \geq 1$
Franck	$-\ln\left[(1 - e^{-\theta t})/(1 - e^{-\theta})\right]$	$\theta \in \boldsymbol{R}^*$

Theorem

$\theta \leq \theta^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$

Archimedean copula



- Clayton copula
- Mixture distributions are ordered with respect to the convex oder

Structural model

Hull, Predescu and White(2005)

- Consider n firms
- Let $V_{i,t},\ i = 1 \dots n$ be their asset dynamics

$$V_{i,t} = \rho V_t + \sqrt{1-\rho^2}\bar{V}_{i,t},\quad i = 1 \dots n$$

- $V,\ \bar{V}_i,\ i = 1 \dots n$ are independent standard Wiener processes
- Default times as first passage times:

$$\tau_i = \inf\{t \in \boldsymbol{R}^+ | V_{i,t} \leq f(t)\},\quad i = 1 \dots n,\ f : \boldsymbol{R} \to \boldsymbol{R} \text{ continuous}$$

- $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \dots n$ are conditionally independent given $\sigma(V_t,\ t \in [0, T])$

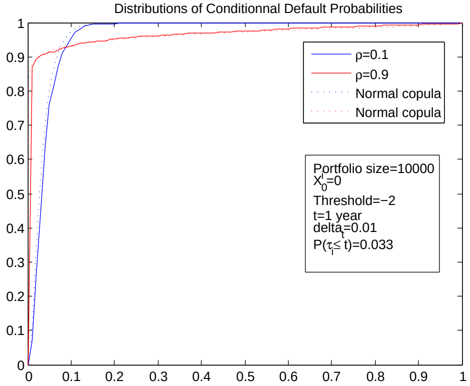
Structural model

Theorem

For any fixed time horizon T*, denote by* $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \ldots n$ *and* $D_i^* = 1_{\{\tau_i^* \leq T\}}$, $i = 1 \ldots n$ *the default indicators corresponding to (resp.)* ρ *and* ρ^**, then:*

$$\rho \leq \rho^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

Structural model



- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p}$
- $\frac{1}{n}\sum_{i=1}^{n} D_i^* \xrightarrow{a.s} \tilde{p}^*$
- Empirically, mixture probabilities are ordered with respect to the convex order: $\tilde{p} \leq_{cx} \tilde{p}^*$

Multivariate Poisson model

Duffie(1998), Lindskog and McNeil(2003), Elouerkhaoui(2006)

- $\bar{N}_t^i$ Poisson with parameter $\bar{\lambda}$: idiosyncratic risk
- N_t Poisson with parameter λ: systematic risk
- $(B_j^i)_{i,j}$ Bernoulli random variable with parameter p
- All sources of risk are independent
- $N_t^i = \bar{N}_t^i + \sum_{j=1}^{N_t} B_j^i, \; i = 1 \ldots n$
- $\tau_i = \inf\{t > 0 | N_t^i > 0\}, \; i = 1 \ldots n$

Multivariate Poisson model

- Dependence structure of $(\tau_1, \dots, \tau_n)$ is the Marshall-Olkin copula
- $\tau_i \sim Exp(\bar{\lambda} + p\lambda)$
- $D_i = 1_{\{\tau_i \leq t\}}$, $i = 1 \dots n$ are conditionally independent given N_t
- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)$
- Conditional default probability:

$$\tilde{p} = 1 - (1-p)^{N_t} \exp(-\bar{\lambda} t)$$

Multivariate Poisson model

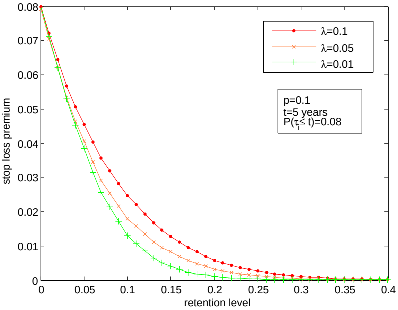
- Comparison of two multivariate Poisson models with parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$
- Supermodular order comparison requires equality of marginals: $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda^*$
- 3 comparison directions:
  - $p = p^*$: $\bar{\lambda}$ v.s λ
  - $\lambda = \lambda^*$: $\bar{\lambda}$ v.s p
  - $\bar{\lambda} = \bar{\lambda}^*$: λ v.s p

Multivariate Poisson model

Theorem ($p = p^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ *and* $(\bar{\lambda}^*, \lambda^*, p^*)$ *be such that* $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*$, *then:*

$$\lambda \leq \lambda^*,\ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$



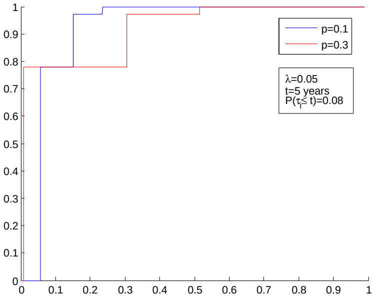
- Computation of $E[(L_t - a)^+]$:
  - 30 names
  - $M_i = 1,\ i = 1 \ldots n$
- When λ increases, the aggregate loss increases with respect to stop-loss order

Multivariate Poisson model

Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, then:

$$p \leq p^*, \ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \dots, D_n) \leq_{sm} (D_1^*, \dots, D_n^*)$$



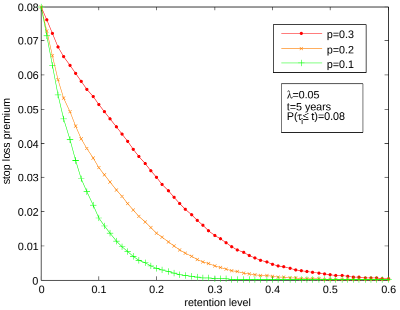
- Convex order for mixture probabilities

Multivariate Poisson model

Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ *and* $(\bar{\lambda}^*, \lambda^*, p^*)$ *be such that* $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, *then:*

$$p \leq p^*,\ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$



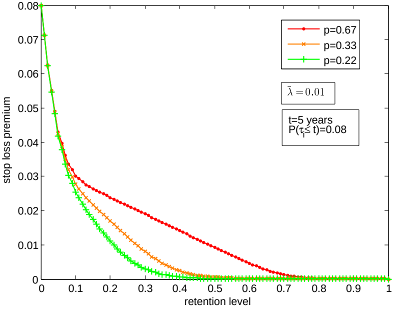
- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1,\ i = 1 \ldots n$
- When p increases, the aggregate loss increases with respect to stop-loss order

Multivariate Poisson model

Theorem ($\bar{\lambda} = \bar{\lambda}^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $p\lambda = p^*\lambda^*$, then:

$$p \leq p^*,\ \lambda \geq \lambda^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$



- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1,\ i = 1 \ldots n$
- When p increases, the aggregate loss increases with respect to stop-loss order

Conclusion

- When considering an exchangeable vector of default indicators, the conditional independence assumption is not restrictive thanks to de Finetti's theorem
- The mixture probability (the factor) can be viewed as the loss of an infinitely granular portfolio
- We completely characterize the supermodular order between exchangeable default indicator vectors in term of the convex ordering of corresponding mixture probabilities
- We show that the mixture probability is the key input to study the impact of dependence on CDO tranche premiums
- Comparison analysis can be performed with the same method within a large number of popular CDO pricing models