Comparison results for exchangeable credit risk portfolios

Areski COUSIN

Université d'Evry

Second International Financial Research Forum - 19 March 2009
Joint work with Jean-Paul Laurent, ISFA, Université de Lyon

Contents

De Finetti theorem and factor representation

- Homogeneity assumption: default indicators $D_1, \dots, D_n$ forms an exchangeable Bernoulli random vector

Definition (Exchangeability)

A random vector $(D_1, \dots, D_n)$ is exchangeable if its distribution function is invariant for every permutations of its coordinates: $\forall \sigma \in S_n$

$$(D_1, \dots, D_n) \overset{d}{=} (D_{\sigma(1)}, \dots, D_{\sigma(n)})$$

- Same marginals

De Finetti theorem and factor representation

- Assume that $D_1, \dots, D_n, \dots$ is an exchangeable sequence of Bernoulli random variables
- Thanks to de Finetti's theorem, there exists a unique random factor $\tilde{p}$ such that
- $D_1, \dots, D_n$ are conditionally independent given $\tilde{p}$
- Denote by $F_{\tilde{p}}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \dots, D_n = d_n) = \int_0^1 p^{\sum_i d_i}(1-p)^{n-\sum_i d_i} F_{\tilde{p}}(dp)$$

- $\tilde{p}$ is characterized by:

$$\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p} \quad \text{as} \quad n \longrightarrow \infty$$

  - $\tilde{p}$ is exactly the loss of the infinitely granular portfolio (Basel 2 terminology)

Stochastic orders

- The convex order compares the dispersion level of two random variables
- Convex order: $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions f
- Stop-loss order: $X \leq_{sl} Y$ if $E[(X-K)^+] \leq E[(Y-K)^+]$ for all $K \in \mathbb{R}$
  - $X \leq_{sl} Y$ and $E[X] = E[Y] \Leftrightarrow X \leq_{cx} Y$

Supermodular order

- The supermodular order captures the dependence level among coordinates of a random vector
- $(X_1, \dots, X_n) \leq_{sm} (Y_1, \dots, Y_n)$ if $E[f(X_1, \dots, X_n)] \leq E[f(Y_1, \dots, Y_n)]$ for all supermodular functions f

Definition (Supermodular function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **supermodular** if for all $x \in \boldsymbol{R}^n$, $1 \leq i < j \leq n$ and $\varepsilon, \delta > 0$ holds

$$f(x_1, \dots, x_i + \varepsilon, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i + \varepsilon, \dots, x_j, \dots, x_n)$$

$$\geq f(x_1, \dots, x_i, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_j, \dots, x_n)$$

Müller(1997)
Stop-loss order for portfolios of dependent risks

$$(D_1, \dots, D_n) \leq_{sm} (D_1^*, \dots, D_n^*) \Rightarrow \sum_{i=1}^{n} M_i D_i \leq_{sl} \sum_{i=1}^{n} M_i D_i^*$$

Main results

- Let us compare two credit portfolios with aggregate loss $L_t = \sum_{i=1}^{n} M_i D_i$ and $L_t^* = \sum_{i=1}^{n} M_i D_i^*$
- Let $D_1, \ldots, D_n$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}$
- Let $D_1^*, \ldots, D_n^*$ exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}^*$

Theorem

$$\tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- In particular, if $\tilde{p} \leq_{cx} \tilde{p}^*$, then:
  - $E[(L_t - a)^+] \leq E[(L_t^* - a)^+]$ for all $a > 0$.
  - $\rho(L_t) \leq \rho(L_t^*)$ for all convex risk measures ρ

Main results

- Let $D_1, \ldots, D_n, \ldots$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}$
- Let $D_1^*, \ldots, D_n^*, \ldots$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}^*$

Theorem (reverse implication)

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^*.$$

Contents

Ordering of CDO tranche premiums

- Analysis of the dependence structure in several popular CDO pricing models
- An increase of the dependence parameter leads to:
  - a decrease of $[0\%, b]$ equity tranche premiums (which guaranties the uniqueness of the market base correlation)
  - an increase of $[a, 100\%]$ senior tranche premiums

Additive factor copula approaches

- The dependence structure of default times is described by some latent variables $V_1, \dots, V_n$ such that:
- $V_i = \rho V + \sqrt{1-\rho^2}\,\bar{V}_i,\ i = 1 \dots n$
- $V, \bar{V}_i,\ i = 1 \dots n$ independent
- $\tau_i = G^{-1}(H_\rho(V_i)),\ i = 1 \dots n$
  - G: distribution function of τ_i
  - H_ρ: distribution function of V_i
- $D_i = 1_{\{\tau_i \le t\}},\ i = 1 \dots n$ are conditionally independent given V
- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid V] = P(\tau_i \le t \mid V) = \tilde{p}$

Additive factor copula approaches

Theorem

For any fixed time horizon t, denote by $D_i = 1_{\{\tau_i \le t\}}$, $i = 1 \ldots n$ *and* $D_i^* = 1_{\{\tau_i^* \le t\}}$, $i = 1 \ldots n$ *the default indicators corresponding to (resp.)* ρ *and* ρ^**, then:*

$$\rho \le \rho^* \Rightarrow \tilde{p} \le_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \le_{sm} (D_1^*, \ldots, D_n^*)$$

- This framework includes popular factor copula models:
  - One factor Gaussian copula - the industry standard for the pricing of CDO tranches
  - Double t: Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2007)
  - Double Variance Gamma: Moosbrucker(2006)

Archimedean copula

Schönbucher and Schubert(2001), Gregory and Laurent(2003), Madan *et al.*(2004), Friend and Rogge(2005)

- V is a positive random variable with Laplace transform φ^{-1}
- $U_1, \ldots, U_n$ are independent Uniform random variables independent of V
- $V_i = \varphi^{-1}\left(-\frac{\ln U_i}{V}\right)$, $i = 1 \ldots n$ (Marshall and Olkin (1988))
  - $(V_1, \ldots, V_n)$ follows a φ-archimedean copula
  - $P(V_1 \leq v_1, \ldots, V_n \leq v_n) = \varphi^{-1}\left(\varphi(v_1) + \ldots + \varphi(v_n)\right)$
- $\tau_i = G^{-1}(V_i)$
  - G: distribution function of τ_i
- $D_i = 1_{\{\tau_i \leq t\}}$, $i = 1 \ldots n$ independent knowing V
- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid V] = P(\tau_i \leq t \mid V)$

Archimedean copula

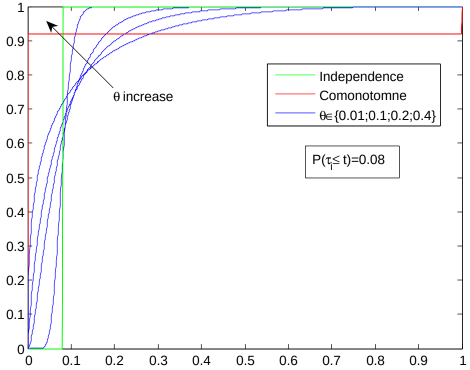
- Conditional default probability: $\tilde{p} = \exp\{-\varphi(G(t)V)\}$

Copula	Generator φ	Parameter
Clayton	$t^{-\theta} - 1$	$\theta \geq 0$
Gumbel	$(-\ln(t))^{\theta}$	$\theta \geq 1$
Franck	$-\ln\left[(1-e^{-\theta t})/(1-e^{-\theta})\right]$	$\theta \in \mathbf{R}^*$

Theorem

$$\theta \leq \theta^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

Archimedean copula



- Clayton copula
- Mixture distributions are ordered with respect to the convex oder

Structural model

Hull, Predescu and White(2005)

- Consider n firms
- Let $V_{i,t},\ i = 1 \dots n$ be their asset dynamics

$$V_{i,t} = \rho V_t + \sqrt{1-\rho^2}\,\bar{V}_{i,t},\quad i = 1 \dots n$$

- $V,\ \bar{V}_i,\ i = 1 \dots n$ are independent standard Wiener processes
- Default times as first passage times:

$$\tau_i = \inf\{t \in \mathbb{R}^+ | V_{i,t} \leq f(t)\},\quad i = 1 \dots n,\ f : \mathbb{R} \to \mathbb{R} \text{ continuous}$$

- $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \dots n$ are conditionally independent given $\sigma(V_t,\ t \in [0, T])$

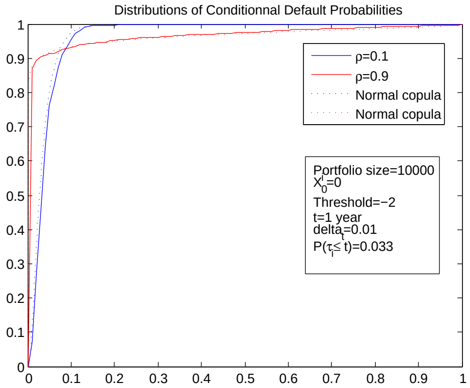
Structural model

Theorem

For any fixed time horizon T, denote by $D_i = 1_{\{\tau_i \le T\}}$, $i = 1 \ldots n$ and $D_i^* = 1_{\{\tau_i^* \le T\}}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) ρ and ρ^*, then:

$$\rho \le \rho^* \Rightarrow (D_1, \ldots, D_n) \le_{sm} (D_1^*, \ldots, D_n^*)$$

Structural model



- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p}$
- $\frac{1}{n}\sum_{i=1}^{n} D_i^* \xrightarrow{a.s} \tilde{p}^*$
- Empirically, mixture probabilities are ordered with respect to the convex order: $\tilde{p} \leq_{cx} \tilde{p}^*$

Multivariate Poisson model

Duffie(1998), Lindskog and McNeil(2003), Elouerkhaoui(2006)

- $\bar{N}_t^i$ Poisson with parameter $\bar{\lambda}$: idiosyncratic risk
- N_t Poisson with parameter λ: systematic risk
- $(B_j^i)_{i,j}$ Bernoulli random variable with parameter p
- All sources of risk are independent
- $N_t^i = \bar{N}_t^i + \sum_{j=1}^{N_t} B_j^i, \; i = 1 \dots n$
- $\tau_i = \inf\{t > 0 | N_t^i > 0\}, \; i = 1 \dots n$

Multivariate Poisson model

- Dependence structure of $(\tau_1, \ldots, \tau_n)$ is the Marshall-Olkin copula
- $\tau_i \sim Exp(\bar{\lambda} + p\lambda)$
- $D_i = 1_{\{\tau_i \leq t\}}, \; i = 1 \ldots n$ are conditionally independent given N_t
- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)$
- Conditional default probability:

$$\tilde{p} = 1 - (1-p)^{N_t} \exp(-\bar{\lambda} t)$$

Multivariate Poisson model

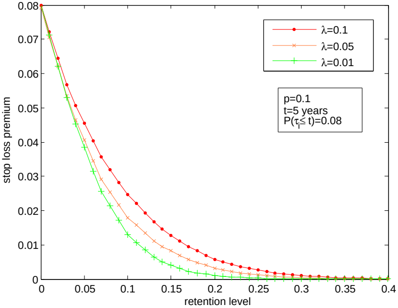
- Comparison of two multivariate Poisson models with parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$
- Supermodular order comparison requires equality of marginals: $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda^*$
- 3 comparison directions:
  - $p = p^*$: $\bar{\lambda}$ v.s λ
  - $\lambda = \lambda^*$: $\bar{\lambda}$ v.s p
  - $\bar{\lambda} = \bar{\lambda}^*$: λ v.s p

Multivariate Poisson model

Theorem ($p = p^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*$, then:

$$\lambda \leq \lambda^*,\ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$



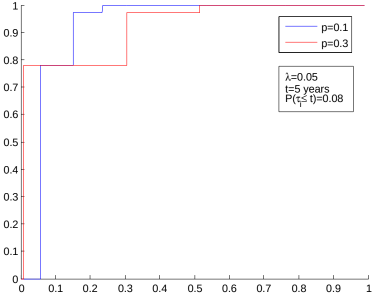
- Computation of $E[(L_t - a)^+]$:
  - 30 names
  - $M_i = 1,\ i = 1 \ldots n$
- When λ increases, the aggregate loss increases with respect to stop-loss order

Multivariate Poisson model

Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, then:

$$p \leq p^*,\ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$



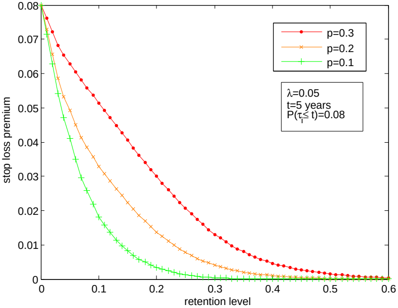
- Convex order for mixture probabilities

Multivariate Poisson model

Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar\lambda, \lambda, p)$ *and* $(\bar\lambda^*, \lambda^*, p^*)$ *be such that* $\bar\lambda + p\lambda = \bar\lambda^* + p^*\lambda$, *then:*

$$p \le p^*,\ \bar\lambda \ge \bar\lambda^* \Rightarrow \tilde p \le_{cx} \tilde p^* \Rightarrow (D_1, \ldots, D_n) \le_{sm} (D_1^*, \ldots, D_n^*)$$



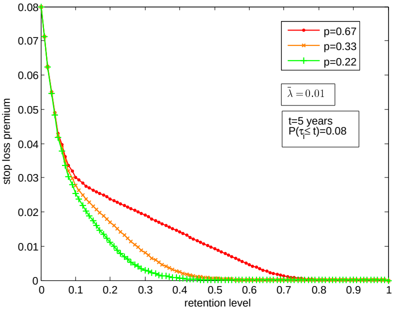
- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1,\ i = 1 \ldots n$
- When p increases, the aggregate loss increases with respect to stop-loss order

Multivariate Poisson model

Theorem ($\bar{\lambda} = \bar{\lambda}^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $p\lambda = p^*\lambda^*$, then:

$$p \leq p^*, \ \lambda \geq \lambda^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \dots, D_n) \leq_{sm} (D_1^*, \dots, D_n^*)$$



- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1, \ i = 1 \dots n$
- When p increases, the aggregate loss increases with respect to stop-loss order

Conclusion

- When considering an exchangeable vector of default indicators, the conditional independence assumption is not restrictive thanks to de Finetti's theorem
- The mixing probability (the factor) can be viewed as the loss of an infinitely granular portfolio
- We completely characterize the supermodular order between exchangeable default indicator vectors in term of the convex ordering of corresponding mixing probabilities
- We show that the mixing probability is the key input to study the impact of dependence on CDO tranche premiums
- Comparison analysis can be performed with the same method within a large class of CDO pricing models

Exchangeability: how realistic is a homogeneous assumption?

- Homogeneity of default marginals is an issue when considering the pricing and hedging of CDO tranches
  - ex: Sudden surge of GMAC spreads in the CDX index in May, 2005
  - This event dramatically impacts the equity tranche compared to the others
- But composition of standard indices are updated every semester, resulting in an increase of portfolio homogeneity
- It may be reasonable to split a credit portfolio in several homogeneous sub-portfolios (by economic sectors for example)
  - Then, for each sub-portfolio, we can find a specific factor and apply the previous comparison analysis
  - The initial credit portfolio can thus be associated with a vector of factors (one by sector)
  - Is it possible to relate comparison between global aggregate losses to comparison between vectors of random factors?

Are comparisons in a static framework restrictive?

- Are comparisons among aggregate losses at fixed horizons too restrictive?
- Computation of CDO tranche premiums only requires marginal loss distributions at several horizons
  - Comparison among aggregate losses at different dates is sufficient
- However, comparison of more complex products such as options on tranche or forward started CDOs are not possible in this framework
- Building a framework in which one can compare directly aggregate loss processes is a subject of future research