Comparison results for exchangeable credit risk portfolios

Areski COUSIN

Université d’Evry

Joint work with Jean-Paul Laurent, ISFA, Université de Lyon
1 Comparison results
- De Finetti theorem and factor representation
- Stochastic orders
- Main results

2 Application to several popular CDO pricing models
- Factor copula approaches
- Structural model
- Multivariate Poisson model
De Finetti theorem and factor representation

- Homogeneity assumption: default indicators $D_1, \ldots, D_n$ forms an exchangeable Bernoulli random vector

**Definition (Exchangeability)**

A random vector $(D_1, \ldots, D_n)$ is exchangeable if its distribution function is invariant for every permutations of its coordinates: $\forall \sigma \in S_n$

$$(D_1, \ldots, D_n) \overset{d}{=} (D_{\sigma(1)}, \ldots, D_{\sigma(n)})$$

- Same marginals
Assume that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables.

Thanks to de Finetti’s theorem, there exists a unique random factor $\tilde{p}$ such that $D_1, \ldots, D_n$ are conditionally independent given $\tilde{p}$.

Denote by $F_{\tilde{p}}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum i d_i} (1 - p)^{n - \sum i d_i} F_{\tilde{p}}(dp)$$

$\tilde{p}$ is characterized by:

$$\frac{1}{n} \sum_{i=1}^n D_i \xrightarrow{a.s.} \tilde{p} \text{ as } n \to \infty$$

$\tilde{p}$ is exactly the loss of the infinitely granular portfolio (Basel 2 terminology).
Stochastic orders

- The convex order compares the dispersion level of two random variables.
- Convex order: $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions $f$.
- Stop-loss order: $X \leq_{sl} Y$ if $E[(X - K)^+] \leq E[(Y - K)^+]$ for all $K \in \mathbb{R}$.
- $X \leq_{sl} Y$ and $E[X] = E[Y] \iff X \leq_{cx} Y$. 
Supermodular order

- The supermodular order captures the dependence level among coordinates of a random vector
- \((X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n)\) if \(E[f(X_1, \ldots, X_n)] \leq E[f(Y_1, \ldots, Y_n)]\) for all supermodular functions \(f\)

**Definition (Supermodular function)**

A function \(f : \mathbb{R}^n \to \mathbb{R}\) is supermodular if for all \(x \in \mathbb{R}^n, 1 \leq i < j \leq n\) and \(\varepsilon, \delta > 0\) holds

\[
f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n) \\
\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)
\]

Müller (1997)

*Stop-loss order for portfolios of dependent risks*

\[
(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \Rightarrow \sum_{i=1}^{n} M_i D_i \leq_{sl} \sum_{i=1}^{n} M_i D_i^*
\]
Main results

- Let us compare two credit portfolios with aggregate loss $L_t = \sum_{i=1}^{n} M_i D_i$ and $L_t^* = \sum_{i=1}^{n} M_i D_i^*$.
- Let $D_1, \ldots, D_n$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}$.
- Let $D_1^*, \ldots, D_n^*$ exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}^*$.

**Theorem**

$$\tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

- In particular, if $\tilde{p} \leq_{cx} \tilde{p}^*$, then:
  - $E[(L_t - a)^+] \leq E[(L_t^* - a)^+]$ for all $a > 0$.
  - $\rho(L_t) \leq \rho(L_t^*)$ for all convex risk measures $\rho$. 

**Comparison results**
Application to several popular CDO pricing models
De Finetti theorem and factor representation
Stochastic orders
Conclusion

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Comparison results for exchangeable credit risk portfolios
Main results

- Let $D_1, \ldots, D_n, \ldots$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}$
- Let $D_1^*, \ldots, D_n^*, \ldots$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}^*$

**Theorem (reverse implication)**

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^*.$$
1 Comparison results
   • De Finetti theorem and factor representation
   • Stochastic orders
   • Main results

2 Application to several popular CDO pricing models
   • Factor copula approaches
   • Structural model
   • Multivariate Poisson model
Ordering of CDO tranche premiums

- Analysis of the dependence structure in several popular CDO pricing models
- An increase of the dependence parameter leads to:
  - a decrease of \([0\%, b]\) equity tranche premiums (which guarantees the uniqueness of the market base correlation)
  - an increase of \([a, 100\%]\) senior tranche premiums
The dependence structure of default times is described by some latent variables $V_1, \ldots, V_n$ such that:

- $V_i = \rho V + \sqrt{1 - \rho^2} \bar{V}_i, \ i = 1 \ldots n$
- $V, \bar{V}_i, \ i = 1 \ldots n$ independent
- $\tau_i = G^{-1}(H_\rho(V_i)), \ i = 1 \ldots n$
  - $G$: distribution function of $\tau_i$
  - $H_\rho$: distribution function of $V_i$
- $D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n$ are conditionally independent given $V$
- $\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} E[D_i | V] = P(\tau_i \leq t | V) = \tilde{p}$
Additive factor copula approaches

**Theorem**

*For any fixed time horizon* $t$, *denote by* $D_i = 1\{\tau_i \leq t\}$, $i = 1 \ldots n$ *and* $D^*_i = 1\{\tau^*_i \leq t\}$, $i = 1 \ldots n$ *the default indicators corresponding to (resp.)* $\rho$ *and* $\rho^*$, *then:*

$$\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D^*_1, \ldots, D^*_n)$$

- This framework includes popular factor copula models:
  - One factor Gaussian copula - the industry standard for the pricing of CDO tranches
  - Double t: Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2007)
  - Double Variance Gamma: Moosbrucker(2006)
Archimedean copula


- $V$ is a positive random variable with Laplace transform $\varphi^{-1}$
- $U_1, \ldots, U_n$ are independent Uniform random variables independent of $V$
- $V_i = \varphi^{-1} \left( -\frac{\ln U_i}{V} \right), \ i = 1 \ldots n$ (Marshall and Olkin (1988))
  - $(V_1, \ldots, V_n)$ follows a $\varphi$-archimedean copula
  - $P(V_1 \leq v_1, \ldots, V_n \leq v_n) = \varphi^{-1} (\varphi(v_1) + \ldots + \varphi(v_n))$
- $\tau_i = G^{-1}(V_i)$
  - $G$: distribution function of $\tau_i$
- $D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n$ independent knowing $V$
- $\frac{1}{n} \sum_{i=1}^{n} D_i \overset{a.s.}{\rightarrow} \mathbb{E}[D_i | V] = P(\tau_i \leq t | V)$
Conditional default probability: \( \tilde{p} = \exp\{-\varphi(G(t)V)\} \)

<table>
<thead>
<tr>
<th>Copula</th>
<th>Generator ( \varphi )</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>( t^{-\theta} - 1 )</td>
<td>( \theta \geq 0 )</td>
</tr>
<tr>
<td>Gumbel</td>
<td>( (-\ln(t))^{\theta} )</td>
<td>( \theta \geq 1 )</td>
</tr>
<tr>
<td>Franck</td>
<td>(- \ln \left[ \frac{(1 - e^{-\theta t})}{(1 - e^{-\theta})} \right] )</td>
<td>( \theta \in \mathbb{R}^* )</td>
</tr>
</tbody>
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**Theorem**

\( \theta \leq \theta^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \)
Archimedean copula

- Clayton copula
- Mixture distributions are ordered with respect to the convex order
Hull, Predescu and White (2005)

- Consider \( n \) firms
- Let \( V_{i,t}, i = 1 \ldots n \) be their asset dynamics
  \[
  V_{i,t} = \rho V_t + \sqrt{1 - \rho^2} \bar{V}_{i,t}, \quad i = 1 \ldots n
  \]
- \( V, \bar{V}_i, i = 1 \ldots n \) are independent standard Wiener processes
- Default times as first passage times:
  \[
  \tau_i = \inf \{ t \in \mathbb{R}^+ | V_{i,t} \leq f(t) \}, \quad i = 1 \ldots n, \quad f : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous}
  \]
- \( D_i = 1_{\{\tau_i \leq \tau\}}, i = 1 \ldots n \) are conditionally independent given \( \sigma(V_t, t \in [0, T]) \)
Theorem

For any fixed time horizon $T$, denote by $D_i = 1\{\tau_i \leq \tau\}, \ i = 1 \ldots n$ and $D_i^* = 1\{\tau_i^* \leq \tau\}, \ i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$
Structural model

Distributions of Conditionnal Default Probabilities

- $\frac{1}{n} \sum_{i=1}^{n} D_i \overset{a.s.}{\rightarrow} \bar{p}$
- $\frac{1}{n} \sum_{i=1}^{n} D_i^* \overset{a.s.}{\rightarrow} \bar{p}^*$
- Empirically, mixture probabilities are ordered with respect to the convex order: $\bar{p} \leq_{cx} \bar{p}^*$

Portfolio size=10000
$X_0=0$
Threshold=$-2$
$t=1$ year
$\delta t=0.01$
$P(\tau_i \leq t)=0.033$

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Multivariate Poisson model


- $\tilde{N}_t^i$ Poisson with parameter $\bar{\lambda}$: idiosyncratic risk
- $N_t$ Poisson with parameter $\lambda$: systematic risk
- $(B_j^i)_{i,j}$ Bernoulli random variable with parameter $p$
- All sources of risk are independent
- $N_t^i = \tilde{N}_t^i + \sum_{j=1}^{N_t} B_j^i, \; i = 1 \ldots n$
- $\tau_i = \inf\{t > 0|N_t^i > 0\}, \; i = 1 \ldots n$
Multivariate Poisson model

- Dependence structure of \((\tau_1, \ldots, \tau_n)\) is the Marshall-Olkin copula
- \(\tau_i \sim \text{Exp}(\bar{\lambda} + p\lambda)\)
- \(D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n\) are conditionally independent given \(N_t\)
- \(\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)\)
- Conditional default probability:

\[
\tilde{p} = 1 - (1 - p)^{N_t} \exp(-\bar{\lambda} t)
\]
Multivariate Poisson model

- Comparison of two multivariate Poisson models with parameter sets 
  \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\)

- Supermodular order comparison requires equality of marginals:
  \(\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda^*\)

- 3 comparison directions:
  - \(p = p^*\): \(\bar{\lambda} \text{ v.s } \lambda\)
  - \(\lambda = \lambda^*\): \(\bar{\lambda} \text{ v.s } p\)
  - \(\bar{\lambda} = \bar{\lambda}^*\): \(\lambda \text{ v.s } p\)
Theorem \((p = p^*)\)

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*\), then:

\[
\lambda \leq \lambda^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \bar{p} \leq_{cx} \bar{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]

- Computation of \(E[(L_t - a)^+]\):
  - 30 names
  - \(M_i = 1, \; i = 1 \ldots n\)
- When \(\lambda\) increases, the aggregate loss increases with respect to stop-loss order
Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, then:

$p \leq p^*$, $\bar{\lambda} \geq \bar{\lambda}^*$ \implies \tilde{p} \leq_{cx} \tilde{p}^* \implies (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$

Convex order for mixture probabilities
Theorem \((\lambda = \lambda^*)\)

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda\), then:

\[ p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \]

- Computation of \(E[(L_t - K)^+]\):
  - 30 names
  - \(M_i = 1, \quad i = 1 \ldots n\)
- When \(p\) increases, the aggregate loss increases with respect to stop-loss order
Theorem \((\bar{\lambda} = \bar{\lambda}^*)\)

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(p\lambda = p^*\lambda^*\), then:

\[
p \leq p^*, \lambda \geq \lambda^* \Rightarrow \tilde{p} \leq p^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]

- Computation of \(E[(L_t - K)^+]\):
  - 30 names
  - \(M_i = 1, \ i = 1 \ldots n\)
- When \(p\) increases, the aggregate loss increases with respect to stop-loss order
Conclusion

- When considering an exchangeable vector of default indicators, the conditional independence assumption is not restrictive thanks to de Finetti’s theorem.
- The mixing probability (the factor) can be viewed as the loss of an infinitely granular portfolio.
- We completely characterize the supermodular order between exchangeable default indicator vectors in term of the convex ordering of corresponding mixing probabilities.
- We show that the mixing probability is the key input to study the impact of dependence on CDO tranche premiums.
- Comparison analysis can be performed with the same method within a large class of CDO pricing models.
Exchangeability: how realistic is a homogeneous assumption?

- Homogeneity of default marginals is an issue when considering the pricing and hedging of CDO tranches
  - ex: Sudden surge of GMAC spreads in the CDX index in May, 2005
  - This event dramatically impacts the equity tranche compared to the others
- But composition of standard indices are updated every semester, resulting in an increase of portfolio homogeneity
- It may be reasonable to split a credit portfolio in several homogeneous sub-portfolios (by economic sectors for example)
  - Then, for each sub-portfolio, we can find a specific factor and apply the previous comparison analysis
  - The initial credit portfolio can thus be associated with a vector of factors (one by sector)
  - Is it possible to relate comparison between global aggregate losses to comparison between vectors of random factors?
Are comparisons in a static framework restrictive?

- Are comparisons among aggregate losses at fixed horizons too restrictive?
- Computation of CDO tranche premiums only requires marginal loss distributions at several horizons
  - Comparison among aggregate losses at different dates is sufficient
- However, comparison of more complex products such as options on tranche or forward started CDOs are not possible in this framework
- Building a framework in which one can compare directly aggregate loss processes is a subject of future research