Beyond the Gaussian copula: stochastic and local correlation for CDOs

Petit déjeuner de la finance
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A comparative analysis of CDO pricing models
Beyond the Gaussian copula: stochastic and local correlation
disponibles sur www.defaultrisk.com

Beyond the Gaussian copula

- *One factor Gaussian copula*
  - Factors models, semi-analytical computations
  - Ordering of risks, Base correlation
  - Gaussian extensions, correlation sensitivities
  - Stochastic recovery rates
- *Model dependence/Choice of copula*
  - Student *t*, double *t*, Clayton, Marshall-Olkin, Stochastic correlation
  - Calibration methodology, empirical results
  - Distribution of conditional default probabilities
- *Beyond the Gaussian copula*
  - Marginal compound correlation
  - Stochastic correlation and state dependent correlation
  - Local correlation

Semi explicit pricing, conditional default probabilities

- Factor approaches to joint default times distributions:
  - *V: low dimensional factor*
  - *Conditionally on V, default times are independent.*
  - *Conditional default and survival probabilities:*

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right), \quad q_t^{i|V} = Q\left(\tau_i > t \mid V\right).$$

- Why factor models ?
  - *Tackle with large dimensions (i-Traxx, CDX)*
- Need of tractable dependence between defaults:
  - *Parsimonious modelling*
  - *Semi-explicit computations for CDO tranches*
  - *Large portfolio approximations*

Semi explicit pricing, conditional default probabilities

- Semi-explicit pricing for CDO tranches
  - Laurent & Gregory [2003]
- *Default payments are based on the* accumulated losses *on the pool of credits:*

$$L(t) = \sum_{i=1}^{n} LGD_i 1_{\{\tau_i \le t\}}, \quad LGD_i = N_i (1 - \delta_i)$$

- *Tranche premiums only involve call options on the accumulated losses*

$$E\left[\left(L(t) - K \right)^+ \right]$$

- *This is equivalent to knowing the distribution of L(t)*

Semi explicit pricing, conditional default probabilities

- Characteristic function: $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
  - *By conditioning upon V and using conditional independence:*

$$\varphi_{L(t)}(u) = E\left[\prod_{1\leq j\leq n}\left(1 - p_t^{j|V} + p_t^{j|V}\varphi_{1-\delta_j}(uN_j)\right)\right]$$

  - *Distribution of L(t) can be obtained by FFT*
    - Or other recursion technique
- Only need of conditional default probabilities $p_t^{i|V}$
- $p_t^{i|V}$ losses on a large homogeneous portfolio
  - *Approximation techniques for pricing CDOs*

Semi explicit pricing, conditional default probabilities

- One factor Gaussian copula:
  - $V, \bar{V}_i,\ i = 1, \dots, n$ *independent Gaussian,*

$$V_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
  - F_i *marginal distribution function of default times*
  - *Conditional default probabilities:*

$$p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$$

One factor Gaussian copula

- CDO margins (bps pa)
  - *With respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads 100 bp*
  - *5 years maturity*

	equity	mezzanine	senior
0%	5341	560	0.03
10%	3779	632	4.6
30%	2298	612	20
50%	1491	539	36
70%	937	443	52
100%	167	167	91

One factor Gaussian copula

- Equity tranche premiums are decreasing wrt ρ
  - *General result*
  - *Equity tranche premium is <u>always</u> decreasing with correlation parameter*
    - See Burtschell et al [2005] for more details about stochastic orders
  - *Guarantees uniqueness of « base correlation »*
  - *Monotonicity properties extend to Student t, Clayton and Marshall-Olkin copulas*

One factor Gaussian copula: extreme cases

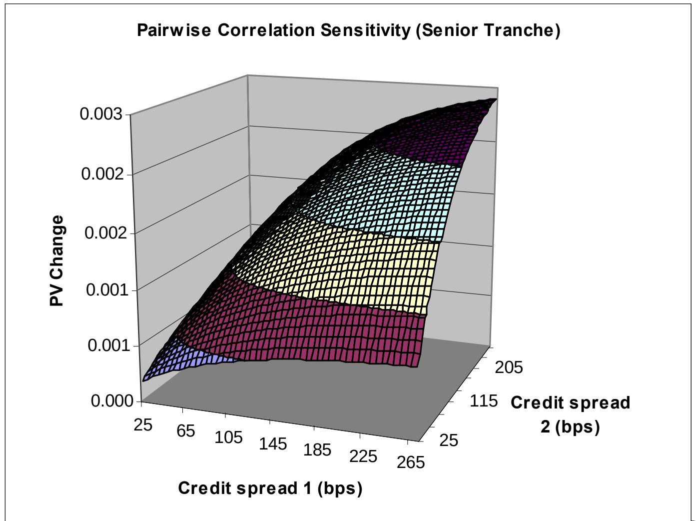
- $\rho = 100\%$
  - *Equity tranche premiums decrease with correlation*
  - *Does* $\rho = 100\%$ *correspond to some lower bound?*
  - $\rho = 100\%$ *corresponds to « comonotonic » default dates:*
  - $\rho = 100\%$ *is a <u>model free</u> lower bound for the equity tranche premium*
- $\rho = 0\%$
  - *Does* $\rho = 0\%$ *correspond to the higher bound on the equity tranche premium?*
  - $\rho = 0\%$ *corresponds to the independence case between default dates*
  - *The answer is no, negative dependence can occur*
  - *Base correlation does not always exists*

One factor Gaussian copula and extensions

- Gaussian extensions

$$\begin{pmatrix} 1 & \rho_{12} & & & & & & \\ \rho_{21} & 1 & & & & & & \\ & & 1 & & \rho_{ij}+\delta & & & \\ & & & . & & & & \\ & & \rho_{ij}+\delta & & . & & & \\ & & & & & 1 & & \\ & & & & & & 1 & . \\ & & & & & & . & 1 \end{pmatrix}$$

  - *Pairwise correlation sensitivities for CDO tranches*
  - *Can be computed analytically*
    - *See Gregory & Laurent, « In the Core of Correlation », Risk*
  - *Positive sensitivities (senior tranches)*



One factor Gaussian copula and extensions

- Gaussian extensions
  - *Intra & intersector correlations*
  - *i, name, s(i) sector*
  - *$W_{k(i)}$ factor for sector k(i)*
  - *W global factor*
  - *Allows for ratings agencies correlation matrices*
  - *Analytical computations still available for CDOs*
  - *Increasing intra or intersector correlations decrease equity tranche premiums*
  - *Does not explain the skew*

$$V_i = \rho_{s(i)} W_{k(i)} + \sqrt{1-\rho_{s(i)}^2}\,\bar{V}_i$$

$$W_{s(i)} = \lambda_{s(i)} W + \sqrt{1-\lambda_{s(i)}^2}\,\bar{W}_{s(i)}$$

$$\begin{pmatrix}
1 & \beta_1 & \beta_1 & & & & & & & \\
\beta_1 & 1 & \beta_1 & & & & & & \gamma & \\
\beta_1 & \beta_1 & 1 & & & & & & & \\
 & & & 1 & & & & & & \\
 & & & & \cdot & & & & & \\
 & & & & & \cdot & & & & \\
 & & & & & & 1 & & & \\
 & & & & & & & 1 & \beta_m & \beta_m \\
 & \gamma & & & & & & \beta_m & 1 & \beta_m \\
 & & & & & & & \beta_m & \beta_m & 1
\end{pmatrix}$$

One factor Gaussian copula and extensions

- Gaussian extensions
  - *Intra & intersector correlations*
  - *i, name, s(i) sector*
  - ρ *systemic correlation*
  - *Accounting for sector diversification in risk assessment*
  - *Risk measures based on unexpected losses,* $\alpha = 99.9\%$

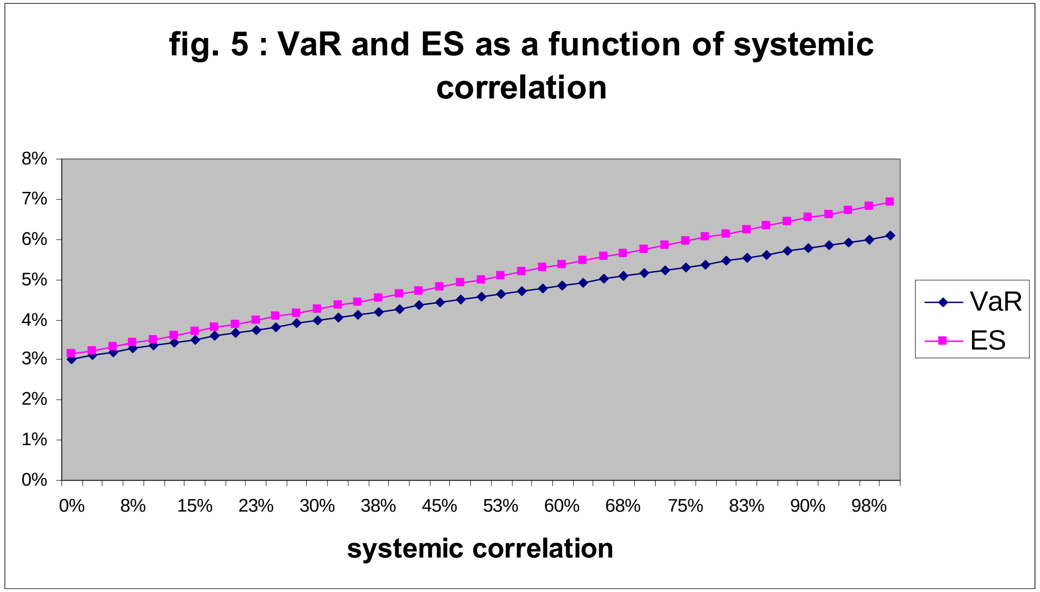
$$V_i = \rho_{s(i)} W_{k(i)} + \sqrt{1-\rho_{s(i)}^2}\,\bar{V}_i$$

$$W_{s(i)} = \rho W + \sqrt{1-\rho^2}\,\bar{W}_{s(i)}$$

	ζ (VaR)	κ (Expected Shortfall)
$\rho = 100\%$ (Basel II)	6,1%	6,9%
$\rho = 50\%$ (multifactor model)	4,6%	5,0%
Relative variation	-25%	-27%

One factor Gaussian copula and extensions

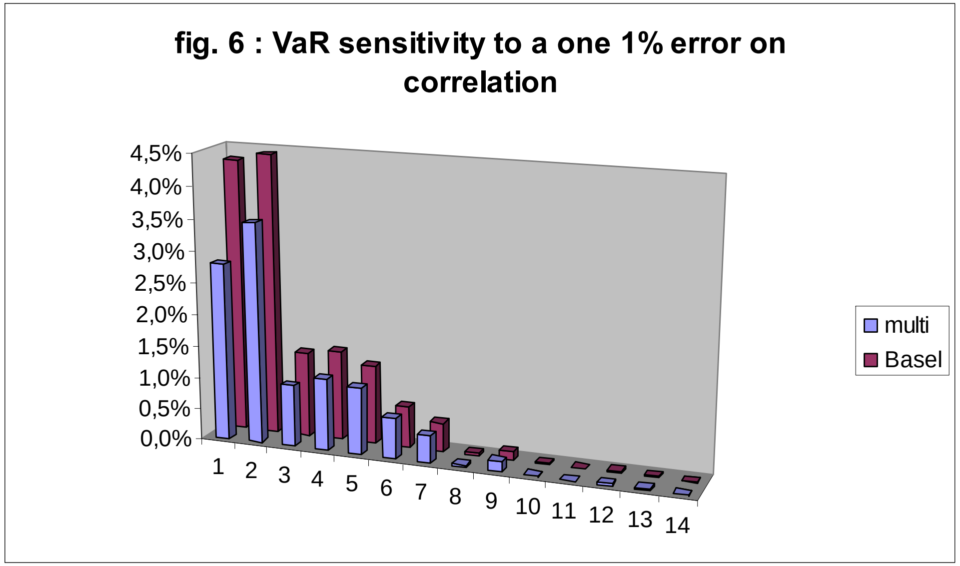
- VaR, Expected Shortfall and systemic correlation



  - *Risk measures change almost linearly wrt to systemic correlation*
  - *Basel II: no sector diversification*
  - *Sector diversification lessens capital requirements*
    - See "**Aggregation and credit risk measurement in retail banking**", Chabaane et al [2003]

One factor Gaussian copula and extensions

- VaR and intrasector correlation



- *Elasticity of VaR wrt intrasector correlation parameters*

$$\frac{\rho_J}{\zeta} \times \frac{\partial \zeta}{\partial \rho_J}$$

- *Lines 1 and 2 correspond to subportfolios with highest credit quality*

One factor Gaussian copula and extensions

- Correlation between default dates and recovery rates
  - *One factor Gaussian copula for default dates* $\Psi_i = \sqrt{\rho}\Psi + \sqrt{1-\rho}\bar{\Psi}_i$
  - *Losses Given Default also have a one factor structure:*

$$\xi_i = \sqrt{\beta}\xi + \sqrt{1-\beta}\bar{\xi}_i$$

  - *Merton type LGD:* $\max\left(0, 1 - e^{\mu + \sigma \xi_i}\right)$
  - *A two factor Gaussian model with factors* Ψ, ξ
  - *Correlation between defaults & recoveries and amongst recoveries*
    - See **Credit Risk Assessment and Stochastic LGD's: an Investigation of Correlation Effects** *in Recovery Risk: The Next Challenge in Credit Risk Management*, Risk Books

One factor Gaussian copula and extensions

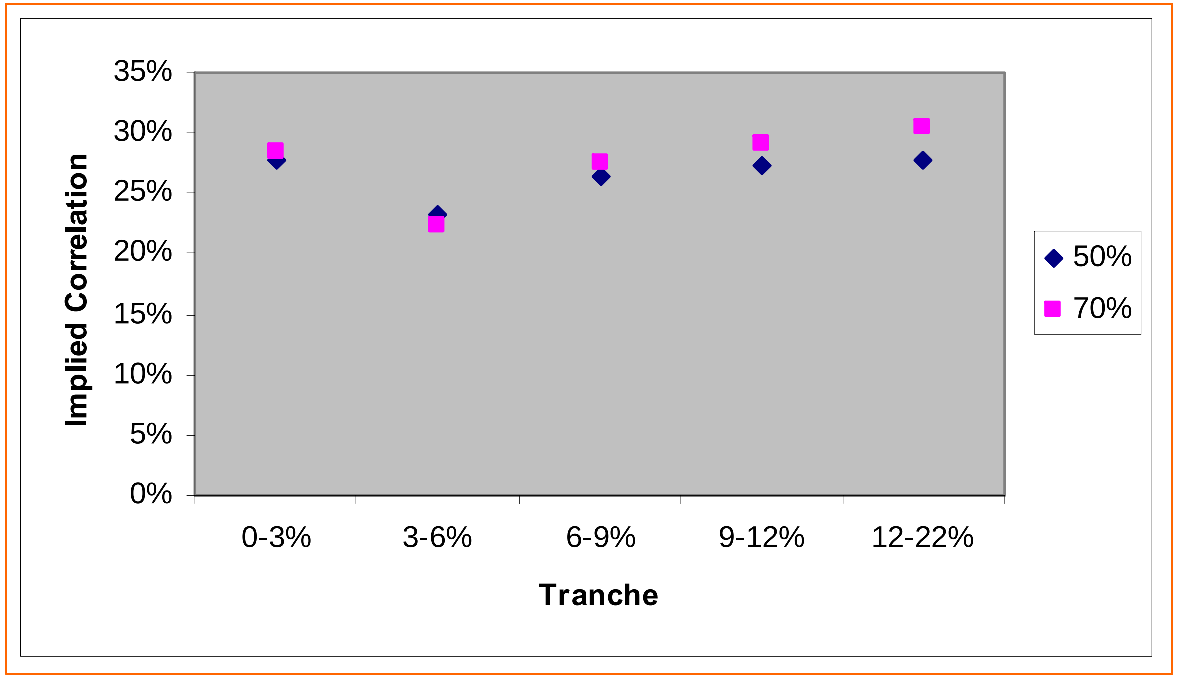
- Correlation between default dates and recovery rates
  - *VaR and ES as a function of correlation parameters*

η \ β	0%	20%	40%	60%	80%	100%
0%	158,9% *154,8%*	161,0% *160,2%*	164,2% *165,4%*	162,5% *164,7%*	159,3% *162,4%*	145,9% *152,1%*
20%	157,5% *153,9%*	175,4% *175,6%*	182,6% *183,7%*	186,8% *188,6%*	186,0% *192,5%*	172,8% *179,8%*
40%	160,2% *156,0%*	194,1% *196,6%*	207,9% *211,6%*	211,8% *218,7%*	212,6% *219,5%*	205,7% *217,2%*
60%	158,2% *155,2%*	207,4% *210,3%*	227,0% *231,1%*	238,9% *243,0%*	240,8% *249,2%*	234,1% *243,4%*
80%	159,6% *156,0%*	223,1% *229,4%*	244,1% *249,4%*	257,4% *265,1%*	264,5% *271,2%*	260,5% *273,4%*
100%	158,1% *153,9%*	238,9% *246,4%*	262,7% *268,0%*	276,5% *287,3%*	283,3% *296,3%*	286,8% *296,6%*

  - *Taking into account correlation between default events and LGD leads to a substantial increase in VaR and Expected Shortfall*

One factor Gaussian copula and extensions

- Correlation between default dates and recovery rates
  - *Correlation smile implied from the correlated recovery rates*
  - *Not as important as what is found in the market*



Model dependence / choice of copula

- Stochastic corrrelation copula
  - $V, \bar{V}_i,\ i = 1, \ldots, n$ *independent Gaussian variables*
  - $B_i = 1$ *correlation* ρ , $B_i = 0$ *correlation* β

$$V_i = B_i\left(\rho V + \sqrt{1-\rho^2}\,\bar{V}_i\right) + (1-B_i)\left(\beta V + \sqrt{1-\beta^2}\,\bar{V}_i\right)$$

$$\tau_i = F_i^{-1}\left(\Phi\left(V_i\right)\right)$$

$$p_t^{i|V} = p\Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right) + (1-p)\Phi\left(\frac{-\beta V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\beta^2}}\right)$$

Model dependence / choice of copula

- Student *t* copula
  - *Embrechts, Lindskog & McNeil, Greenberg et al, Mashal et al, O'Kane & Schloegl, Gilkes & Jobst*

$$\begin{cases} X_i = \rho V + \sqrt{1-\rho^2}\,\bar{V}_i \\ V_i = \sqrt{W} \times X_i \\ \tau_i = F_i^{-1}\left(t_\nu\left(V_i\right)\right) \end{cases}$$

  - $V, \bar{V}_i$ *independent Gaussian variables*
  - $\frac{\nu}{W}$ *follows a* χ_ν^2 *distribution*
  - *Conditional default probabilities (two factor model)*

$$p_t^{i|V,W} = \Phi\left(\frac{-\rho V + W^{-1/2} t_\nu^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}} \right)$$

Model dependence / choice of copula

- *Clayton* copula
  - *Schönbucher & Schubert, Rogge & Schönbucher, Friend & Rogge, Madan et al*

$$V_i = \psi\left(-\frac{\ln U_i}{V}\right) \qquad \tau_i = F_i^{-1}\left(V_i\right) \qquad \psi(s) = \left(1+s\right)^{-1/\theta}$$

    - Marshall-Olkin construction of archimedean copulas
  - *V: Gamma distribution with parameter* θ
  - $U_1, \ldots, U_n$ *independent uniform variables*
  - *Conditional default probabilities (one factor model)*

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

Model dependence / choice of copula

- Double *t* model (Hull & White)

$$V_i = \rho_i \left(\frac{\nu-2}{\nu}\right)^{1/2} V + \sqrt{1-\rho_i^2}\left(\frac{\bar{\nu}-2}{\bar{\nu}}\right)^{1/2} \bar{V}_i$$

  - $V, \bar{V}_i$ *are independent Student t variables*
    - with ν and $\bar{\nu}$ degrees of freedom

$$\tau_i = F_i^{-1}\left(H_i\left(V_i\right)\right)$$

  - *where* H_i *is the distribution function of* V_i

$$p_t^{i|V} = t_{\bar{\nu}}\left(\left(\frac{\bar{\nu}}{\bar{\nu}-2}\right)^{1/2} \frac{H_i^{-1}\left(F_i(t)\right) - \rho_i\left(\frac{\nu-2}{\nu}\right)^{1/2} V}{\sqrt{1-\rho_i^2}}\right)$$

Model dependence / choice of copula

- Shock models (multivariate exponential copulas)
  - *Duffie & Singleton, Giesecke, Elouerkhaoui, Lindskog & McNeil, Wong*
- Modelling of default dates: $V_i = \min\left(V, \bar{V}_i\right)$
  - $V, \bar{V}_i$ *exponential with parameters* $\alpha, 1-\alpha$
  - *Default dates* $\tau_i = S_i^{-1}\left(\exp - \min\left(V, \bar{V}_i\right)\right)$
    - S_i marginal survival function
  - *Conditionally on* V, τ_i *are independent.*
- Conditional default probabilities

$$q_t^{i|V} = 1_{V > -\ln S_i(t)} S_i(t)^{1-\alpha}$$

Model dependence / choice of copula

- Calibration procedure
  - *One parameter copulas*
  - *Fit Clayton, Student t, double t, Marshall Olkin parameters onto CDO equity tranches*
    - Computed under one factor Gaussian model
    - Or given market quotes on equity trances
  - *Reprice mezzanine and senior CDO tranches*
    - Given the previous parameter

Model dependence / choice of copula

- CDO margins (bps pa)
  - *With respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads 100 bp*
  - *5 years maturity*

	equity	mezzanine	senior
0%	5341	560	0.03
10%	3779	632	4.6
30%	2298	612	20
50%	1491	539	36
70%	937	443	52
100%	167	167	91

Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
θ	0	0.05	0.18	0.36	0.66	∞
ρ_6^2			14%	39%	63%	100%
ρ_{12}^2			22%	45%	67%	100%
ρ $t(4)$-$t(4)$	0%	12%	34%	55%	73%	100%
ρ $t(5)$-$t(4)$	0%	13%	36%	56%	74%	100%
ρ $t(4)$-$t(5)$	0%	12%	34%	54%	73%	100%
ρ $t(3)$-$t(4)$	0%	10%	32%	53%	75%	100%
ρ $t(4)$-$t(3)$	0%	11%	33%	54%	73%	100%
α	0	28%	53%	69%	80%	100%

Table 5: correspondence between parameters

Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	560	633	612	539	443	167
Clayton	560	637	628	560	464	167
Student (6)			637	550	447	167
Student (12)			621	543	445	167
$t(4)$-$t(4)$	560	527	435	369	313	167
$t(5)$-$t(4)$	560	545	454	385	323	167
$t(4)$-$t(5)$	560	538	451	385	326	167
$t(3)$-$t(4)$	560	495	397	339	316	167
$t(4)$-$t(3)$	560	508	406	342	291	167
MO	560	284	144	125	134	167

Table 6: mezzanine tranche (bps pa)

Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0.03	4.6	20	36	52	91
Clayton	0.03	4.0	18	33	50	91
Student (6)			17	34	51	91
Student (12)			19	35	52	91
t(4)-*t*(4)	0.03	11	30	45	60	91
t(5)-*t*(4)	0.03	10	29	45	59	91
t(4)-*t*(5)	0.03	10	29	44	59	91
t(3)-*t*(4)	0.03	12	32	47	71	91
t(4)-*t*(3)	0.03	12	32	47	61	91
MO	0.03	25	49	62	73	91

Table 7: senior tranche (bps pa)

Gaussian, Clayton and Student *t* CDO premiums are close

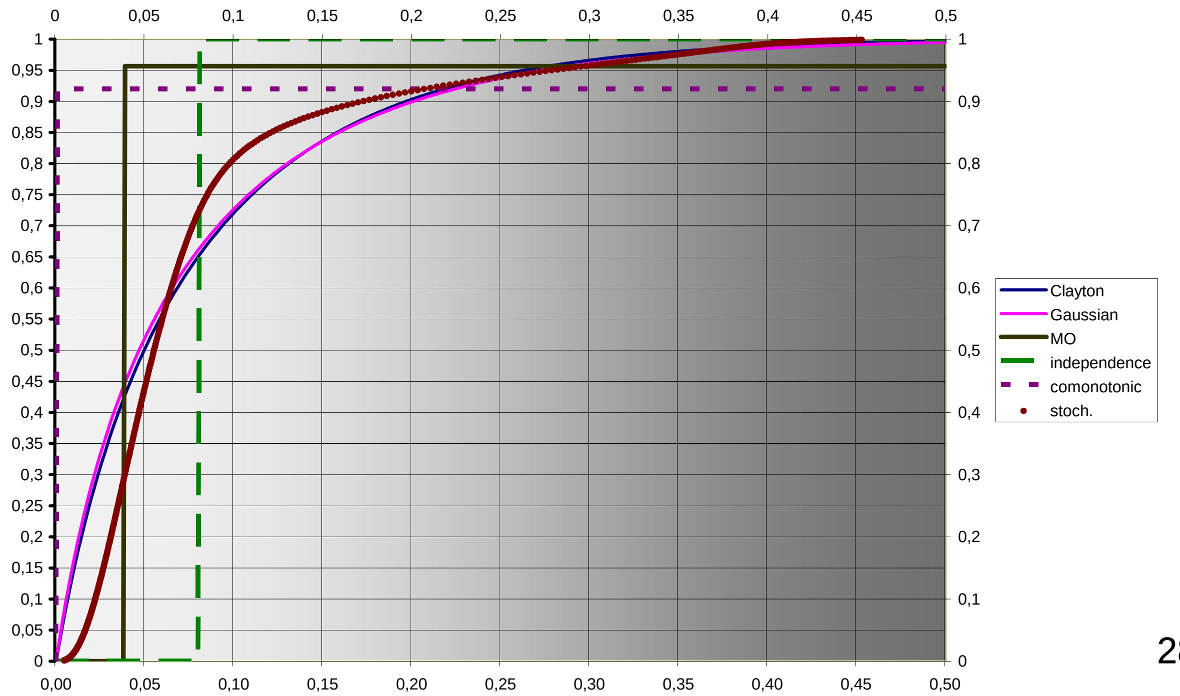
Model dependence / choice of copula

- Why Clayton and Gaussian copulas provide same SL premiums?
  - *Loss distributions depend on the **distribution** of conditional default probabilities*

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

$$p_t^{i|V} = \Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right)$$

  - *Distribution of conditional default probabililities are close for Gaussian and Clayton*



Matching the correlation skew

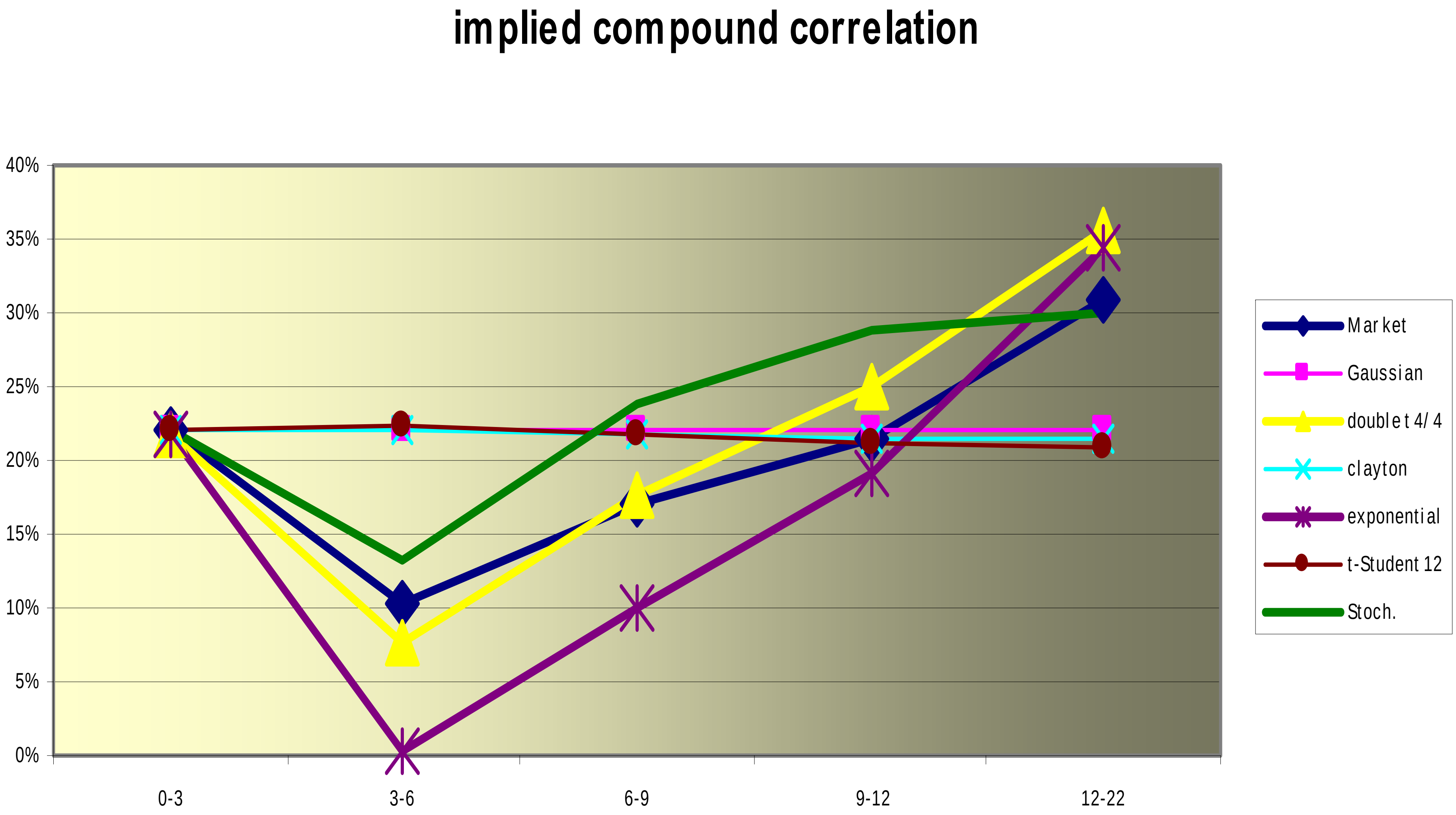
Tranches	Market	Gaussian	Clayton	Student (12)	t(4)-t(4)	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[3-6%]	101	163	163	164	82	122	14
[6-9%]	33	48	47	47	34	53	11
[9-12%]	16	17	16	15	22	29	11
[12-22%]	9	3	2	2	13	8	11

Table 17: CDO tranche premiums iTraxx (bps pa)

Tranches	Market	Gaussian	Clayton	Student (12)	t(4)-t(4)	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[0-6%]	466	503	504	504	456	479	418
[0-9%]	311	339	339	340	305	327	272
[0-12%]	233	253	253	254	230	248	203
[0-22%]	128	135	135	135	128	135	113

Table 18: “equity tranche” CDO tranche premiums iTraxx (bps pa)

Matching the correlation skew



Beyond the Gaussian copula: stochastic and local correlation

- Stochastic correlation
  - *Latent variables* $V_i = \tilde{\rho}_i V + \sqrt{1-\tilde{\rho}_i^2}\,\bar{V}_i, \quad i = 1,\ldots,n$

$$\tilde{\rho}_i = (1-B_s)(1-B_i)\rho + B_s$$

$\tilde{\rho}_i$, stochastic correlation,

$Q(B_s = 1) = q_s)$, systemic state,

$Q(B_i = 1) = q$, idiosyncratic state

  - *Conditional default probabilities*

$$p_t^{\cdot|V,B_s=0} = (1-q)\Phi\left(\frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1-\rho^2}}\right) + qF(t),\ F(t) \text{ default probability}$$

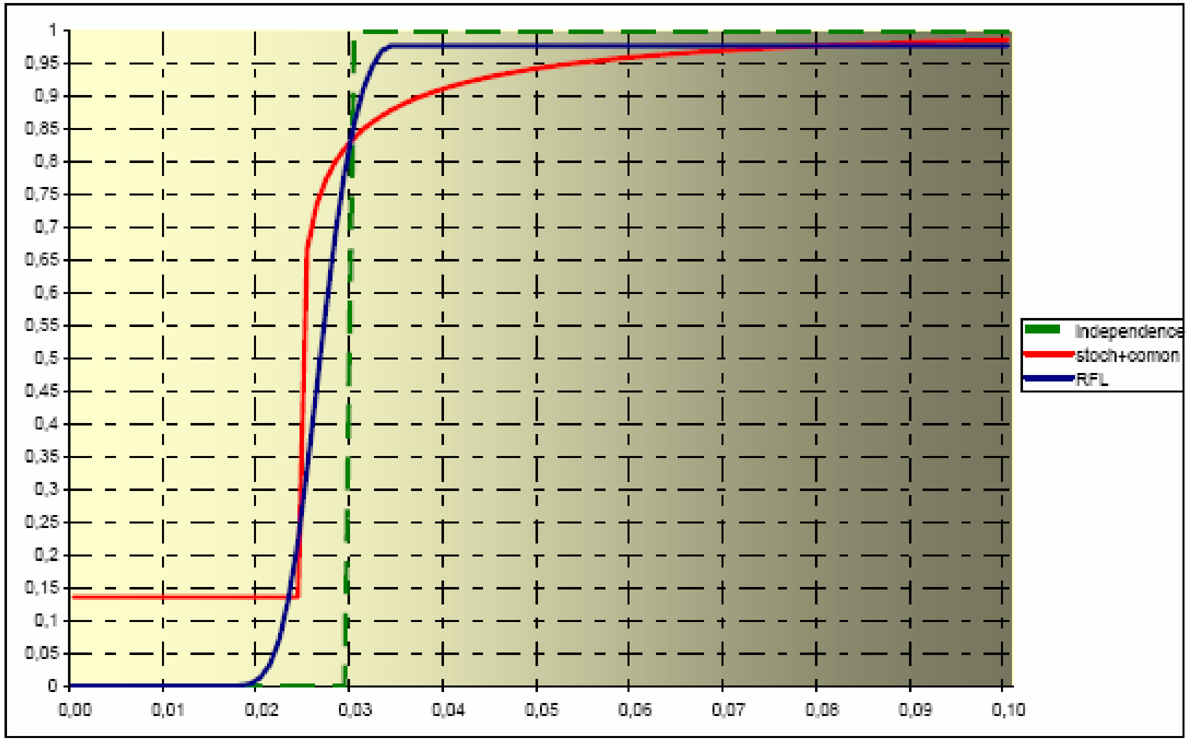
$$p_t^{\cdot|V,B_s=1} = 1_{V \le \Phi^{-1}(F(t))}, \text{ comonotonic}$$

Beyond the Gaussian copula: stochastic and local correlation

- Stochastic correlation $\tilde{\rho}_i = (1 - B_s)(1 - B_i)\rho + B_s$
  - *Semi-analytical techniques for pricing CDOs available*
  - *Large portfolio approximation can be derived*
  - *Allows for Monte Carlo*
  - $\nearrow \rho, \searrow q_s, \searrow q$ *leads to increase senior tranche premiums*
- State dependent correlation $V_i = m_i(V)V + \sigma_i(V)\bar{V}_i, \quad i = 1, \ldots, n$
  - *Local correlation* $V_i = -\rho(V)V + \sqrt{1 - \rho^2(V)}\bar{V}_i$
    - Turc et al
  - *Random factor loadings* $V_i = m + \left(l 1_{V<e} + h 1_{V \geq e}\right)V + \nu \bar{V}_i$
    - Andersen & Sidenius

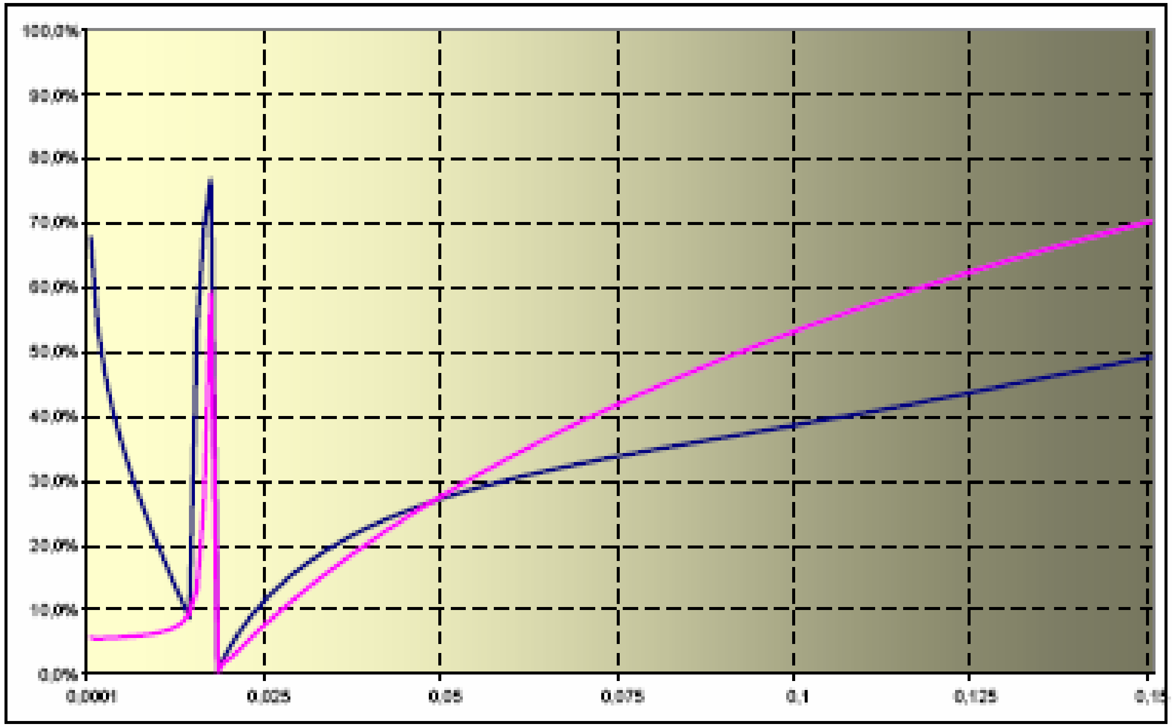
Beyond the Gaussian copula: stochastic and local correlation

- Distribution functions of conditional default probabilities
  - *stochastic correlation vs RFL*



  - *With respect to level of aggregate losses*
  - *Also correspond to loss distributions on large portfolios*

- Marginal compound correlations:
  - *With respect to attachment – detachment point*
  - *Compound correlation of a* $[\alpha,\alpha]$ *tranche*

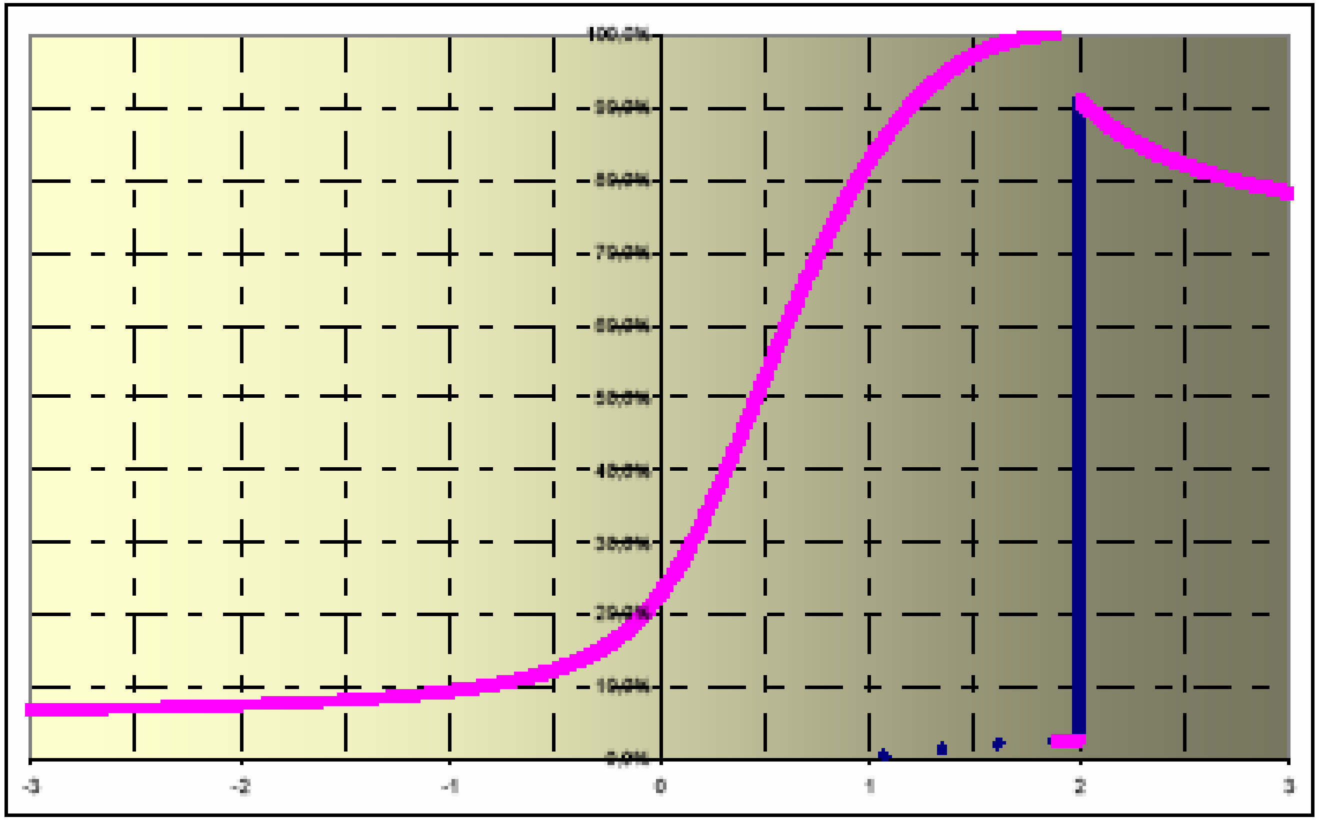


  - *Stochastic correlation vs RFL*

- Marginal compound correlation
  - *Can be obtained from the distribution function of conditional default probabilities*
  - *Need to solve a second order equation*
  - *There might be zero, one or two marginal compound correlations*
  - *Associated with the same conditional default probabilities*
  - *Always a zero marginal compound correlation at the expected loss*

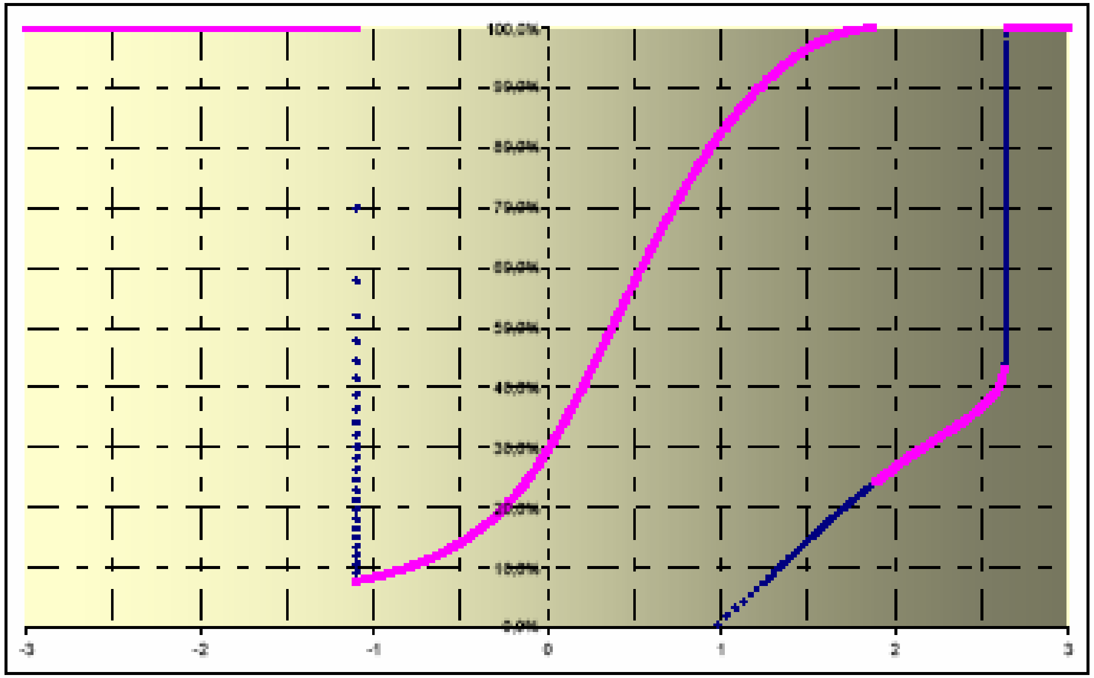
- Local correlation
  - *Can be obtained from the conditional default probability distribution*
  - *Need to solve for a functional equation*
  - *Fixed point algorithm*
  - *Step one: solving for a second order equation similar to the one giving marginal compound correlation*
  - *Local correlation at step one: rescaled marginal compound correlation*
  - *Same issues of uniqueness and existence*

- Local correlation associated with RFL (as a function of the factor)



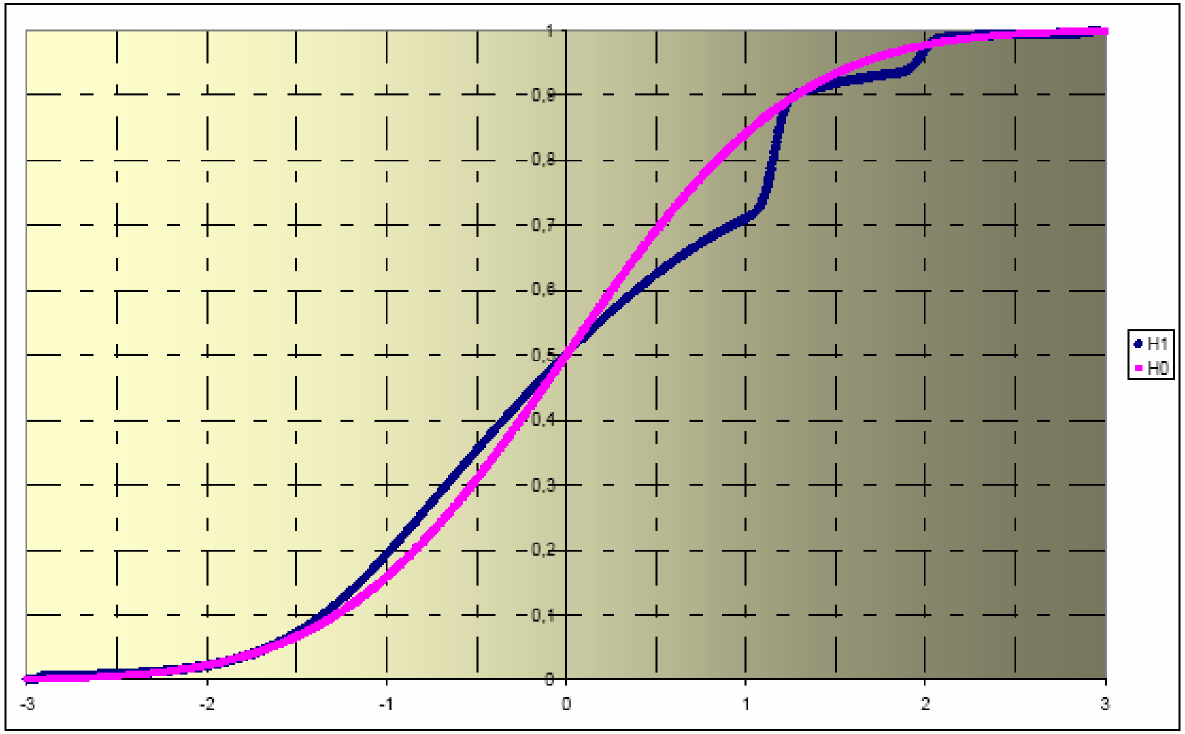
  - *Jump at threshold 2, low correlation level 5%, high correlation level 85%*
  - *Possibly two local correlations*

- Local correlation associated with stochastic correlation model
  - *With respect to factor V*



  - *Correlations of 1 for high-low values of V (comonotonic state)*
  - *Possibly two local correlations leading to the same prices*
  - *As for RFL, rather irregular pattern*

- Checking for the convergence of the fixed point algorithm



- *Good news: convergence at step one*

- Market fits: stochastic correlation model

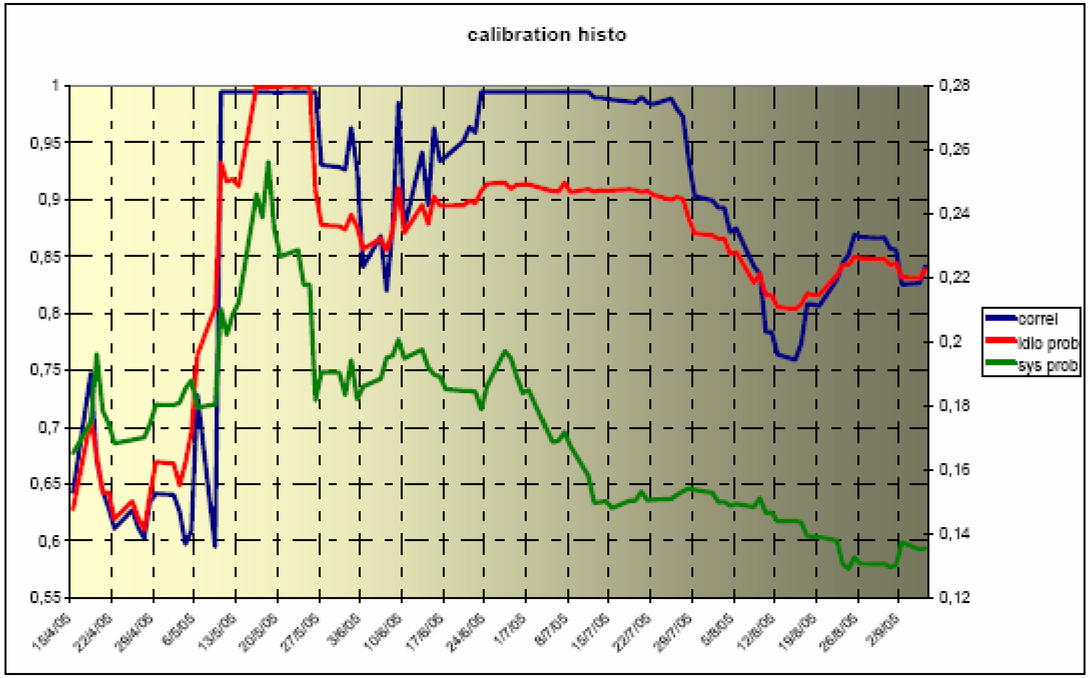
Tranche	Market	Model	Market ρ	Model ρ
Index	36			
[0-3%]	24%	25%	16%	14%
[3-6%]	83	84	4%	4%
[6-9%]	27	27	12%	12%
[9-12%]	14	14	17%	17%
[12-22%]	9	9	28%	28%
[22-100%]	4	2	63%	56%

Table 2. Fit of model characterised by equation (2.8) to iTraxx market data on 31-August-2005. All values in bp pa unless otherwise stated. As usual the equity tranche is quoted as an up-front premium, in addition to the contractual 500 bp pa. $q_s = 0.13$, $q = 0.84$, $\rho^2 = 73.5\%$.

Tranche	Market	Model	Market ρ	Model ρ
Index	50			
[0-3%]	40%	38%	10%	13%
[3-7%]	126	139	2%	2%
[7-10%]	36	39	12%	13%
[10-15%]	20	17	20%	19%
[15-30%]	10	10	34%	34%
[30-100%]	2	3	59%	65%

Table 3. Fit of model characterised by equation (2.8) to CDX market data on 31-August-2005. All values in bp pa unless otherwise stated. As usual the equity tranche is quoted as an up-front premium, in addition to the contractual 500 bp pa. $q_s = 0.15$, $q = 0.84$, $\rho^2 = 85.3\%$.

- Calibration history (from 15 April 2005)
  - *Implied correlation, implied idiosyncratic and systemic probabilities*



  - *Trouble in fitting during the crisis*
  - *Since then, decrease in systemic probability*

Conclusion

- Analysis of dependence through Gaussian models
  - *CDO premiums, Risk measures*
  - *Stochastic orders, base correlations*
  - *Analytical techniques, large portfolio approximations*
- Matching the skew with second generation models
  - *RFL, double t*
  - *Conditional default probability distributions are the drivers*
  - *Technique can be extended to structural or intensity models*
- Beyond the Gaussian copula
  - *Stochastic, local & marginal compound correlation*
- Pricing bespoke portfolios, CDO squared with a consistent model