Hedging CDOs in Markovian contagion models

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Jean-Paul LAURENT

Professor, ISFA Actuarial School, University of Lyon & scientific consultant BNP Paribas

http://laurent.jeanpaul.free.fr

Presentation related to the papers

***Hedging default risks of CDOs in Markovian contagion models* (2008)**

Available on www.defaultrisk.com

with Areski Cousin (Univ. Lyon) and Jean-David Fermanian (BNP Paribas)

and

***Hedging issues for CDOs* (with Areski Cousin)**

Some risks associated with CDOs

- Default risk
  - Default bond price jumps to recovery value at default time.
  - Drives the CDO cash-flows
- Credit spread risk
  - Changes in defaultable bond prices prior to default
    - Due to shifts in credit quality or in risk premiums
  - Changes in the marked to market of tranches
- Interactions between credit spread and default risks
  - Increase of credit spreads increases the probability of future defaults
  - Arrival of defaults may lead to jump in credit spreads
    - Contagion effects: Jarrow & Yu (2001)
    - Not consistent with the reduced-form approach

Mathematical Framework

- n obligors
- Default times: $\tau_1, \ldots, \tau_n$
  - (Ω, A, P) Probability space
- Default indicator processes: $N_i(t) = 1_{\{\tau_i \le t\}}, i = 1, \ldots, n$
- $H_{i,t} = \sigma\left(N_i(s), s \le t\right), i = 1, \ldots n;\ H_t = \bigvee_{i=1}^{n} H_{i,t}$
  - Natural filtration of default times
  - Ordered default times: $\tau^1, \ldots, \tau^n$
  - No simultaneous defaults: $\tau^1 < \ldots < \tau^n, P - a.s.$
- $\alpha_1^P, \ldots, \alpha_n^P \quad (P, H_t)$ intensities
  - $t \to N_i(t) - \int_0^t \alpha_i^P(s)ds \quad (P, H_t)$ martingales

Mathematical Framework

- Instantaneous digital CDS
  - Traded at t
  - Stylized cash-flow at t+dt:

$$\underbrace{dN_i(t)}_{\substack{\text{default}\\\text{payment}}} - \underbrace{\alpha_i(t)dt}_{\substack{\text{premium}\\\text{payment}}}$$

- Default free interest rate: r
- Payoffs of self-financed strategies:

$$\underbrace{V_0}_{\substack{\text{initial}\\\text{investment}}} e^{rT} + \sum_{i=1}^{n} \int_0^T \underbrace{\delta_i(s)}_{\substack{\text{holdings}\\\text{in CDS } i}} e^{r(T-s)} \left(dN_i(s) - \alpha_i(s)ds\right)$$

  - $\delta_1(\bullet), \ldots, \delta_n(\bullet)$ H_t – predictable processes

Mathematical Framework

- Absence of arbitrage opportunities: $\left\{ \underbrace{\alpha_i(t)}_{\substack{\text{CDS} \\ \text{premium}}} > 0 \right\} \overset{P-a.s.}{=} \left\{ \underbrace{\alpha_i^P(t)}_{\substack{(P-H_t) \\ \text{intensity}}} > 0 \right\}$
- As a consequence: $\exists! Q \sim P$,
  - such that $\alpha_1, \ldots, \alpha_n$ are the (Q, H_t) intensities of default times
- $M : H_T$ – measurable, Q –integrable payoff
- Integral representation theorem of point processes (Brémaud)

$$M = E^Q[M] + \sum_{i=1}^{n} \int_0^T \underbrace{\theta_i(s)}_{\substack{H_s- \\ \text{predictable}}} \left(\underbrace{dN_i(s) - \alpha_i(s) ds}_{\substack{\text{CDS} \\ \text{cash-flow}}} \right)$$

Mathematical Framework

- Integral representation theorem implies completeness of the credit market
  - Perfect replication of claims which depend only upon the default history
    - With CDS on underlying names and default-free asset
    - CDO tranches
  - *Q*: unique martingale measure
  - Replication price of *M* at time *t*: $V_t = E^Q\left[Me^{-r(T-t)} \middle| H_t\right]$
  - Note that the holdings of CDS only depend upon default history
    - Credit spread risk is not taken into account

Mathematical Framework

- Need of additional assumptions to effectively compute dynamic hedging strategies:

$$\begin{cases} \alpha_i(t) = \alpha\big(t, N(t)\big),\ i = 1,\ldots,n \\ N(t) = \sum_{i=1}^{n} N_i(t), \text{ number of defaults at time } t \end{cases}$$

  - CDS spreads only depend upon the current credit status
    - ➢Markov property
  - CDS spreads only depend on the number of defaults
    - ➢Mean-field
  - All names have the same short-term credit spread
    - ➢Homogeneity

Mathematical Framework

- $N(t)=\sum_{i=1}^{n} 1_{\{\tau_i \le t\}}$ number of default process
- is a continuous time *Q*- Markov chain
  - Pure death process
  - Generator of the Chain $\Lambda(t)=\begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & 0 & & & 0 \\ 0 & & \bullet & \bullet & & & 0 \\ 0 & & & & \bullet & & 0 \\ 0 & & & & & \bullet & 0 \\ 0 & & & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
  - $\lambda(t,N(t))$ is the intensity of the pure jump process $N(t)$
    - ➢is also the aggregate loss intensity

$$\lambda(t,N(t))=\underbrace{\left(n-N(t)\right)}_{\substack{\text{number of}\\ \text{non-defaulted}\\ \text{names}}}\times\underbrace{\alpha\left(t,N(t)\right)}_{\substack{\text{individual}\\ \text{pre-default}\\ \text{intensity}}}$$

Mathematical Framework

- Replication price for a CDO tranche $V_t = V_{CDO}\left(t, N(t)\right)$
- Only depends on the number of defaults
  - And of the individual characteristics of the tranche
    - Seniority, maturity, features of premium payments
- Thanks to the "homogeneity" between names:
  - All hedge ratios with respect to individual CDS are equal
  - Only hedge with the CDS index + risk-free asset
- Replicating hedge ratio:

$$\delta\left(t, N(t)\right) = \frac{V_{CDO}\left(t, N(t)+1\right) - V_{CDO}\left(t, N(t)\right)}{V_{CDS\ \text{Index}}\left(t, N(t)+1\right) - V_{CDS\ \text{Index}}\left(t, N(t)\right)}$$

Empirical results

- Calibration of loss intensities
  - From marginal distributions of aggregate losses
  - Or onto CDO tranche quotes
  - Use of forward Kolmogorov equations
    - For the Markov chain
  - Easy to solve for a pure death process
- Loss intensities with respect to the number of defaults
  - For simplicity, assumption of time homogeneous intensities
  - Increase in intensities: contagion effects
  - Compare flat and steep base correlation structures

3%	6%	9%	12%	22%
18%	28%	36%	42%	58%

Table 8. Base correlations with respect to attachment points.

Number of names: 125
Default-free rate: 4%
5Y credit spreads: 20 bps
Recovery rate: 40%

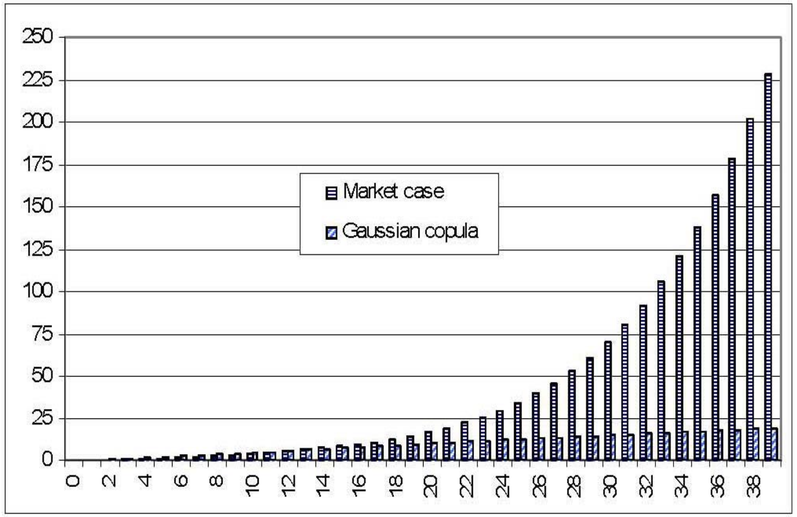

Figure 6. Loss intensities for the Gaussian copula and market case examples. Number of defaults on the x – axis.

Empirical results

- Dynamics of the credit default swap index in the Markov chain

Nb Defaults	Weeks			
	0	**14**	**56**	**84**
0	20	19	17	16
1	0	31	23	20
2	0	95	57	43
3	0	269	150	98
4	0	592	361	228
5	0	1022	723	490
6	0	1466	1193	905
7	0	1870	1680	1420
8	0	2243	2126	1945
9	0	2623	2534	2423
10	0	3035	2939	2859

Table 9. Dynamics of credit default swap index spread $s_{IS}(i,k)$ in basis points per annum.

  - The first default leads to a jump from 19 bps to 31 bps
  - The second default is associated with a jump from 31 bps to 95 bps
  - Explosive behavior associated with upward base correlation curve

Empirical results

- What about the credit deltas?
  - In a homogeneous framework, deltas with respect to CDS are all the same
  - Perfect dynamic replication of a CDO tranche with a credit default swap index and the default-free asset
  - Credit delta with respect to the credit default swap index
  - = change in PV of the tranche / change in PV of the CDS index

Nb Defaults	OutStanding Nominal	Weeks			
		0	14	56	84
0	3.00%	0.541	0.617	0.823	0.910
1	2.52%	0	0.279	0.510	0.690
2	2.04%	0	0.072	0.166	0.304
3	1.56%	0	0.016	0.034	0.072
4	1.08%	0	0.004	0.006	0.012
5	0.60%	0	0.002	0.002	0.002
6	0.12%	0	0.001	0.000	0.000
7	0.00%	0	0	0	0

Table 11. Delta of the default leg of the $[0,3\%]$ equity tranche with respect to the credit default swap index ($\delta_d(i,k)$).

Empirical results

- Dynamics of credit deltas:

Nb Defaults	OutStanding Nominal	Weeks			
		0	14	56	84
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2	2.04%	0	0.072	0.166	0.304
3	1.56%	0	0.016	0.034	0.072
4	1.08%	0	0.004	0.006	0.012
5	0.60%	0	0.002	0.002	0.002
6	0.12%	0	0.001	0.000	0.000
7	0.00%	0	0	0	0

Table 11. Delta of the default leg of the $[0,3\%]$ equity tranche with respect to the credit default swap index ($\delta_d(i,k)$).

  - Deltas are between 0 and 1
  - Gradually decrease with the number of defaults
    - ➢Concave payoff, negative gammas
  - When the number of defaults is > 6, the tranche is exhausted
  - Credit deltas increase with time
    - ➢Consistent with a decrease in time value

Empirical results

- Market and theoretical deltas at inception
  - Market deltas computed under the Gaussian copula model
    - ➢ Base correlation is unchanged when shifting spreads
    - ➢ "Sticky strike" rule
    - ➢ Standard way of computing CDS index hedges in trading desks

	[0-3%]	[3-6%]	[6-9%]	[9-12%]	[12-22%]
market deltas	27	4.5	1.25	0.6	0.25
model deltas	21.5	4.63	1.63	0.9	NA

- Smaller equity tranche deltas for in the Markov chain model
    - ➢ How can we explain this?

Empirical results

- Smaller equity tranche deltas in the Markov chain model
  - Default is associated with an increase in dependence
    - ➢Contagion effects

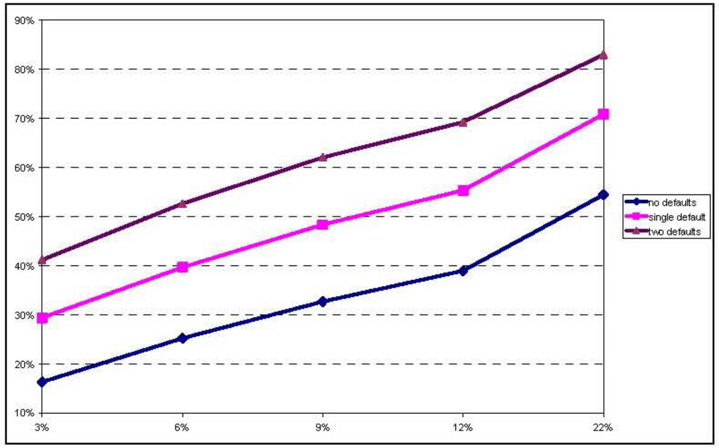


Figure 8. Dynamics of the base correlation curve with respect to the number of defaults. Detachment points on the x–axis. Base correlations on the y–axis.

  - Increasing correlation leads to a decrease in the PV of the equity tranche
    - ➢Sticky implied tree deltas
  - Recent market shifts go in favour of the contagion model

Empirical results

- The current crisis is associated with joint upward shifts in credit spreads
  - Systemic risk
- And an increase in base correlations

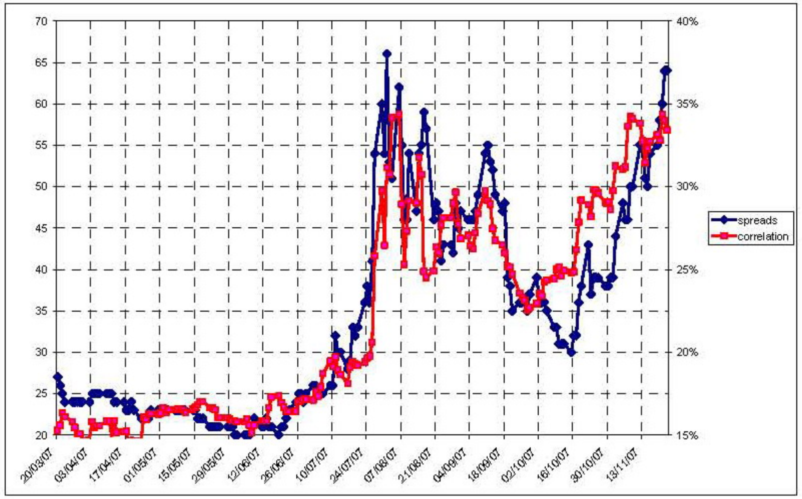


Figure 9. Credit spreads on the five years iTraxx index (Series 7) in bps on the left axis. Implied correlation on the equity tranche on the right axis

- Sticky implied tree deltas are well suited in regimes of fear
  - Derman: “regimes of volatility” (1999)

Empirical results

- Comparing with results provided by:
  - Arnsdorf and Halperin "*BSLP: Markovian Bivariate Spread-Loss Model for Portfolio Credit Derivatives*" Working Paper, JP Morgan (2007), Figure 7

	[0-3%]	[3-6%]	[6-9%]	[9-12%]	[12-22%]
market deltas	26.5	4.5	1.25	0.65	0.25
model deltas	21.9	4.81	1.64	0.79	0.38

  - Computed in March 2007 on the iTraxx tranches
  - Two dimensional Markov chain, shift in credit spreads

	[0-3%]	[3-6%]	[6-9%]	[9-12%]	[12-22%]
market deltas	27	4.5	1.25	0.6	0.25
model deltas	21.5	4.63	1.63	0.9	0.6

  - Note that our results, related to default deltas, are quite similar
    - ➢ Equity tranche deltas are smaller in contagion models than Gaussian copula credit deltas

Empirical results

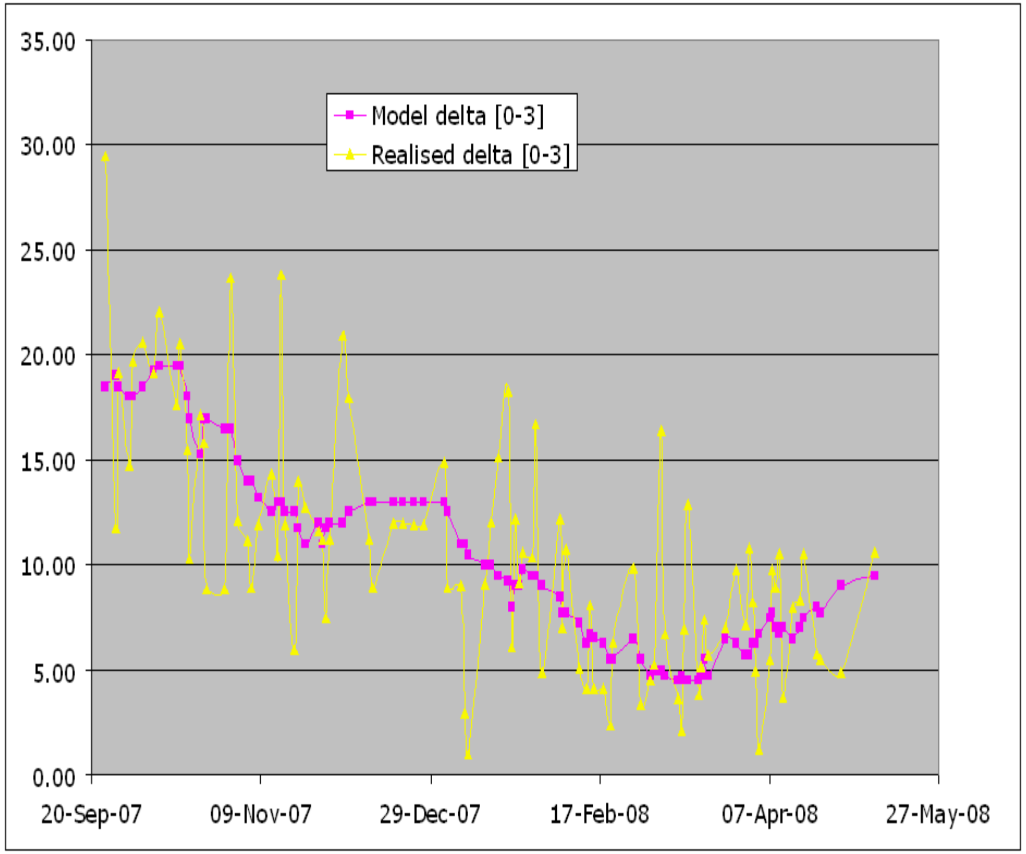
- Cont and Kan: “*Dynamic hedging of portfolio credit derivatives*” (2008)
- **Spread deltas**
  - Gaussian copula model
  - Local intensity corresponds to our contagion model
  - BSLP corresponds to Arnsdorf and Halperin (2007)
  - GPL: generalized Poisson loss model of Brigo *et al*. (2006)
- This shows some kind of robustness
- Picture becomes more complicated when considering other hedging criteria…

Tranche	Gauss	Local	BSLP	GPL
0 - 3	24.48	24.52	24.79	24.48
3 - 6	5.54	5.45	5.30	5.54
6 - 9	1.79	1.80	1.80	1.79
9 - 12	0.87	0.85	0.88	0.87
12 - 22	0.35	0.35	0.32	0.35
22 - 100	0.08	0.08	0.09	0.08

Spread deltas computed for 5Y Europe iTraxx on 20 September 2006

Empirical results

- Back-test study on iTraxx Series 8 equity tranche
- Comparison of realized spread deltas on the equity tranche and model (implied tree) deltas
- Good hedging performance compared with the Gaussian copula model
  - **During the credit crisis**
  - **Discrepancy with results of Cont and Kan (2008)?**



Source: S. Amraoui BNP Paribas

Empirical results

- Cont and Kan (2008) show rather poor performance of “jump to default” deltas
  - Even in the recent crisis period
- However, unsurprisingly, the credit deltas (“jump to default”) seem to be rather sensitive to the calibration of contagion parameters on quoted CDO tranches
- Right pictures represent aggregate loss intensities
  - Huge contagion effects for the first six defaults in Cont *et al*. (2008)
  - Much smaller contagion effects for the first defaults in Laurent *et al*. (2007)

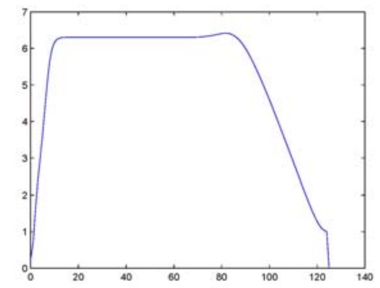


Figure 2: Dependence of default intensity on number of defaults for $t = 1$ year: ITRAXX Europe Series 6, March 15 2007..

Cont, Minca and Savescu (2008)

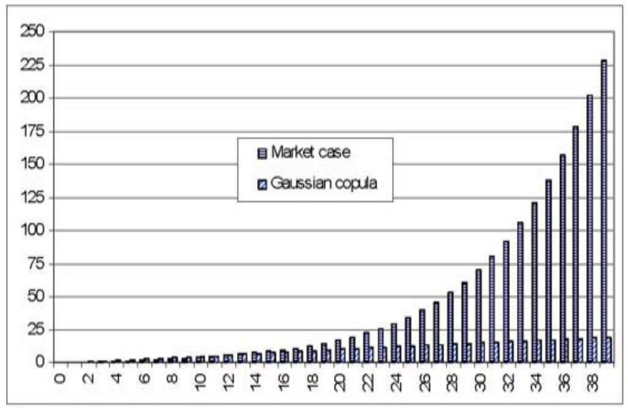


Figure 6. Loss intensities for the Gaussian copula and market case examples. Number of defaults on the $x-$axis.

Laurent, Cousin and Fermanian (2007)

Empirical results

- Frey and Backhaus: "*Dynamic hedging of synthetic CDO tranches with spread risk and default contagion*" (2007)

Tranche	[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
Spread	26 %	84 bp	24 bp	14 bp	11 bp
Tranche Correlation	17.30 %	3.22 %	9.93 %	15.81 %	27.46 %
Gauss Cop. Δ	$\boxed{0.61}$	0.23	0.06	0.03	0.07

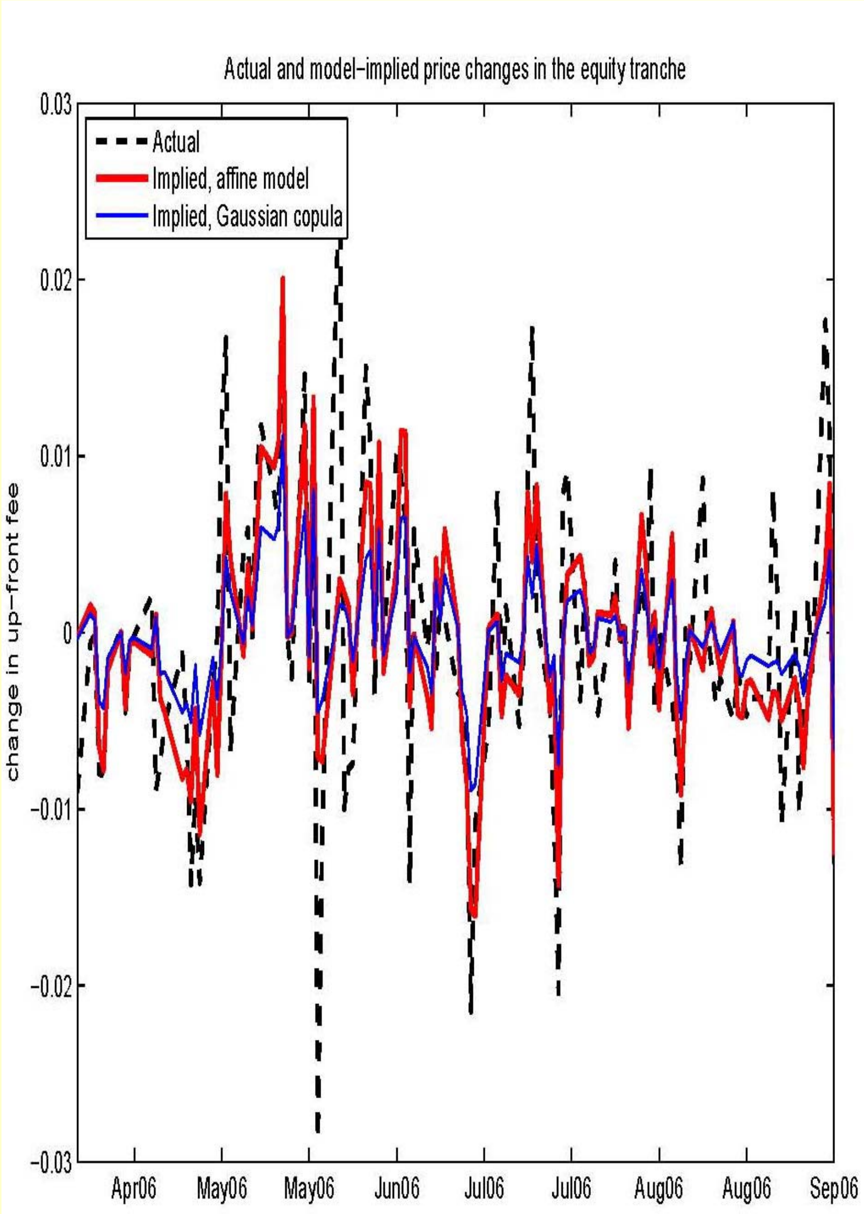
VOD: Value on default

	VOD in the Markov model	VOD in the Copula model
$[0, 3]$	$\boxed{0.344}$	1.002
$[3, 6]$	0.138	0.171
$[6, 9]$	0.058	0.023
$[9, 12]$	0.039	0.008
$[12, 22]$	0.107	0.010

Much smaller deltas in the contagion model than in Gaussian copula model

Empirical results

- Laurent: “A note on the risk management of CDO” (2007)
  - provides a theoretical framework for hedging credit spread risk only while default risk is diversified at the portfolio level
  - no default contagion, correlation between defaults are related to “correlation” between credit spreads
- Feldhütter: “An empirical investigation of an intensity-based model for pricing CDO tranches” (2008)
  - comparison of hedging performance of a Duffie and Garleanu (2001) reduced-form model and one factor Gaussian copula
  - Use of information at time *t*+1 to compute hedge ratios at time *t*
  - Higher deltas for the equity tranche in the affine model compared with the 1F Gaussian copula (market deltas)



Empirical results

- Consistent results with the affine model of Eckner (2007) based on December 2005 CDX data

Tranches	[0-3%]	[3-7%]	[7-10%]	[10-15%]	[15-30%]
market deltas	18.5	5.5	1.5	0.8	0.4
AJD deltas	21.7	6.0	1.1	0.4	0.1
contagion model deltas	17.9	6.3	2.5	1.3	0.8

  - Market deltas, “intensity” model credit deltas in Eckner (2007) and contagion model deltas
  - Goes into the opposite direction when comparing with the contagion model
- Note that Feldhütter (2008) and Eckner (2007) are pre-crisis
- And are according to a “sticky delta rule” (Derman) which is reflects irrational exuberance or greed
  - And might be appropriate for the pre-crisis period

Conclusion

- Main theoretical features of the complete market model
  - No simultaneous defaults
    - **Unlike multivariate Poisson models**
  - Credit spreads are driven by defaults
    - ➢ Contagion model
      - **Jumps in credit spreads at default times**
    - ➢ Credit spreads are deterministic between two defaults
  - Bottom-up approach
    - ➢ Aggregate loss intensity is derived from individual loss intensities
  - Correlation dynamics is also driven by defaults
    - ➢ Defaults lead to an increase in dependence

Conclusion

- What did we learn from the previous approaches?
  - Thanks to stringent assumptions:
    - credit spreads driven by defaults
    - homogeneity
    - Markov property
  - It is possible to compute a dynamic hedging strategy
    - Based on the CDS index
  - That fully replicates the CDO tranche payoffs
    - Model matches market quotes of liquid tranches
    - Very simple implementation
    - Credit deltas are easy to understand
  - Improve the computation of default hedges
    - Since it takes into account credit contagion
    - Provide some meaningful results in the current credit crisis

Additional selected references

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