Comparative analysis of CDO pricing models

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Joint work with X. Burtschell & J. Gregory
A comparative analysis of CDO pricing models
Beyond the Gaussian copula: stochastic and local correlation
Available on www.defaultrisk.com

Comparative analysis of CDO pricing models

- 1 Factor based copulas
  - Collective & individual models of credit losses
  - Semi-explicit pricing
- 2 One factor Gaussian copula
  - Ordering of risks, Base correlation
  - correlation sensitivities
  - Stochastic recovery rates
- 3 Model dependence / choice of copula
  - Student *t*, double *t*, Clayton, Marshall-Olkin, Stochastic correlation
  - Distribution of conditional default probabilities
- 4 Beyond the Gaussian copula
  - Stochastic correlation and state dependent correlation
  - Marginal and local correlation

1 Factor based copulas

- CDO valuation, credit risk assessment
  - *Only need of loss distributions for different time horizons*
  - *Aggregate loss at time t on a given portfolio:* $L(t)$
  - *Marginal loss distribution for time horizon t*

$$l \to F_{L(t)}(l) = Q\left(L(t) \leq l\right)$$

  - *VaR and quantile based risk measures for risk assessment*

$$\int_0^1 F_{L(t)}^{-1}(\alpha) v(\alpha) d\alpha$$

  - *Pricing of CDOs only involve options on aggregate loss*

$$E^Q\left[\left(L(t) - K\right)^+\right]$$

    - K attachment – detachment points

1 Factor based copulas

- Modelling approaches
  - *Direct modelling of* $L(t)$: *collective model*
    - Dealing with heterogeneous portfolios
    - non stationary, non Markovian
    - Aggregation of portfolios, bespoke portfolios?
    - Risk management of correlation risk?
  - *Modelling of default indicators of names: individual model*

$$L(t) = \sum_{i=1}^{n} LGD_i 1_{\tau_i \le t}$$

  - *Numerical approaches*
    - e.g. smoothing of base correlation of liquid tranches

1 Factor based copulas

- Individual model / factor based copulas
  - *Allows to deal with non homogeneous portfolios*
  - *Arbitrage free prices*
    - non standard attachment –detachment points
    - Non standard maturities
  - *Consistent pricing of bespoke, CDO^2, zero-coupon CDOs*
  - *Computations*
    - Semi-explicit pricing, computation of Greeks, LHP
  - *But…*
    - Poor dynamics of aggregate losses (forward starting CDOs)
    - Risk management, credit deltas, theta effects
    - Calibration onto liquid tranches (matching the skew)

1 Factor based copulas

- Factor approaches to joint default times distributions:
  - *V: low dimensional factor*
  - *Conditionally on V, default times are independent.*
  - *Conditional default and survival probabilities:*

$$p_t^{i|V} = Q\left(\tau_i \le t \mid V\right), \quad q_t^{i|V} = Q\left(\tau_i > t \mid V\right).$$

- Why factor models ?
  - *Tackle with large dimensions (i-Traxx, CDX)*
- Need of tractable dependence between defaults:
  - *Parsimonious modelling*
  - *Semi-explicit computations for CDO tranches*
  - *Large portfolio approximations*

1 Factor based copulas

- Semi-explicit pricing for CDO tranches
  - Laurent & Gregory [2003]
  - *Default payments are based on the* accumulated losses *on the pool of credits:*

$$L(t) = \sum_{i=1}^{n} LGD_i 1_{\{\tau_i \le t\}}, \quad LGD_i = N_i (1 - \delta_i)$$

  - *Tranche premiums only involve call options on the accumulated losses*

$$E\left[\left(L(t) - K \right)^+ \right]$$

  - *This is equivalent to knowing the distribution of L(t)*

1 Factor based copulas

- Characteristic function: $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
  - *By conditioning upon V and using conditional independence:*

$$\varphi_{L(t)}(u) = E\left[\prod_{1 \le j \le n}\left(1 - p_t^{j|V} + p_t^{j|V}\varphi_{1-\delta_j}(uN_j)\right)\right]$$

  - *Distribution of L(t) can be obtained by FFT*
    - Similar approaches: recursion, inversion of Laplace transforms
- Only need of conditional default probabilities $p_t^{i|V}$
- $p_t^{i|V}$ losses on a large homogeneous portfolio
  - *Approximation techniques for pricing CDOs*

Comparative analysis of CDO pricing models

- *2 One factor Gaussian copula*
  - Ordering of risks, Base correlation
  - correlation sensitivities
  - Stochastic recovery rates
- *3 Model dependence/Choice of copula*
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2 One factor Gaussian copula

- One factor Gaussian copula:
  - $V, \bar{V}_i,\ i = 1, \ldots, n$ *independent Gaussian,*

$$V_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
  - F_i *marginal distribution function of default times*
  - *Conditional default probabilities:*

$$p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$$

2 One factor Gaussian copula

- Equity tranche premiums are decreasing wrt ρ
  - *General result (use of stochastic orders theory)*
  - *Equity tranche premium is always decreasing with correlation parameter*
  - *Guarantees uniqueness of « base correlation »*
  - *Monotonicity properties extend to Student t, Clayton and Marshall-Olkin copulas*

2 One factor Gaussian copula

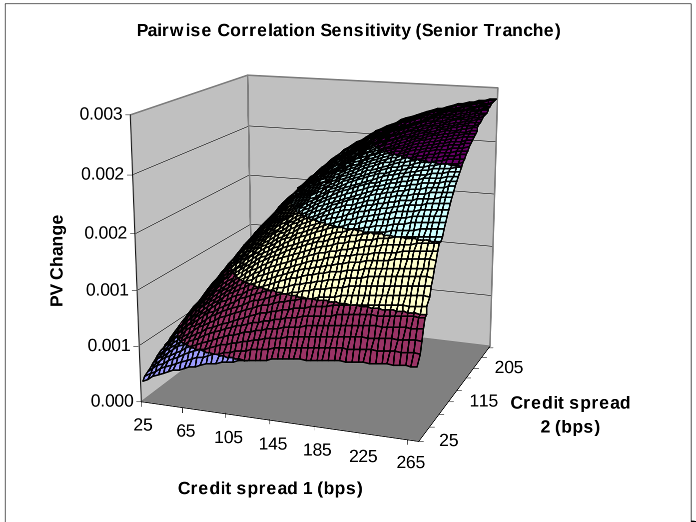
- $\rho = 100\%$
  - *Equity tranche premiums decrease with correlation*
  - *Does* $\rho = 100\%$ *correspond to some lower bound?*
  - $\rho = 100\%$ *corresponds to « comonotonic » default dates:*
  - $\rho = 100\%$ *is a* *model free* *lower bound for the equity tranche premium*
- $\rho = 0\%$
  - *Does* $\rho = 0\%$ *correspond to the higher bound on the equity tranche premium?*
  - $\rho = 0\%$ *corresponds to the independence case between default dates*
  - *The answer is no, negative dependence can occur*
  - *Base correlation does not always exists*

2 One factor Gaussian copula

- Pair-wise correlations

$$\begin{pmatrix} 1 & \rho_{12} & & & & & \\ \rho_{21} & 1 & & & & & \\ & & 1 & & \rho_{ij}+\delta & & \\ & & & \cdot & & & \\ & & \rho_{ij}+\delta & & \cdot & & \\ & & & & & 1 & \\ & & & & & & 1 \quad \cdot \\ & & & & & & \cdot \quad 1 \end{pmatrix}$$

  - *Pair-wise correlation sensitivities for CDO tranches*
  - *Can be computed analytically*
    - *See Gregory & Laurent, « In the Core of Correlation », Risk*
  - *Higher correlation sensitivities for riskier names (senior tranche)*



2 One factor Gaussian copula

- Intra Inter sector correlations
  - *i, name, s(i) sector*
  - $W_{s(i)}$ *factor for sector s(i)*
  - *W global factor*
  - *Allows for ratings agencies correlation matrices*
  - *Analytical computations still available for CDOs*
  - *Increasing intra or intersector correlations decrease equity tranche premiums*
  - *Does not explain the skew*

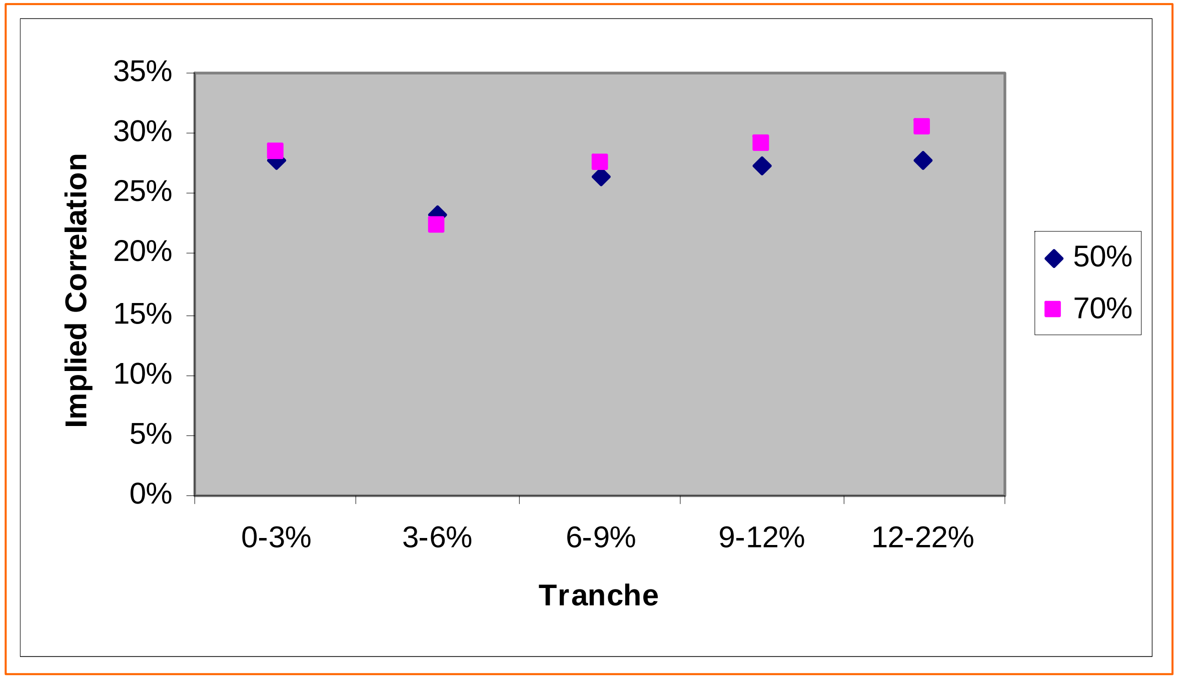
$$V_i = \rho_{s(i)} W_{s(i)} + \sqrt{1-\rho_{s(i)}^2}\,\bar{V}_i$$

$$W_{s(i)} = \lambda_{s(i)} W + \sqrt{1-\lambda_{s(i)}^2}\,\bar{W}_{s(i)}$$

$$\begin{pmatrix}
1 & \beta_1 & \beta_1 & & & & & & & \\
\beta_1 & 1 & \beta_1 & & & & & & \gamma & \\
\beta_1 & \beta_1 & 1 & & & & & & & \\
 & & & 1 & & & & & & \\
 & & & & \cdot & & & & & \\
 & & & & & \cdot & & & & \\
 & & & & & & 1 & & & \\
 & & & & & & & 1 & \beta_m & \beta_m \\
 & \gamma & & & & & & \beta_m & 1 & \beta_m \\
 & & & & & & & \beta_m & \beta_m & 1
\end{pmatrix}$$

2 One factor Gaussian copula

- Correlation between default dates and recovery rates
  - *Correlation smile implied from the correlated recovery rates*
  - *Not as important as what is found in the market*



3 Model dependence / choice of copula

- Stochastic correlation copula
  - $V, \bar{V}_i, \ i = 1, \ldots, n$ *independent Gaussian variables*
  - $B_i = 1$ *correlation* ρ, $B_i = 0$ *correlation* β

$$V_i = B_i\left(\rho V + \sqrt{1-\rho^2}\,\bar{V}_i\right) + (1 - B_i)\left(\beta V + \sqrt{1-\beta^2}\,\bar{V}_i\right)$$

$$\tau_i = F_i^{-1}\left(\Phi\left(V_i\right)\right)$$

$$p_t^{i|V} = p\Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right) + (1-p)\Phi\left(\frac{-\beta V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\beta^2}}\right)$$

3 Model dependence / choice of copula

- Student *t* copula

$$\begin{cases} X_i = \rho V + \sqrt{1-\rho^2}\,\bar{V}_i \\ V_i = \sqrt{W} \times X_i \\ \tau_i = F_i^{-1}\left(t_\nu\left(V_i\right)\right) \end{cases}$$

  - $V, \bar{V}_i$ *independent Gaussian variables*
  - $\dfrac{\nu}{W}$ *follows a* χ_ν^2 *distribution*
  - *Conditional default probabilities (two factor model)*

$$p_t^{i|V,W} = \Phi\left(\frac{-\rho V + W^{-1/2} t_\nu^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right)$$

3 Model dependence / choice of copula

- *Clayton* copula

$$V_i = \psi\left(-\frac{\ln U_i}{V}\right) \qquad \tau_i = F_i^{-1}(V_i) \qquad \psi(s) = (1+s)^{-1/\theta}$$

  - *V: Gamma distribution with parameter* θ
  - $U_1, \ldots, U_n$ *independent uniform variables*
  - *Conditional default probabilities (one factor model)*

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

3 Model dependence / choice of copula

- Double *t* model (Hull & White)

$$V_i = \rho_i \left(\frac{\nu - 2}{\nu}\right)^{1/2} V + \sqrt{1-\rho_i^2}\left(\frac{\bar{\nu}-2}{\bar{\nu}}\right)^{1/2} \bar{V}_i$$

  - $V, \bar{V}_i$ *are independent Student t variables*
    - with ν and $\bar{\nu}$ degrees of freedom

$$\tau_i = F_i^{-1}\left(H_i\left(V_i\right)\right)$$

  - *where* H_i *is the distribution function of* V_i

$$p_t^{i|V} = t_{\bar{\nu}}\left(\left(\frac{\bar{\nu}}{\bar{\nu}-2}\right)^{1/2} \frac{H_i^{-1}\left(F_i(t)\right) - \rho_i \left(\frac{\nu-2}{\nu}\right)^{1/2} V}{\sqrt{1-\rho_i^2}}\right)$$

3 Model dependence / choice of copula

- Shock models (multivariate exponential copulas)
  - *Marshall-Olkin copulas*
- Modelling of default dates: $V_i = \min\left(V, \bar{V}_i\right)$
  - $V, \bar{V}_i$ *exponential with parameters* $\alpha, 1-\alpha$
  - *Default dates* $\tau_i = S_i^{-1}\left(\exp - \min\left(V, \bar{V}_i\right)\right)$
    - S_i marginal survival function
  - *Conditionally on* V, τ_i *are independent.*
- Conditional default probabilities

$$q_t^{i|V} = 1_{V > -\ln S_i(t)} S_i(t)^{1-\alpha}$$

3 Model dependence / choice of copula

- Calibration procedure
  - *One parameter copulas*
  - *Fit Clayton, Student t, double t, Marshall Olkin parameters onto CDO equity tranches*
    - Computed under one factor Gaussian model
  - *Reprice mezzanine and senior CDO tranches*
    - Given the fitted parameter
    - Look for departures from the Gaussian copula
    - Look for ability to explain the correlation skew

3 Model dependence / choice of copula

- CDO margins (bps pa)
  - *With respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads 100 bps*
  - *5 years maturity*

	equity	mezzanine	senior
0%	5341	560	0.03
10%	3779	632	4.6
30%	2298	612	20
50%	1491	539	36
70%	937	443	52
100%	167	167	91

3 Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	560	633	612	539	443	167
Clayton	560	637	628	560	464	167
Student (6)			637	550	447	167
Student (12)			621	543	445	167
t(4)-*t*(4)	560	527	435	369	313	167
t(5)-*t*(4)	560	545	454	385	323	167
t(4)-*t*(5)	560	538	451	385	326	167
t(3)-*t*(4)	560	495	397	339	316	167
t(4)-*t*(3)	560	508	406	342	291	167
MO	560	284	144	125	134	167

Table 6: mezzanine tranche (bps pa)

3 Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0.03	4.6	20	36	52	91
Clayton	0.03	4.0	18	33	50	91
Student (6)			17	34	51	91
Student (12)			19	35	52	91
t(4)-*t*(4)	0.03	11	30	45	60	91
t(5)-*t*(4)	0.03	10	29	45	59	91
t(4)-*t*(5)	0.03	10	29	44	59	91
t(3)-*t*(4)	0.03	12	32	47	71	91
t(4)-*t*(3)	0.03	12	32	47	61	91
MO	0.03	25	49	62	73	91

Table 7: senior tranche (bps pa)

Gaussian, Clayton and Student *t* CDO premiums are close

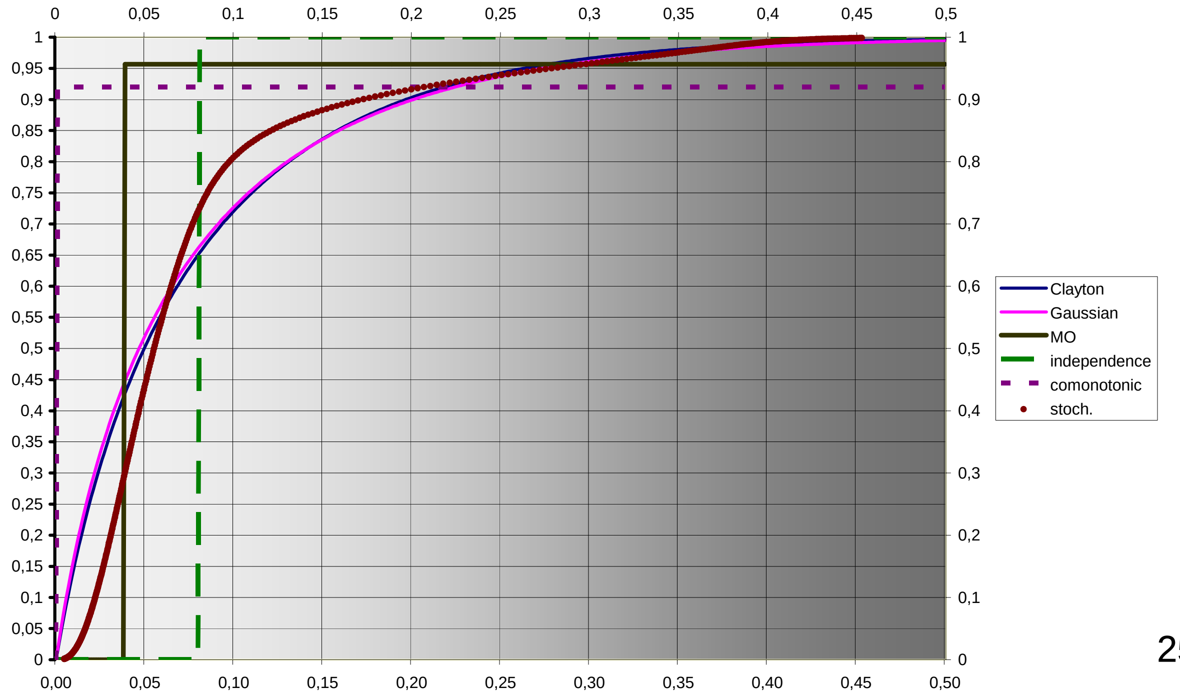
3 Model dependence / choice of copula

- Why do Clayton and Gaussian copulas provide same premiums?
  - *Loss distributions depend on the **distribution** of conditional default probabilities*

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

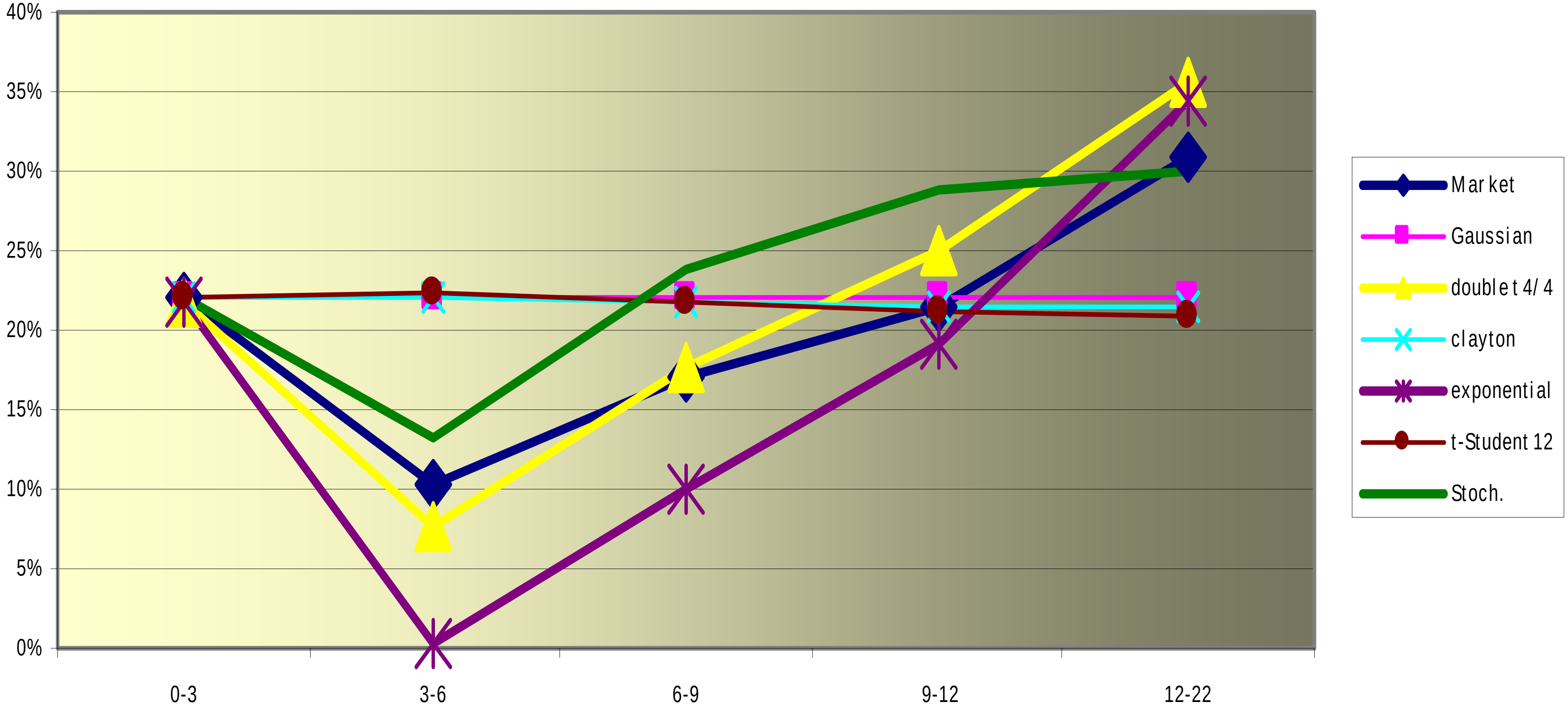
$$p_t^{i|V} = \Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right)$$

  - *Distribution of conditional default probabilities are close for Gaussian and Clayton*



3 Model dependence / choice of copula

implied compound correlation



4 Beyond the Gaussian copula

- Stochastic correlation
  - *Latent variables* $V_i = \tilde{\rho}_i V + \sqrt{1-\tilde{\rho}_i^2}\,\bar{V}_i, \quad i = 1,\ldots,n$

$$\tilde{\rho}_i = (1-B_s)(1-B_i)\rho + B_s$$

$\tilde{\rho}_i$, stochastic correlation,

$Q(B_s = 1) = q_s)$, systemic state,

$Q(B_i = 1) = q$, idiosyncratic state

  - *Conditional default probabilities*

$$p_t^{\cdot|V,B_s=0} = (1-q)\Phi\left(\frac{\Phi^{-1}\left(F(t)\right)-\rho V}{\sqrt{1-\rho^2}}\right) + qF(t),\ F(t) \text{ default probability}$$

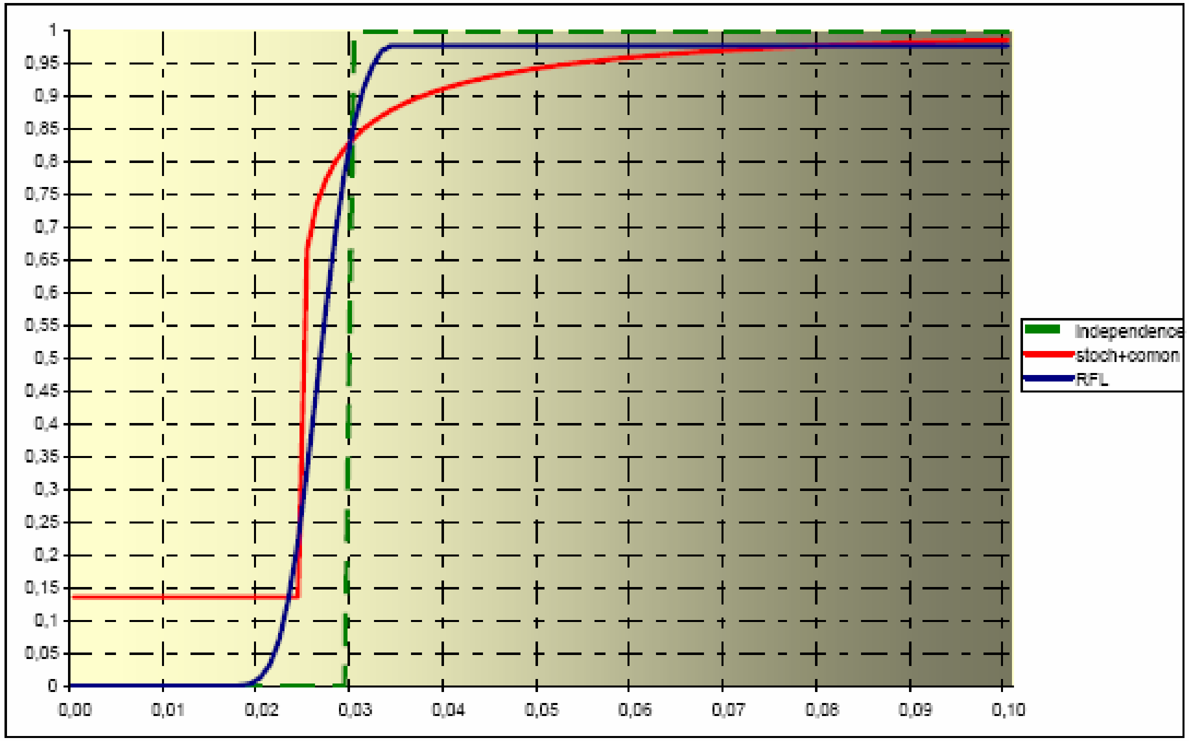
$$p_t^{\cdot|V,B_s=1} = 1_{V\leq\Phi^{-1}\left(F(t)\right)}, \text{ comonotonic}$$

4 Beyond the Gaussian copula

- Stochastic correlation $\tilde{\rho}_i = (1 - B_s)(1 - B_i)\rho + B_s$
  - *Semi-analytical techniques for pricing CDOs available*
  - *Large portfolio approximation can be derived*
  - *Allows for Monte Carlo*
  - $\nearrow \rho, \searrow q_s, \searrow q$ *leads to increase senior tranche premiums*
- State dependent correlation $V_i = m_i(V)V + \sigma_i(V)\bar{V}_i, \quad i = 1, \dots, n$
  - *Local correlation* $V_i = -\rho(V)V + \sqrt{1 - \rho^2(V)}\bar{V}_i$
    - Turc et al
  - *Random factor loadings* $V_i = m + \left(l 1_{V<e} + h 1_{V \geq e}\right)V + \nu \bar{V}_i$
    - Andersen & Sidenius

4 Beyond the Gaussian copula

- Distribution functions of conditional default probabilities
  - *stochastic correlation vs RFL*



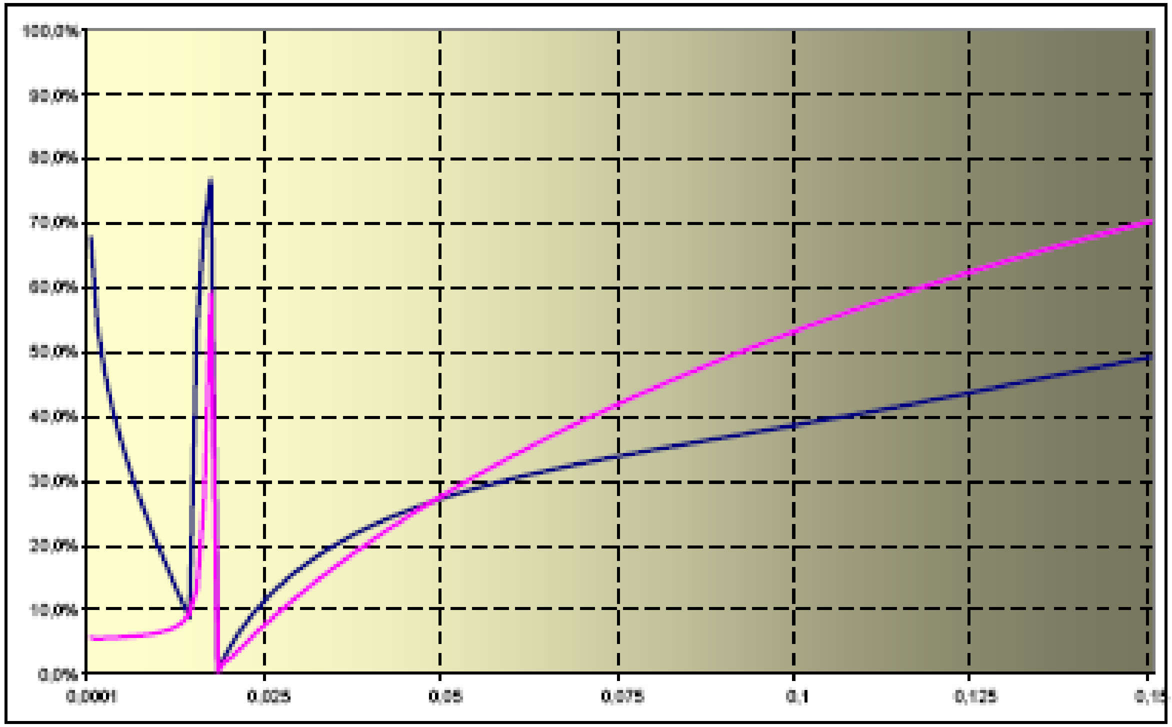
  - *With respect to level of aggregate losses*
  - *Also correspond to loss distributions on large portfolios*

4 Beyond the Gaussian copula

- Marginal compound correlation
  - *Compound correlation of a* $[\alpha,\alpha]$ *tranche*
    - Digital call on aggregate loss
  - *obtained from conditional default probability distribution*
  - *Need to solve a second order equation*
  - *zero, one or two marginal compound correlations*

4 Beyond the Gaussian copula

- Marginal compound correlations:
  - *With respect to attachment – detachment point*



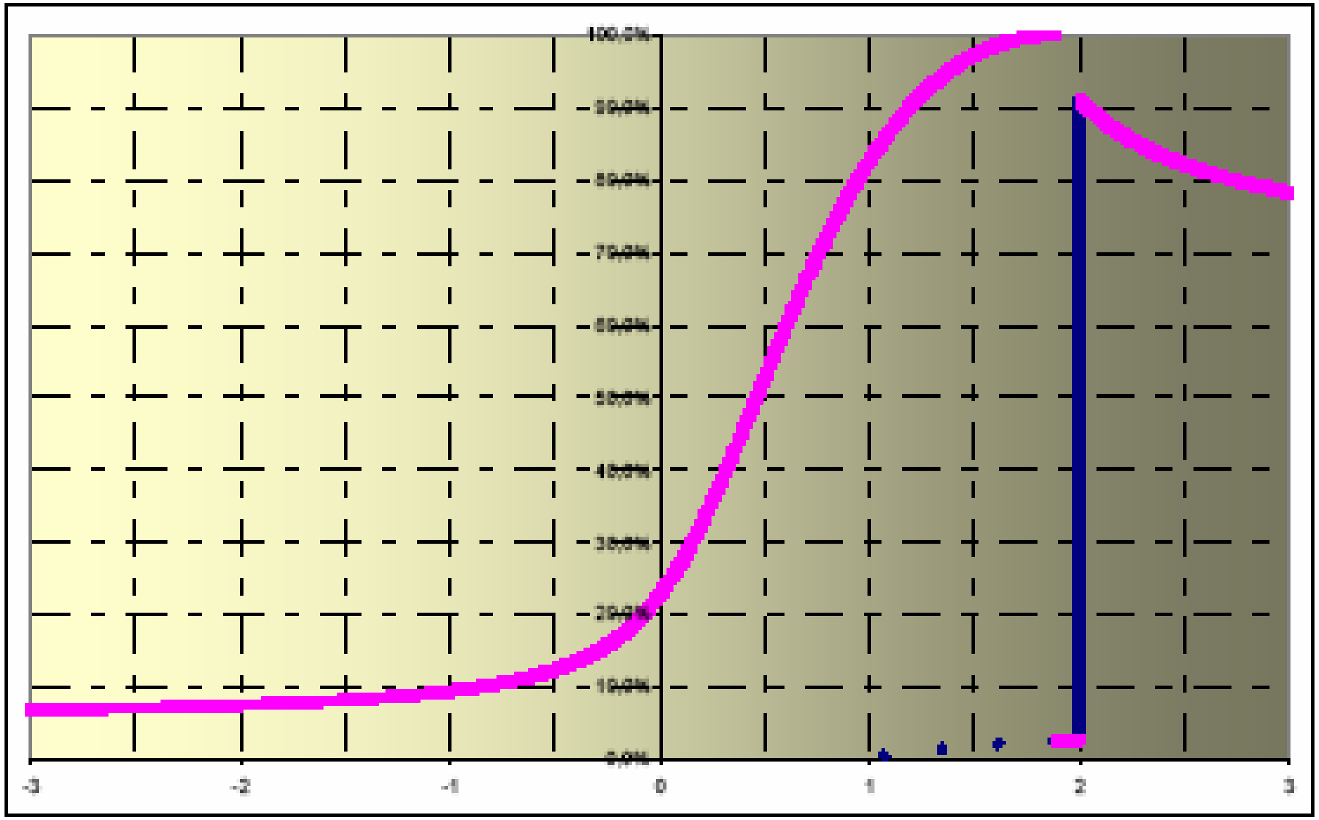
  - *Stochastic correlation vs RFL*
  - *zero marginal compound correlation at the expected loss*

4 Beyond the Gaussian copula

- Local correlation
  - *obtained from conditional default probability distribution*
  - *Fixed point algorithm*
  - *Local correlation at step one: rescaled marginal compound correlation*
  - *Same issues of uniqueness and existence as marginal compound correlation*

4 Beyond the Gaussian copula

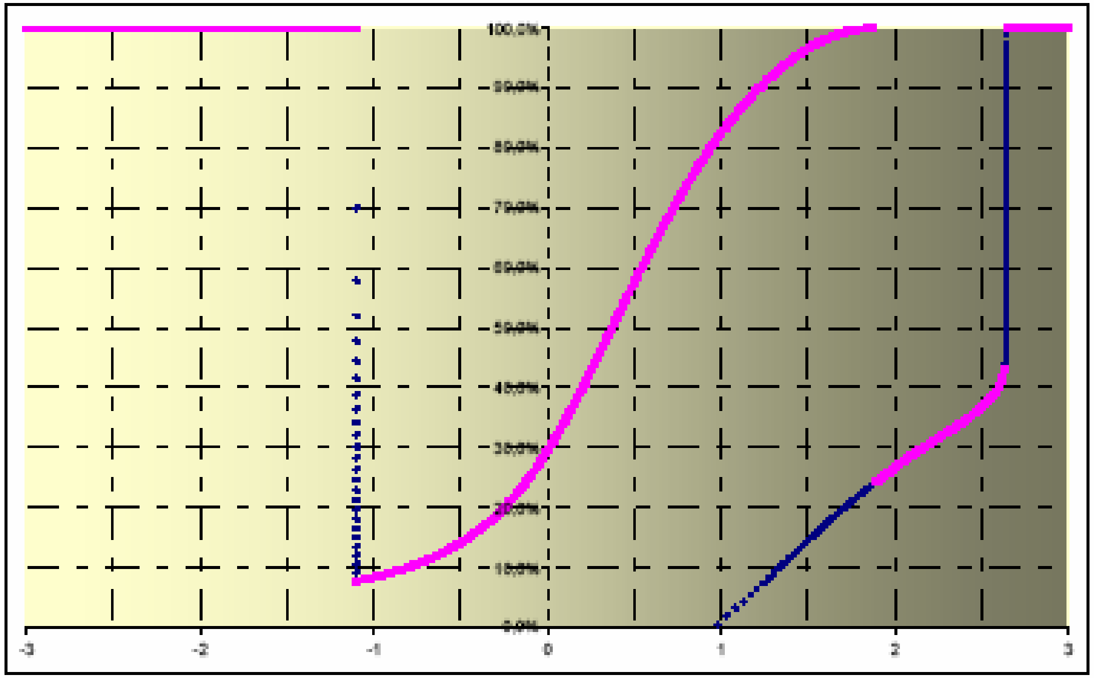
- Local correlation associated with RFL (as a function of the factor)



  - *Jump at threshold 2, low correlation level 5%, high correlation level 85%*
  - *Possibly two local correlations*

4 Beyond the Gaussian copula

- Local correlation associated with stochastic correlation model
  - *With respect to factor V*



  - *Correlations of 1 for high-low values of V (comonotonic state)*
  - *Possibly two local correlations leading to the same prices*
  - *As for RFL, rather irregular pattern*

Conclusion

- Analysis of dependence through factor models
  - *Usefulness of stochastic orders*
  - *Correlation sensitivities, base correlations*
- Matching the correlation skew
  - *Conditional default probability distributions are the drivers*
- Beyond the Gaussian copula
  - *Stochastic, local & marginal compound correlation*
- Further work
  - *Matching term structure of correlation skews*
  - *Integrating factor copulas and intensity approaches*