Implementing a new powerful framework for enhanced pricing & risk management of a credit derivatives correlation book

Risk Management
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Accurately Valuing Basket Default Swaps and CDO's using Factor Models

- Analytical valuation of CDO tranches and basket default swaps
- Factors and conditional independence framework
- Choosing the right copula for default times
- Assessing the contributions of different names to the pricing
- Effective risk management of CDO's and basket default swaps

What are we looking for ?

- A framework where:
  - *One can easily deal with a large number of names,*
  - *Tackle with different time horizons,*
  - *Compute quickly and accurately:*
    - Basket credit derivatives premiums
    - CDO margins on different tranches
    - Deltas with respect to shifts in credit curves and correlation parameters
- Main technical assumption:
  - *Default times are independent conditionnally on a low dimensional factor*

Probabilistic Tools: Survival Functions

- $i = 1, \ldots, n$ names.
- $\tau_1, \ldots, \tau_n$ default times.
- Marginal distribution function: $F_i(t) = Q(\tau_i \leq t)$
- Marginal survival function: $S_i(t) = Q(\tau_i > t)$
  - *Given from CDS quotes.*
- Joint survival function:

$$S(t_1, \ldots, t_n) = Q(\tau_1 > t_1, \ldots, \tau_n > t_n)$$

Probabilistic Tools: Factor Copulas

- Factor approaches to joint distributions:
  - *V: low dimensional factor, not observed « latent factor ».*
  - *Conditionally on V, default times are independent.*
  - *Conditional default probabilities:*

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right), \quad q_t^{i|V} = Q\left(\tau_i > t \mid V\right).$$

  - *Conditional joint distribution:*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid V) = \prod_{1 \leq i \leq n} p_{t_i}^{i|V}$$

  - *Joint survival function (implies integration wrt V):*

$$Q(\tau_1 > t_1, \ldots, \tau_n > t_n) = E\left[\prod_{i=1}^{n} q_{t_i}^{i|V}\right]$$

Probabilistic Tools: Gaussian Copulas

- One factor Gaussian copula:
  - $V, \bar{V}_i, \; i = 1, \ldots, n$ *independent Gaussian,*

    $$V_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
  - *Conditional default probabilities:* $p_t^{i|V} = \Phi\left(\dfrac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$
- Joint survival function:

  $$S(t_1, \ldots, t_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\rho_i v - \Phi^{-1}(F_i(t_i))}{\sqrt{1-\rho_i^2}}\right)\right) \varphi(v)dv$$

- Can be extended to Student *t* copulas (two factors).

Probabilistic Tools : Clayton copula

- *Schönbucher & Schubert.*
- Conditional default probabilities:

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

- *V: Gamma distribution with parameter* θ

- Joint survival function:

$$S(t_1, \ldots, t_n) = \int \prod_{i=1}^{n} \left(1 - p_{t_i}^{i|V}\right) \frac{1}{\Gamma(1/\theta)} e^{-V} V^{(1-\theta)/\theta} dV$$

Probabilistic Tools: Simultaneous Defaults

- *Duffie & Singleton, Wong*
- Modelling of defaut dates: $\tau_i = \min(\bar{\tau}_i, \tau)$
  - $Q(\tau_i = \tau_j) \geq Q\left(\tau \leq \min(\bar{\tau}_i, \bar{\tau}_j)\right) > 0$ *simultaneous defaults.*
  - *Conditionally on* τ, τ_i *are independent.*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid \tau) = \prod_{1 \leq i \leq n} Q(\tau_i \leq t_i \mid \tau)$$

- Conditional default probabilities:

$$p_t^{i|\tau} = 1_{\tau > t} Q(\bar{\tau}_i \leq t) + 1_{\tau \leq t}$$

Probabilistic Tools: Affine Jump Diffusion

- *Duffie, Pan & Singleton ;Duffie & Garleanu.*
- $n+1$ *independent affine jump diffusion processes:*

$$X_1, \ldots, X_n, X_c$$

- Conditional default probabilities:

$$Q(\tau_i > t \mid V) = q_t^{i|V} = V\alpha_i(t)$$

$$V = \exp\left(-\int_0^t X_c(s)ds\right), \quad \alpha_i(t) = E\left[\exp\left(-\int_0^t X_i(s)ds\right)\right].$$

- Survival function:

$$Q(\tau_1 > t, \ldots, \tau_n > t) = E\left[V^n\right] \times \prod_{i=1}^n \alpha_i(t).$$

- *Explicitely known.*

Pricing of Basket credit derivatives

- First to default time $\tau^1 = \min(\tau_1, \dots, \tau_n)$
- First to default swap:
  - *Credit protection at first to default time*
- Survival function of first to default time

$$Q(\tau^1 > t) = Q(\tau_1 > t, \dots, \tau_n > t) = E\left[\prod_{i=1}^{n} q_t^{i|V}\right]$$

- Semi-analytical expressions of:
  - *First to default, second to default, … last to default swap premiums*
  - *Paper « basket defaults swaps, CDO's and Factor Copulas » available on www.defaultrisk.com*
  - *« I will survive », technical paper, RISK magazine, june 2003*

Pricing of Basket Defaut Swaps: model dependence

- First to default swap premium vs number of names
  - *From n=1 to n=50 names*
  - *Unit nominal*
  - *Credit spreads = 80 bp*
  - *Recovery rates = 40 %*
  - *Maturity = 5 years*
  - *Basket premiums in bp*
- Comparison between Gaussian, Clayton and Marshall-Olkin copulas:
  - *Gaussian correlation parameter= 30%*

names	Gaussian	Clayton	MO
1	80	80	80
5	332	336	244
10	567	574	448
15	756	762	652
20	917	921	856
25	1060	1060	1060
30	1189	1183	1264
35	1307	1294	1468
40	1417	1397	1672
45	1521	1492	1875
50	1618	1580	2079
kendall	19%	8%	33%

Pricing of Basket Defaut Swaps: model dependence

- From first to last to default swap premiums
  - *10 names, unit nominal*
  - *Spreads of names uniformly distributed between 60 and 150 bp*
  - *Recovery rate = 40%*
  - *Maturity = 5 years*
  - *Gaussian correlation: 30%*
- Same FTD premiums imply consistent prices for protection at all ranks
- Model with simultaneous defaults provides very different results

Rank	Clayton	Gaussian
1	723	723
2	277	274
3	122	123
4	55	56
5	24	25
6	10	11
7	3.6	4.3
8	1.2	1.5
9	0.28	0.39
10	0.04	0.06
kendall	9%	19%

Risk Management of Basket Credit Derivatives

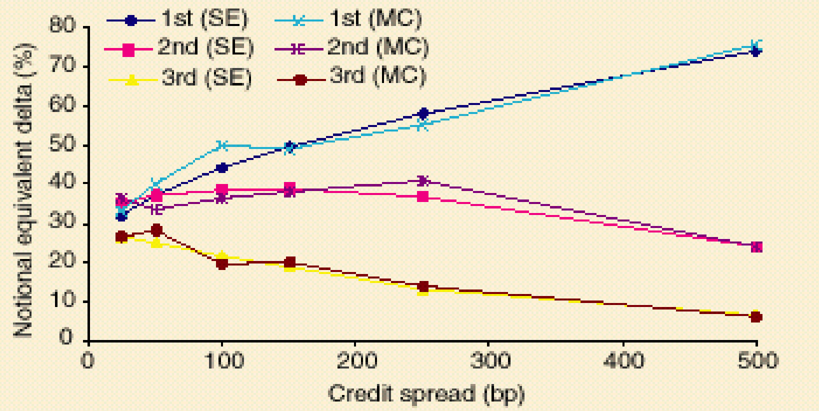
- Example: six names portfolio
- Changes in credit curves of individual names
- Amount of individual CDS to hedge the basket
- Semi-analytical more accurate than 10^5 Monte Carlo simulations.
- Much quicker: about 25 Monte Carlo simulations.

A. Comparison of the semi-explicit formulas with Monte Carlo simulations

	First to default		Second to default		Third to default	
	SE	MC	SE	MC	SE	MC
0%	1,075.1	1,075.9	214.8	214.7	28.2	27.7
20%	927.0	925.9	247.2	247.5	61.4	61.8
30%	859.9	857.9	256.8	257.6	77.6	78.0
40%	796.6	795.2	263.3	264.2	92.7	93.0
60%	679.6	678.0	268.8	268.9	119.5	119.8
80%	573.1	571.7	266.2	266.1	141.0	140.9
100%	500.0	500.0	250.0	250.0	150.0	150.0

Premiums in basis points per annum as a function of correlation for a five-year maturity basket with credit spreads of 25, 50, 100, 150, 250 and 500bp and equal recovery rates of 40%

1. Deltas calculated using semi-explicit formulas and Monte Carlo approaches

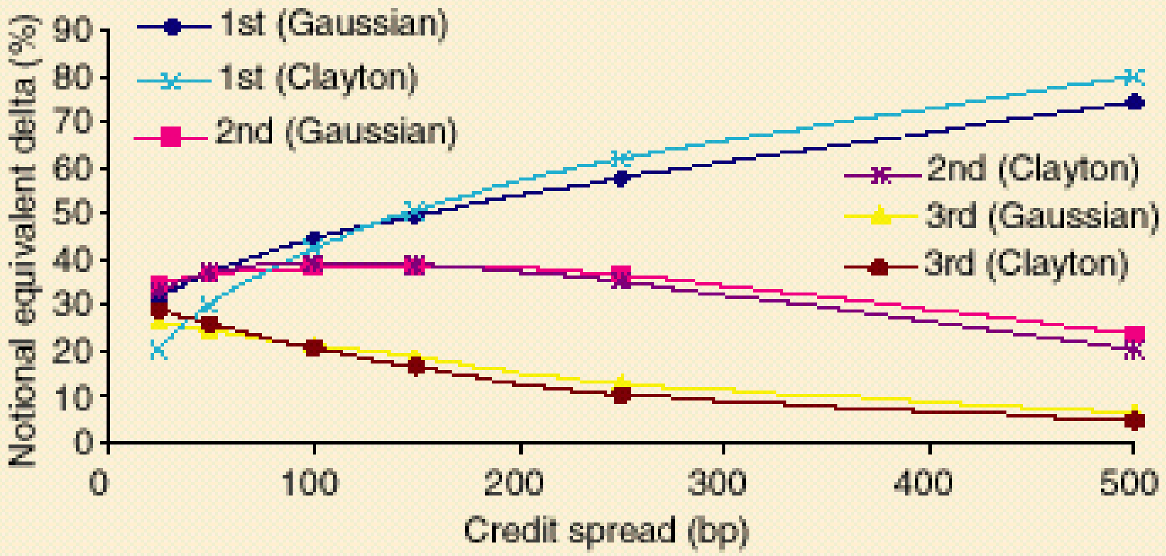


Comparison of deltas calculated using the analytical formulas and 105 Monte Carlo simulations for the example given in table A. The Monte Carlo deltas are calculated by applying a 10bp parallel shift to each curve

Risk Management of Basket Credit Derivatives

- Changes in credit curves of individual names
  - *Dependence upon the choice of copula for defaults*

2. Deltas using Gaussian and Clayton copula



Comparison of deltas calculated using Gaussian (30% correlation) and Clayton copulas ($\theta = 0.27$)

CDO Tranches

«Everything should be made as simple as possible, not simpler»

- *Use of loss distributions over different time horizons*
- *Computation of loss distributions from FFT*
- *Explicit margin computations for tranches*
- *Model risk and margin computations*
- *Risk management: explicit greeks*

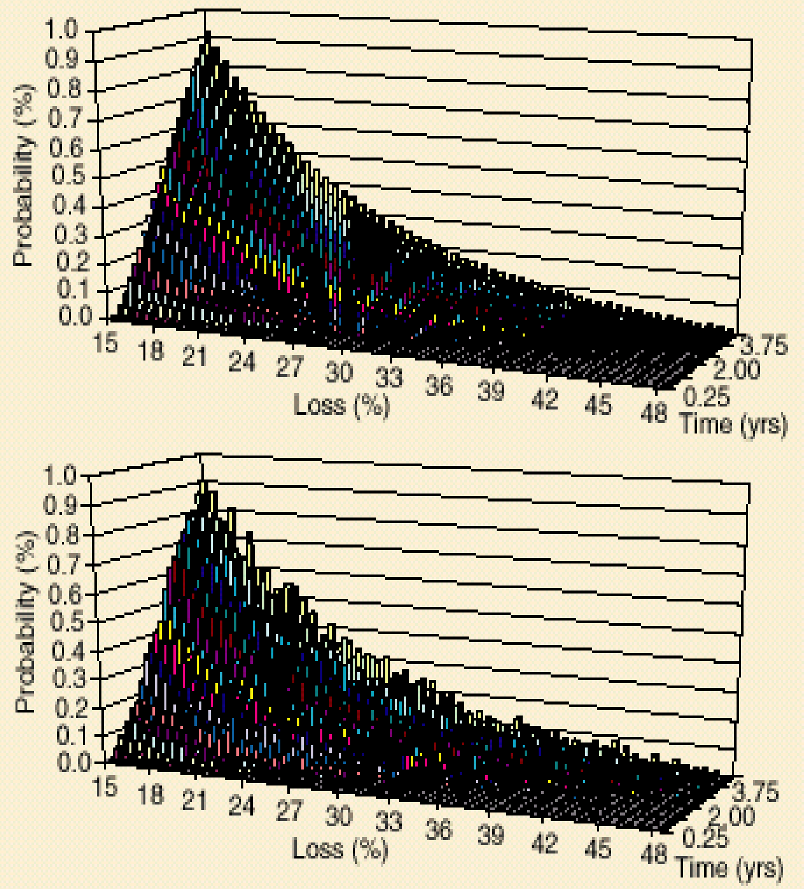
Credit Loss Distributions

- Accumulated loss at t: $L(t) = \sum_{1 \le i \le n} N_i(1-\delta_i)N_i(t)$
    - Where $N_i(t) = 1_{\tau_i \le t},\ N_i(1-\delta_i)$ loss given default.
- Characteristic function: $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
- By conditioning: $\varphi_{L(t)}(u) = E\left[\prod_{1 \le j \le n}\left(1 - p_t^{j|V} + p_t^{j|V}\varphi_{1-\delta_j}(uN_j)\right)\right]$
- Distribution of $L(t)$ is obtained by FFT.

Credit Loss Distributions

- One hundred names, same nominal.
- Recovery rates: 40%
- Credit spreads uniformly distributed between 60 and 250 bp.
- Gaussian copula, correlation: 50%
- 10^5 Monte Carlo simulations

3. Loss distribution



Loss distribution over time for the table B example with 50% correlation for the semi-explicit approach (top) and Monte Carlo simulation (bottom)

Valuation of Synthetic CDO's

- Tranches with thresholds $0 \leq A \leq B \leq \sum N_j$
  - *Mezzanine: pays whenever losses are between A and B*
- Cumulated payments at time *t on mezzanine*

$$M(t) = (L(t) - A))\, 1_{[A,B]}(L(t)) + (B - A) 1_{]B,\infty[}(L(t))$$

- Explicit margin computations of different tranches
  - *Taking into account discounting effects*
  - *Accrued premiums*
- Contribution of names to the PV of the default leg
  - *See « basket defaults swaps, CDO's and Factor Copulas »*

Valuation of Synthetic CDO's

B. Pricing of five-year maturity CDO tranches

	Equity (0–3%)		Mezzanine (3–14%)		Senior (14–100%)	
	SE	MC	SE	MC	SE	MC
0%	8,219.4	8,228.5	816.2	814.3	0.0	0.0
20%	4,321.1	4,325.3	809.4	806.9	13.7	13.7
40%	2,698.8	2,696.7	734.3	731.4	33.4	33.2
60%	1,750.6	1,738.5	641.0	637.8	54.1	53.7
80%	1,077.5	1,067.9	529.5	526.9	77.0	76.6
100%	410.3	406.6	371.2	367.0	110.4	109.6

Premiums in basis points per annum as a function of correlation for 5-year maturity CDO tranches on a portfolio with credit spreads uniformly distributed between 60 and 250bp. The recovery rates are 40%

- *One factor Gaussian copula*
- *CDO tranches margins with respect to correlation parameter*

Valuation of Synthetic CDO's

- CDO margins (bp)
  - *Gaussian copula*
  - *Attachement points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads uniformaly distributed between 60 and 150 bp*
  - *5 years maturity*

ρ	equity	mezzanine	senior
0 %	6176	694	0.05
10 %	4046	758	5.8
30 %	2303	698	23
50 %	1489	583	40
70 %	933	470	56

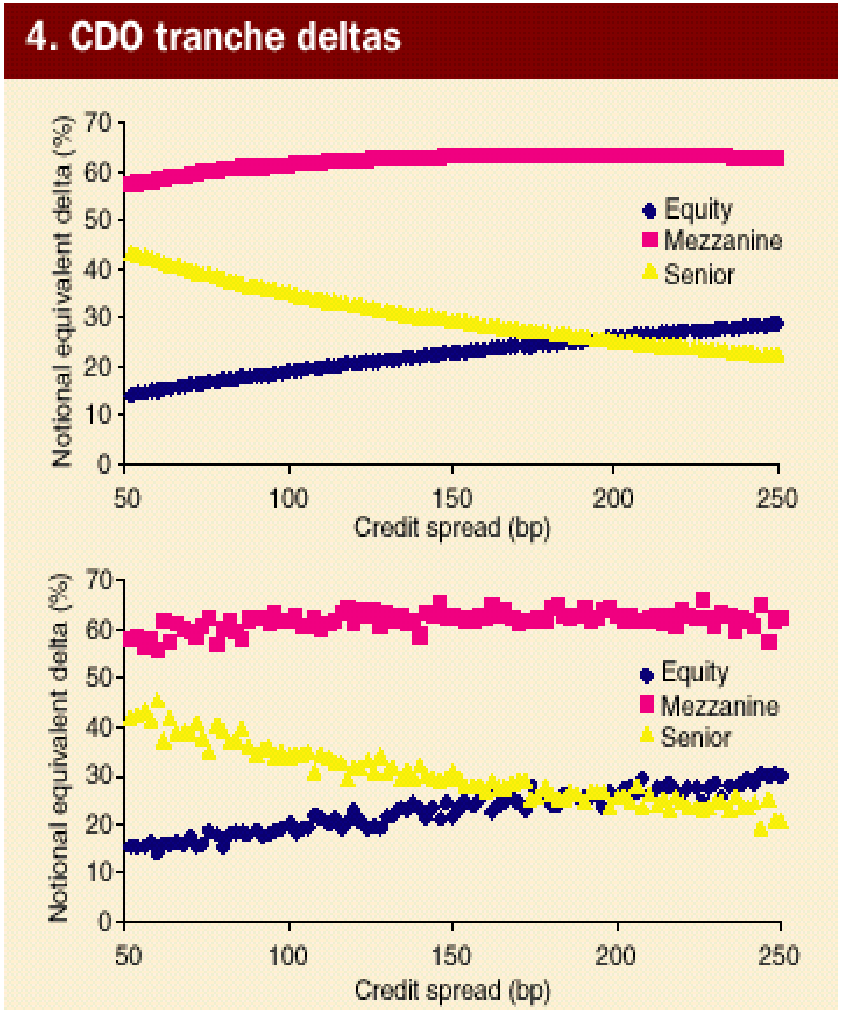
Valuation of Synthetic CDO's

- CDO margins (bp)
  - *Gaussian correlation = 10%*
  - *Parameters of Clayton and Marshall Olkin copulas are set for matching of equity tranches.*
- For the pricing of CDO tranches, the Clayton and Gaussian copula models are close.
- Very different results with Marshall-Olkin copula

	Gaussian	Clayton	MO
equity	4060	4060	4060
mezzanine	786	785	314
senior	6	5	30
kendall	6%	3%	not constant

Risk Management of CDO's

- Hedging of CDO tranches with respect to credit curves of individual names
- Amount of individual CDS to hedge the CDO tranche
- Semi-analytic : some seconds
- Monte Carlo more than one hour and still shaky



CDO tranche deltas using the analytical method (top) and Monte Carlo (bottom) for a correlation of 50%

Conclusion

- Factor models of default times:
  - *Deal easily with a large range of names and dependence structures*
  - *Simple computation of basket credit derivatives and CDO's*
    - Prices and risk parameters
  - *Parsimonious modelling*
  - *From deal to book risk management*
- Other dependence structures between default dates need to be investigated.