Credit Correlation Modelling
Comparative analysis of CDO pricing models

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A comparative analysis of CDO pricing models *available on* *www.defaultrisk.com*

Comparative analysis of CDO pricing models

- Purpose of the presentation
  - *Some insights about current issues in CDO modelling*
  - *Gaussian copula*
    - One factor Gaussian copula
    - Ordering of risks
    - Correlation sensitivities and Gaussian extensions
  - *Model dependence/Choice of copula*
    - Student *t*, double *t*, Clayton, Marshall-Olkin, Stochastic correlation
    - Calibration issues
    - Distribution of conditional default probabilities
  - *Matching the correlation skew*
  - *Further issues*

Comparative analysis of CDO pricing models

- Agenda
  - *Conditional default probabilities and pricing of CDOs*
  - *One factor Gaussian copula*
    - Dependence to the correlation parameter
    - Gaussian extensions, correlation sensitivities
  - *Model dependence/Choice of copula*
    - Student *t*, double *t*, Clayton, Marshall-Olkin, Stochastic correlation
    - Calibration
    - Empirical results
  - *Matching the correlation skew*

Semi explicit pricing, conditional default probabilities

- Factor approaches to joint default times distributions:
  - *V: low dimensional factor*
  - *Conditionally on V, default times are independent.*
  - *Conditional default and survival probabilities:*

$$p_t^{i|V} = Q(\tau_i \le t \mid V), \quad q_t^{i|V} = Q(\tau_i > t \mid V).$$

- Why factor models ?
  - *Tackle with large dimensions (125 names in I-TRAXX)*
- Need tractable dependence between defaults:
  - *Parsimonious modelling*
  - *Semi-explicit computations for CDO tranches*

Semi explicit pricing, conditional default probabilities

- Semi-explicit pricing for CDO tranches
  - Laurent & Gregory [2003]
  - *Default payments are based on the* accumulated losses *on the pool of credits:*

$$L(t) = \sum_{i=1}^{n} LGD_i 1_{\{\tau_i \le t\}}, \quad LGD_i = N_i(1-\delta_i)$$

  - *Tranche premiums only involves call options on the accumulated losses*

$$E\left[\left(L(t)-K\right)^{+}\right]$$

Semi explicit pricing, conditional default probabilities

- Characteristic function: $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
  - *By conditioning upon V and using conditional independence:*

$$\varphi_{L(t)}(u) = E\left[\prod_{1\le j\le n}\left(1 - p_t^{j|V} + p_t^{j|V}\varphi_{1-\delta_j}(uN_j)\right)\right]$$

  - *Distribution of L(t) can be obtained by FFT*
    - Or other inversion technique
- Only need of conditional (on factor) probabilities $p_t^{i|V}$

Semi explicit pricing, conditional default probabilities

- One factor Gaussian copula:
  - $V, \bar{V}_i, \; i = 1, \ldots, n$ *independent Gaussian,*

$$V_i = \rho_i V + \sqrt{1-\rho_i^2}\,\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
  - F_i *marginal distribution function of default times*
  - *Conditional default probabilities:*

$$p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$$

One factor Gaussian copula

- CDO margins (bps pa)
  - *With respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads 100 bp*
  - *5 years maturity*

	equity	mezzanine	senior
0%	5341	560	0.03
10%	3779	632	4.6
30%	2298	612	20
50%	1491	539	36
70%	937	443	52
100%	167	167	91

One factor Gaussian copula

- Equity tranche premiums are decreasing wrt ρ
  - *General result ?*
  - *Supermodular function f is such that:*

$$f : \mathbb{R}^n \to \mathbb{R} \qquad \Delta_i^{\varepsilon} f(x) = f(x + \varepsilon e_i) - f(x)$$

$$\forall x \in \mathbb{R}^n, \forall \varepsilon, \delta > 0 \qquad \Delta_i^{\varepsilon}\Delta_j^{\delta} f(x) \geq 0$$

  - *Supermodular order (increase in dependence)*

$$X = (X_1, \ldots, X_n) \qquad Y = (Y_1, \ldots, Y_n)$$

$$X \leq_{sm} Y \;\Leftrightarrow\; E\left[f(X)\right] \leq E\left[f(Y)\right], \; \forall f \;\; \text{supermodular}$$

One factor Gaussian copula

- « Supermodular » order of Gaussian vectors
  - *Let X and Y be Gaussian vectors with zero mean*

$$\Sigma^X = \begin{pmatrix} 1 & \sigma_{12}^X & & & & \\ \sigma_{21}^X & 1 & & & & \\ & & 1 & & \sigma_{ij}^X & \\ & & & 1 & & \\ & & \sigma_{ji}^X & & 1 & \\ & & & & & 1 \end{pmatrix} \qquad \Sigma^Y = \begin{pmatrix} 1 & \sigma_{12}^Y & & & & \\ \sigma_{21}^Y & 1 & & & & \\ & & 1 & & \sigma_{ij}^Y & \\ & & & 1 & & \\ & & \sigma_{ji}^Y & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$$\Sigma^X \leq \Sigma^Y \overset{\text{def}}{\Leftrightarrow} \sigma_{ij}^X \leq \sigma_{ij}^Y, \;\; \forall i, j$$

  - *Müller & Scarsini (2000), Müller (2001)*

$$\Sigma^X \leq \Sigma^Y \Leftrightarrow X \leq_{sm} Y$$

One factor Gaussian copula

- « Stop-Loss » order
  - *Accumulated losses:* $L(t), L'(t)$

$$L(t) \leq_{sl} L'(t) \overset{\text{def}}{\Leftrightarrow} E\left[\left(L(t)-K\right)^+\right] \leq E\left[\left(L'(t)-K\right)^+\right], \forall K \geq 0$$

  - *Supermodular order of latent variables implies stop-loss order of accumulated losses*
  - *Thus,equity tranche premium is* <u>*always*</u> *decreasing with correlation*
  - *Guarantees uniqueness of « base correlation »*
  - *Monotonicity properties extend to Student t, Clayton (Wei & Hu* [2002]*) and Marshall-Olkin copulas*

One factor Gaussian copula

- Second issue
  - *Equity tranche premium decrease with correlation*
  - *Does* $\rho = 100\%$ *correspond to some lower bound?*
  - $\rho = 100\%$ *corresponds to « comonotonic » default dates:*

$$(\tau_1, \ldots, \tau_n) \text{ comonotonic } \Leftrightarrow (\tau_1, \ldots, \tau_n) \overset{d}{=} \left(F_1^{-1}(U), \ldots, F_n^{-1}(U)\right)$$

    - where U is uniform

$$\left(\tau_1, \ldots, \tau_n\right) \leq_{sm} \left(F_1^{-1}(U), \ldots, F_n^{-1}(U)\right)$$

    - Tchen (1980)
  - $\rho = 100\%$ *is a* <u>*model free*</u> *lower bound for the equity tranche premium*

One factor Gaussian copula

- Third issue
  - *Does $\rho = 0\%$ corresponds to the higher bound on the equity tranche premium?*
  - *$\rho = 0\%$ corresponds to the independence case between default dates*
  - *The answer is no, negative dependence can occur*
  - *Base correlation does not always exists*
    - Even in Gaussian copula models
  - *Factor models are usually associated with positive dependence*
    - i.e. independent default dates are smaller with respect to supermodular order

One factor Gaussian copula

- Gaussian extensions
  - *Pairwise correlation sensitivities*

$$\begin{pmatrix} 1 & \rho_{12} & & & & & & \\ \rho_{21} & 1 & & & & & & \\ & & 1 & & \rho_{ij}+\delta & & & \\ & & & . & & & & \\ & & \rho_{ij}+\delta & & . & & & \\ & & & & & 1 & & \\ & & & & & & 1 & . \\ & & & & & & . & 1 \end{pmatrix}$$

  - *Intra and intersector correlations*

$$\begin{pmatrix} 1 & \beta_1 & \beta_1 & & & & & & \\ \beta_1 & 1 & \beta_1 & & & & & \gamma & \\ \beta_1 & \beta_1 & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & . & & & & \\ & & & & & . & & & \\ & & & & & & 1 & & \\ & & & & & & 1 & \beta_m & \beta_m \\ & \gamma & & & & & \beta_m & 1 & \beta_m \\ & & & & & & \beta_m & \beta_m & 1 \end{pmatrix}$$

  - *In the core of correlation, Gregory & Laurent, Risk october 2004*

Model dependence / choice of copula

- Stochastic corrrelation copula
  - *$V, \bar{V}_i,\ i = 1, \dots, n$ independent Gaussian variables*
  - *$B_i = 1$ correlation ρ, $B_i = 0$ correlation β*

$$V_i = B_i\left(\rho V + \sqrt{1-\rho^2}\bar{V}_i\right) + (1-B_i)\left(\beta V + \sqrt{1-\beta^2}\bar{V}_i\right)$$

$$V_i = \left(B_i\rho + (1-B_i)\beta\right)V + \sqrt{1-\left(B_i\rho + (1-B_i)\beta\right)^2}\,\bar{V}_i$$

$$\tau_i = F_i^{-1}\left(\Phi(V_i)\right)$$

$$p_t^{i|V} = p\Phi\left(\frac{-\rho V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho^2}}\right) + (1-p)\Phi\left(\frac{-\beta V + \Phi^{-1}(F_i(t))}{\sqrt{1-\beta^2}}\right)$$

Model dependence / choice of copula

- Student *t* copula
  - *Embrechts, Lindskog & McNeil, Greenberg et al, Mashal et al, O'Kane & Schloegl, Gilkes & Jobst*

$$\begin{cases} X_i = \rho V + \sqrt{1-\rho^2}\bar{V}_i \\ V_i = \sqrt{W} \times X_i \\ \tau_i = F_i^{-1}\left(t_\nu(V_i)\right) \end{cases}$$

  - *$V, \bar{V}_i$ independent Gaussian variables*
  - *$\frac{\nu}{W}$ follows a χ_ν^2 distribution*
  - *Conditional default probabilities (two factor model)*

$$p_t^{i|V,W} = \Phi\left(\frac{-\rho V + W^{-1/2}t_\nu^{-1}(F_i(t))}{\sqrt{1-\rho^2}}\right)$$

Model dependence / choice of copula

- *Clayton* copula
  - *Schönbucher & Schubert, Rogge & Schönbucher, Friend & Rogge, Madan et al*

$$V_i = \psi\left(-\frac{\ln U_i}{V}\right) \qquad \tau_i = F_i^{-1}(V_i) \qquad \psi(s) = (1+s)^{-1/\theta}$$

    - Marshall-Olkin construction of archimedean copulas
  - *V: Gamma distribution with parameter θ*
  - *$U_1, \dots, U_n$ independent uniform variables*
  - *Conditional default probabilities (one factor model)*

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

Model dependence / choice of copula

- Double *t* model (Hull & White)

$$V_i = \rho_i\left(\frac{\nu-2}{\nu}\right)^{1/2} V + \sqrt{1-\rho_i^2}\left(\frac{\bar{\nu}-2}{\bar{\nu}}\right)^{1/2}\bar{V}_i$$

  - *$V, \bar{V}_i$ are independent Student t variables*
    - with ν and $\bar{\nu}$ degrees of freedom

$$\tau_i = F_i^{-1}\left(H_i(V_i)\right)$$

  - *where H_i is the distribution function of V_i*

$$p_t^{i|V} = t_{\bar{\nu}}\left(\left(\frac{\bar{\nu}}{\bar{\nu}-2}\right)^{1/2}\frac{H_i^{-1}(F_i(t)) - \rho_i\left(\frac{\nu-2}{\nu}\right)^{1/2}V}{\sqrt{1-\rho_i^2}}\right)$$

Model dependence / choice of copula

- Shock models (multivariate exponential copulas)
  - *Duffie & Singleton, Giesecke, Elouerkhaoui, Lindskog & McNeil, Wong*
- Modelling of default dates: $V_i = \min\left(V, \bar{V}_i\right)$
  - $V, \bar{V}_i$ *exponential with parameters* $\alpha, 1-\alpha$
  - *Default dates* $\tau_i = S_i^{-1}\left(\exp - \min\left(V, \bar{V}_i\right)\right)$
    - S_i marginal survival function
  - *Conditionally on* V, τ_i *are independent.*
- Conditional default probabilities

$$q_t^{i|V} = 1_{V > -\ln S_i(t)} S_i(t)^{1-\alpha}$$

Model dependence / choice of copula

- Calibration issues
  - *One parameter copulas*
  - *Well suited for homogeneous portfolios*
  - *Dependence is « monotonic » in the parameter*
- Calibration procedure
  - *Fit Clayton, Student t, double t, Marshall Olkin parameters onto CDO equity tranches*
    - Computed under one factor Gaussian model
    - Or given market quotes on equity trances
  - *Reprice mezzanine and senior CDO tranches*
    - Given the previous parameters

Model dependence / choice of copula

- CDO margins (bps pa)
  - *With respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads 100 bp*
  - *5 years maturity*

	equity	mezzanine	senior
0%	5341	560	0.03
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Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
θ	0	0.05	0.18	0.36	0.66	∞
ρ_6^2			14%	39%	63%	100%
ρ_{12}^2			22%	45%	67%	100%
ρ t(4)-t(4)	0%	12%	34%	55%	73%	100%
ρ t(5)-t(4)	0%	13%	36%	56%	74%	100%
ρ t(4)-t(5)	0%	12%	34%	54%	73%	100%
ρ t(3)-t(4)	0%	10%	32%	53%	75%	100%
ρ t(4)-t(3)	0%	11%	33%	54%	73%	100%
α	0	28%	53%	69%	80%	100%

Table 5: correspondence between parameters

Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	560	633	612	539	443	167
Clayton	560	637	628	560	464	167
Student (6)			637	550	447	167
Student (12)			621	543	445	167
t(4)-*t*(4)	560	527	435	369	313	167
t(5)-*t*(4)	560	545	454	385	323	167
t(4)-*t*(5)	560	538	451	385	326	167
t(3)-*t*(4)	560	495	397	339	316	167
t(4)-*t*(3)	560	508	406	342	291	167
MO	560	284	144	125	134	167

Table 6: mezzanine tranche (bps pa)

Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0.03	4.6	20	36	52	91
Clayton	0.03	4.0	18	33	50	91
Student (6)			17	34	51	91
Student (12)			19	35	52	91
t(4)-*t*(4)	0.03	11	30	45	60	91
t(5)-*t*(4)	0.03	10	29	45	59	91
t(4)-*t*(5)	0.03	10	29	44	59	91
t(3)-*t*(4)	0.03	12	32	47	71	91
t(4)-*t*(3)	0.03	12	32	47	61	91
MO	0.03	25	49	62	73	91

Table 7: senior tranche (bps pa)

Gaussian, Clayton and Student *t* CDO premiums are close

Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0%	0%	0%	0%	0%	100%
Clayton	0%	0%	2%	15%	35%	100%
Student (6)			5%	12%	25%	100%
Student (12)			1%	4%	13%	100%
t(4)-*t*(4)	0%	0%	1%	10%	48%	100%
t(5)-*t*(4)	0%	0%	0%	0%	0%	100%
t(4)-*t*(5)	0%	100%	100%	100%	100%	100%
t(3)-*t*(4)	0%	100%	100%	100%	100%	100%
t(4)-*t*(3)	0%	0%	0%	0%	0%	100%
MO	0%	28%	53%	69%	80%	100%

Table 8: coefficient of lower tail dependence (%)

Tail dependence is poorly related to CDO tranche premiums

Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0%	1%	6%	16%	33%	100%
Clayton	0%	3%	8%	15%	25%	100%
Student (6)			9%	25%	44%	100%
Student (12)			14%	30%	47%	100%
MO	0%	16%	36%	53%	67%	100%

Table 9: Kendall's τ (%)

Kendall's tau is poorly related to CDO tranche premiums

Model dependence / choice of copula

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	0.66%	0.91%	1.54%	2.41%	3.59%	8.1%
Clayton	0.66%	0.88%	1.45%	2.24%	3.31%	8.1%
Student (6)			1.41%	2.31%	3.52%	8.1%
Student (12)			1.49%	2.36%	3.56%	8.1%
t(4)-*t*(4)	0.66%	1.22%	2.38%	3.49%	4.67%	8.1%
t(5)-*t*(4)	0.66%	1.16%	2.27%	3.38%	4.57%	8.1%
t(4)-*t*(5)	0.66%	1.18%	2.28%	3.37%	4.54%	8.1%
t(3)-*t*(4)	0.66%	1.34%	2.57%	3.69%	5.02%	8.1%
t(4)-*t*(3)	0.66%	1.31%	2.55%	3.70%	4.87%	8.1%
MO	0.66%	2.63%	4.53%	5.65%	6.53%	8.1%

Table 13: bivariate default probabilities (5 year time horizon)

Bivariate default probabilities are well related to tranche premiums

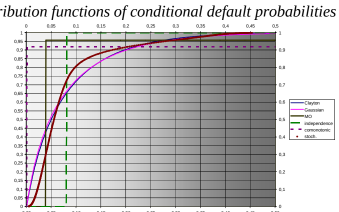
Model dependence / choice of copula

- Why Clayton and Gaussian copulas provide same SL premiums?
  - *Loss distributions only depend on the* ***distribution*** *of conditional default probabilities*

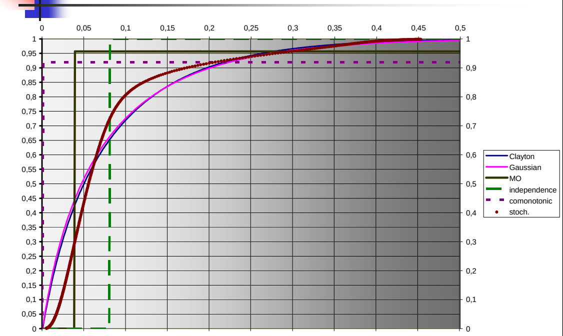
$$p_t^{i|V} = \exp\left(V\left(1-F_i(t)^{-\theta}\right)\right)$$

$$p_t^{i|V} = \Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right)$$

  - *Distribution functions of conditional default probabilities*



Distribution of conditional default probabilities



Matching the correlation skew

Tranches	Market	Gaussian	Clayton	Student (12)	*t*(4)-*t*(4)	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[3-6%]	101	163	163	164	82	122	14
[6-9%]	33	48	47	47	34	53	11
[9-12%]	16	17	16	15	22	29	11
[12-22%]	9	3	2	2	13	8	11

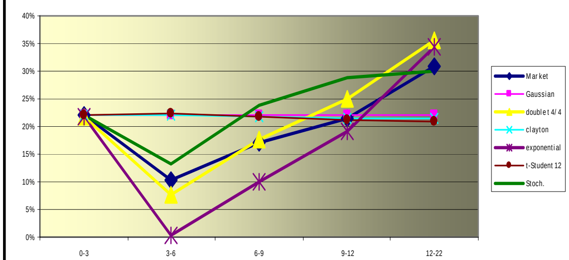
Table 17: CDO tranche premiums iTraxx (bps pa)

Tranches	Market	Gaussian	Clayton	Student (12)	*t*(4)-*t*(4)	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[0-6%]	466	503	504	504	456	479	418
[0-9%]	311	339	339	340	305	327	272
[0-12%]	233	253	253	254	230	248	203
[0-22%]	128	135	135	135	128	135	113

Table 18: "equity tranche" CDO tranche premiums iTraxx (bps pa)

Matching the correlation skew

implied compound correlation



Conclusion

- Matching the skew with second generation models
  - *RFL, stochastic correlation, double t*
  - *Conditional default probability distributions are the drivers*
- Pricing bespoke portfolios, CDO squared with a consistent model
- Not yet fully satisfactory
  - *Matching 5Y/10Y with same set of parameters*
  - *Stability of parameters through time*
  - *Dynamics of the correlation skew / risk management*
  - *Calibration of multiple parameters, possibly name dependent*
- Still more work on the quant agenda.