Hedging Default and Credit Spread Risks within CDOs

Global Derivatives Trading & Risk Management
Paris
23 May 2007

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Presentation related to papers
A note on the risk management of CDOs (2006)
Hedging default risks of CDOs in Markovian contagion models (2007)
Available on www.defaultrisk.com
Hedging Default and Credit Spread Risks within CDOs

• Bullet points
  - Hedging default and credit spread risks in contagion models
  - Dealing with simultaneous defaults
  - Hedging default and credit spread risks within intensity models
  - Parallel and idiosyncratic Gammas

• Purpose of the presentation
  - Not trying to embrace all risk management issues
  - Focus on very specific aspects of default and credit spread risk

• Overlook of the presentation
  - Economic background
  - Tree approach to hedging defaults
  - Hedging credit spread risks for large portfolios
I - Economic Background

• Hedging CDOs context
• About 1 000 papers on defaultrisk.com
• About 10 papers dedicated to hedging issues
  – In interest rate or equity markets, pricing is related to the cost of the hedge
  – In credit markets, pricing is disconnect from hedging
• Need to relate pricing and hedging
• What is the business model for CDOs?
• Risk management paradigms
  – Static hedging, risk-return arbitrage, complete markets
• Static hedging
• Buy a portfolio of credits, split it into tranches and sell the tranches to investors
  ➢ No correlation or model risk for market makers
  ➢ No need to dynamically hedge with CDS
• Only « budget constraint »:
  ➢ Sum of the tranche prices greater than portfolio of credits price
  ➢ Similar to stripping ideas for Treasury bonds
• No clear idea of relative value of tranches
  ➢ Depends of demand from investors
  ➢ Markets for tranches might be segmented
I - Economic Background

• Risk – return arbitrage

• Historical returns are related to ratings, factor exposure
  – CAPM, equilibrium models
  – In search of high alphas
  – Relative value deals, cross-selling along the capital structure

• Depends on the presence of « arbitrageurs »
  – Investors with small risk aversion
    ➢ Trading floors, hedge funds
  – Investors without too much accounting, regulatory, rating constraints
I - Economic Background

• The ultimate step: complete markets
  – As many risks as hedging instruments
  – News products are only designed to save transactions costs and are used for risk management purposes
  – Assumes a high liquidity of the market

• Perfect replication of payoffs by dynamically trading a small number of « underlying assets »
  – Black-Scholes type framework
  – Possibly some model risk

• This is further investigated in the presentation
  – Dynamic trading of CDS to replicate CDO tranche payoffs
I - Economic Background

- **Default risk**
  - Default bond price jumps to recovery value at default time.
  - Drives the CDO cash-flows

- **Credit spread risk**
  - Changes in defaultable bond prices prior to default
    - Due to shifts in credit quality or in risk premiums
  - Changes in the marked to market of tranches

- **Interactions between credit spread and default risks**
  - Increase of credit spreads increase the probability of future defaults
  - Arrival of defaults may lead to jump in credit spreads

  - Contagion effects (Jarrow & Yu)
I - Economic Background

• Credit deltas in copula models
• CDS hedge ratios are computed by bumping the marginal credit curves
  – Local sensitivity analysis
  – Focus on credit spread risk
  – Deltas are copula dependent
  – Hedge over short term horizons
    ➢ Poor understanding of gamma, theta, vega effects
    ➢ Does not lead to a replication of CDO tranche payoffs
• Last but not least: not a hedge against defaults…
• Credit deltas in copula models
  – Stochastic correlation model (Burstchell, Gregory & Laurent, 2007)
Main assumptions and results

- Credit spreads are driven by defaults
  - Contagion model
  - Credit spreads are deterministic between two defaults
- Homogeneous portfolio
  - Only need of the CDS index
  - No individual name effect
- Markovian dynamics
  - Pricing and hedging CDOs within a binomial tree
  - Easy computation of dynamic hedging strategies
  - Perfect replication of CDO tranches
We will start with two names only
Firstly in a static framework
- Look for a First to Default Swap
- Discuss historical and risk-neutral probabilities
Further extending the model to a dynamic framework
- Computation of prices and hedging strategies along the tree
- Pricing and hedging of tranchelets
Multiname case: homogeneous Markovian model
- Computation of risk-neutral tree for the loss
- Computation of dynamic deltas
Technical details can be found in the paper:
- “hedging default risks of CDOs in Markovian contagion models”
II - Tree approach to hedging defaults

• Some notations:
  – $\tau_1, \tau_2$ default times of counterparties 1 and 2,
  – $\mathcal{H}_t$ available information at time $t$,
  – $P$ historical probability,
  – $\alpha_1^P, \alpha_2^P$ : (historical) default intensities:
    $$P\left[\tau_i \in [t, t + dt] | \mathcal{H}_t \right] = \alpha_i^P dt, \ i = 1, 2$$

• Assumption of « local » independence between default events
  – Probability of 1 and 2 defaulting altogether:
    $$P\left[\tau_1 \in [t, t + dt], \tau_2 \in [t, t + dt] | \mathcal{H}_t \right] = \alpha_1^P dt \times \alpha_2^P dt \text{ in } (dt)^2$$
  – Local independence: simultaneous joint defaults can be neglected
II - Tree approach to hedging defaults

- Building up a tree:
  - Four possible states: \((D,D), (D,ND), (ND,D), (ND,ND)\)
  - Under no simultaneous defaults assumption \(p_{(D,D)} = 0\)
  - Only three possible states: \((D,ND), (ND,D), (ND,ND)\)
  - Identifying (historical) tree probabilities:

\[
\begin{align*}
\alpha_1^P dt & \quad (D,ND) \\
\alpha_2^P dt & \quad (ND,D) \\
1-(\alpha_1^P + \alpha_2^P) dt & \quad (ND,ND)
\end{align*}
\]

\[
\begin{align*}
p_{(D,D)} = 0 & \Rightarrow p_{(D,ND)} = p_{(D,D)} + p_{(D,ND)} = p_{(D,D)} = \alpha_1^P dt \\
p_{(D,D)} = 0 & \Rightarrow p_{(ND,D)} = p_{(D,D)} + p_{(ND,D)} = p_{(ND,D)} = \alpha_2^P dt \\
p_{(ND,ND)} = 0 & \Rightarrow 1 - p_{(D,D)} - p_{(D,D)} = \alpha_2^P dt
\end{align*}
\]
II - Tree approach to hedging defaults

- Stylized cash flows of short term digital CDS on counterparty 1:
  - $\alpha_1^O \, dt$ CDS 1 premium
  - $\alpha_1^P \, dt$
  - $1 - \alpha_1^O \, dt$ (D, ND)
  - $\alpha_2^P \, dt$
  - $-\alpha_1^O \, dt$ (ND, D)
  - $1 - (\alpha_1^P + \alpha_2^P) \, dt$
  - $-\alpha_1^O \, dt$ (ND, ND)

- Stylized cash flows of short term digital CDS on counterparty 2:
  - $\alpha_1^P \, dt$
  - $-\alpha_2^O \, dt$ (D, ND)
  - $\alpha_2^P \, dt$
  - $1 - \alpha_2^O \, dt$ (ND, D)
  - $1 - (\alpha_1^P + \alpha_2^P) \, dt$
  - $-\alpha_2^O \, dt$ (ND, ND)
II - Tree approach to hedging defaults

- Cash flows of short term digital first to default swap with premium $\alpha_F^0 dt$:

\[
\begin{align*}
0 & \xleftarrow{\alpha_2^P dt} 1 - \alpha_1^0 dt & (D, ND) \\
0 & \xleftarrow{\alpha_2^P dt} 1 - \alpha_F^0 dt & (ND, D) \\
1 - \left(\alpha_1^P + \alpha_2^P\right) dt & \xleftarrow{\alpha_1^P dt} 1 - \left(\alpha_1^0 + \alpha_2^0\right) dt & (D, ND) \\
1 - \left(\alpha_1^P + \alpha_2^P\right) dt & \xleftarrow{\alpha_2^P dt} \left(\alpha_1^0 + \alpha_2^0\right) dt & (ND, D) \\
1 - \left(\alpha_1^P + \alpha_2^P\right) dt & \xleftarrow{1 - \alpha_1^P dt} \left(-\alpha_1^0 + \alpha_2^0\right) dt & (ND, ND)
\end{align*}
\]

- Cash flows of holding CDS 1 + CDS 2:

\[
\begin{align*}
0 & \xleftarrow{\alpha_2^P dt} 1 - \left(\alpha_1^0 + \alpha_2^0\right) dt & (D, ND) \\
0 & \xleftarrow{\alpha_2^P dt} 1 - \left(\alpha_1^0 + \alpha_2^0\right) dt & (ND, D) \\
1 - \left(\alpha_1^P + \alpha_2^P\right) dt & \xleftarrow{1 - \alpha_1^P dt} \left(-\alpha_1^0 + \alpha_2^0\right) dt & (ND, ND)
\end{align*}
\]

- Perfect hedge of first to default swap by holding 1 CDS 1 + 1 CDS 2
  - Delta with respect to CDS 1 = 1, delta with respect to CDS 2 = 1
II - Tree approach to hedging defaults

- Absence of arbitrage opportunities imply:
  \[ \alpha_F^O = \alpha_1^O + \alpha_2^O \]

- Arbitrage free first to default swap premium
  - Does not depend on historical probabilities \( \alpha_1^P, \alpha_2^P \)

- Three possible states: \((D,ND), (ND,D), (ND,ND)\)

- Three tradable assets: CDS1, CDS2, risk-free asset

- For simplicity, let us assume \( r = 0 \)
Three state contingent claims

- Example: claim contingent on state \((D, ND)\)
- Can be replicated by holding
- \(1\) CDS \(1 + \alpha_0^d dt\) risk-free asset

Replication price = \(\alpha_0^d dt\)
Similarly, the replication prices of the \((ND, D)\) and \((ND, ND)\) claims

$$\alpha_1^P dt \quad 0 \quad (D, ND)$$

$$\alpha_2^O dt \quad \alpha_2^P dt \quad 1 \quad (ND, D)$$

$$1 - (\alpha_1^P + \alpha_2^P) dt \quad 0 \quad (ND, ND)$$

Replication price of:

$$\alpha_1^P dt \quad a \quad (D, ND)$$

$$\alpha_2^O dt \quad \alpha_2^P dt \quad b \quad (ND, D)$$

$$1 - (\alpha_1^P + \alpha_2^P) dt \quad c \quad (ND, ND)$$

Replication price = \(\alpha_1^O dt \times a + \alpha_2^O dt \times b + \left(1 - (\alpha_1^O + \alpha_2^O) dt\right) c\)
II - Tree approach to hedging defaults

- Replication price obtained by computing the expected payoff
  - Along a risk-neutral tree

\[ \alpha_1^o \, dt \times a + \alpha_2^o \, dt \times b + \left(1 - \left(\alpha_1^o + \alpha_2^o\right) dt\right) c \]

- Risk-neutral probabilities
  - Used for computing replication prices
  - Uniquely determined from short term CDS premiums
  - No need of historical default probabilities
• Computation of deltas
  – Delta with respect to CDS 1: \( \delta_1 \)
  – Delta with respect to CDS 2: \( \delta_2 \)
  – Delta with respect to risk-free asset: \( p \)

\[ p \text{ also equal to up-front premium} \]

\[
\begin{align*}
    a &= p + \delta_1 \times (1 - \alpha_1^0 \, dt) + \delta_2 \times (-\alpha_2^0 \, dt) \\
    b &= p + \delta_1 \times (-\alpha_1^0 \, dt) + \delta_2 \times (1 - \alpha_2^0 \, dt) \\
    c &= p + \delta_1 \times (-\alpha_1^0 \, dt) + \delta_2 \times (-\alpha_2^0 \, dt)
\end{align*}
\]

– As for the replication price, deltas only depend upon CDS premiums
II - Tree approach to hedging defaults

- **Dynamic case:**
  - $\alpha_1^o \ dt$ CDS 2 premium after default of name 1
  - $\alpha_2^o \ dt$ CDS 1 premium after default of name 2
  - $\lambda_2^o \ dt$ CDS 1 premium if no name defaults at period 1
  - $\kappa_1^o \ dt$ CDS 2 premium if no name defaults at period 1

- Change in CDS premiums due to contagion effects
  - Usually, $\pi_1^o < \alpha_1^o < \lambda_1^o$ and $\pi_2^o < \alpha_2^o < \lambda_2^o$
Computation of prices and hedging strategies by backward induction

- use of the dynamic risk-neutral tree
- Start from period 2, compute price at period 1 for the three possible nodes
- + hedge ratios in short term CDS 1,2 at period 1
- Compute price and hedge ratio in short term CDS 1,2 at time 0

Example to be detailed:
- computation of CDS 1 premium, maturity = 2
- $p_1 dt$ will denote the periodic premium
- Cash-flow along the nodes of the tree
II - Tree approach to hedging defaults

- Computations CDS on name 1, maturity = 2

  \[
  0 = (1 - p_1) \alpha_1^O + (\kappa_1^O - p_1(1 - \kappa_1^O)) \alpha_2^O \\
  + (1 - p_1) \pi_1^O - p_1(1 - \pi_1^O) \pi_2^O - p_1(1 - \pi_1^O - \pi_2^O)(1 - \alpha_1^O - \alpha_2^O)
  \]

- Premium of CDS on name 1, maturity = 2, time = 0, \( p_1 dt \) solves for:
II - Tree approach to hedging defaults

- Example: stylized zero coupon CDO tranchelets
  - Zero-recovery, maturity 2
  - Aggregate loss at time 2 can be equal to 0,1,2
    - Equity type tranche contingent on no defaults
    - Mezzanine type tranche : one default
    - Senior type tranche : two defaults

\[
\begin{align*}
\alpha_1^0 dt \times \kappa_2^0 dt &+ \alpha_2^0 dt \times \kappa_1^0 dt \\
\text{up-front premium default leg} &
\end{align*}
\]

\[
\begin{align*}
\alpha_1^0 dt &
\end{align*}
\]

\[
\begin{align*}
\alpha_2^0 dt &
\end{align*}
\]

\[
\begin{align*}
1 - (\alpha_1^0 + \alpha_2^0) dt &
\end{align*}
\]

\[
\begin{align*}
\lambda_2^0 dt & 1 \ (D,D) \\
1 - \lambda_2^0 dt & 0 \ (D,ND) \\
\kappa_1^0 dt & 1 \ (D,D) \\
1 - \kappa_1^0 dt & 0 \ (ND,D) \\
\pi_1^0 dt & 0 \ (D,ND) \\
\pi_2^0 dt & 0 \ (ND,D) \\
1 - (\pi_1^0 + \pi_2^0) dt & 0 \ (ND,ND)
\end{align*}
\]
II - Tree approach to hedging defaults

- mezzanine tranche
  - Time pattern of default payments
    - \[ \alpha_1^o \, dt + \alpha_2^o \, dt \]
    - \[ + \left( 1 - \left( \alpha_1^o + \alpha_2^o \right) \, dt \right) \left( \pi_1^o + \pi_2^o \right) \, dt \]
      - up-front premium default leg
      - \[ 1 - \left( \alpha_1^o + \alpha_2^o \right) \, dt \]
      - \[ 1 - \left( \pi_1^o + \pi_2^o \right) \, dt \]

- Possibility of taking into account discounting effects
- The timing of premium payments
- Computation of dynamic deltas with respect to short or actual CDS on names 1,2
• In theory, one could also derive dynamic hedging strategies for index CDO tranches
  – Numerical issues: large dimensional, non recombining trees
  – Homogeneous Markovian assumption is very convenient

➢ CDS premiums at a given time $t$ only depend upon the current number of defaults $N(t)$
  – CDS premium at time 0 (no defaults) $\alpha_1^0 dt = \alpha_2^0 dt = \alpha^0 (t = 0, N(0) = 0)$
  – CDS premium at time 1 (one default) $\lambda_2^0 dt = \kappa_1^0 dt = \alpha^0 (t = 1, N(t) = 1)$
  – CDS premium at time 1 (no defaults) $\pi_1^0 dt = \pi_2^0 dt = \alpha^0 (t = 1, N(t) = 0)$
II - Tree approach to hedging defaults

- Homogeneous Markovian tree

- If we have \( N(1) = 1 \), one default at \( t=1 \)
- The probability to have \( N(2) = 1 \), one default at \( t=2 \)…
- Is \( 1 - \alpha^Q(1,1) \) and does not depend on the defaulted name at \( t=1 \)
- \( N(t) \) is a Markov process
- Dynamics of the number of defaults can be expressed through a binomial tree
II - Tree approach to hedging defaults

- From name per name to number of defaults tree

\[
\begin{align*}
\alpha^Q (0,0) & \quad (D, ND) \\
\alpha^Q (0,0) & \quad (ND, D) \\
1 - 2\alpha^Q_i (0,0) & \quad (ND, ND)
\end{align*}
\]

\[
\begin{align*}
\alpha^Q (1,1) & \quad (D, D) \\
1 - \alpha^Q (1,1) & \quad (D, ND) \\
\alpha^Q (1,0) & \quad (D, ND) \\
1 - \alpha^Q (1,0) & \quad (ND, D) \\
1 - 2\alpha^Q (1,0) & \quad (ND, ND)
\end{align*}
\]

\[
\begin{align*}
n(0) = 0 & \quad N(1) = 1 \\
2\alpha^Q (0,0) & \quad N(2) = 2 \\
1 - 2\alpha^Q_i (0,0) & \quad N(1) = 0
\end{align*}
\]

\[
\begin{align*}
\alpha^Q (1,1) & \quad (D, D) \\
1 - \alpha^Q (1,1) & \quad (D, ND) \\
\alpha^Q (1,0) & \quad (D, ND) \\
1 - \alpha^Q (1,0) & \quad (ND, D) \\
1 - 2\alpha^Q (1,0) & \quad (ND, ND)
\end{align*}
\]

\[
\begin{align*}
n(2) = 1 \\
n(2) = 0
\end{align*}
\]

Number of defaults tree
Easy extension to $n$ names

- Predefault name intensity at time $t$ for $N(t)$ defaults: $\alpha^O(t, N(t))$
- Number of defaults intensity: sum of surviving name intensities:

$$
\lambda(t, N(t)) = (n - N(t)) \alpha^O(t, N(t))
$$

- $\alpha^O(0,0), \alpha^O(1,0), \alpha^O(1,1), \alpha^O(2,0), \alpha^O(2,1), \ldots$ can be easily calibrated
- on marginal distributions of $N(t)$ by forward induction.
II - Tree approach to hedging defaults

• Previous recombining binomial risk-neutral tree provides a framework for the valuation of payoffs depending upon the number of defaults
  – Applies to CDO tranches (homogeneous portfolio)
  – Applies to credit default swap index

• What about the credit deltas?
  – In a homogeneous framework, deltas with respect to CDS are all the same
  – Possibility of perfect dynamic replication of a CDO tranche with a credit default swap index and the default-free asset
  – Credit delta with respect to the credit default swap index
  – = change in PV of the tranche / change in PV of the CDS index
II - Tree approach to hedging defaults

- Example: number of defaults distribution at 5Y generated from a Gaussian copula
  - Correlation parameter: 30%
  - Number of names: 125
  - Default-free rate: 3%
  - 5Y credit spreads: 20 bps
  - Recovery rate: 40%

- Figure shows the corresponding expected losses for a 5Y horizon
II - Tree approach to hedging defaults

- Calibration of loss intensities
  - For simplicity, assumption of time homogeneous intensities
  - Figure below represents loss intensities, with respect to the number of defaults
  - Increase in intensities: contagion effects
II - Tree approach to hedging defaults

- Dynamics of the 5Y CDS index spread
  - In bp pa

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<th>14</th>
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Dynamics of credit deltas ([0,3%] equity tranche)
- With respect to the 5Y CDS index
- For selected time steps

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<th>OutStanding Nominal</th>
<th>Weeks</th>
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- Hedging strategy leads to a perfect replication of equity tranche payoff
- Deltas > 1
### II - Tree approach to hedging defaults

- Credit deltas default leg and premium leg (equity tranche)

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<td>0.003</td>
<td>0.003</td>
<td>0.003</td>
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### II - Tree approach to hedging defaults

- Dynamics of credit deltas ([3,6%] tranche)

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<td>0.014</td>
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### Dynamics of credit deltas ([6,9%] tranche)

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<tr>
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<td>3.00%</td>
<td>0</td>
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<tr>
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<td>0</td>
</tr>
<tr>
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</tr>
<tr>
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<tr>
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<tr>
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</tr>
<tr>
<td>11</td>
<td>3.00%</td>
<td>0</td>
</tr>
<tr>
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<tr>
<td>13</td>
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II - Tree approach to hedging defaults

- Small dependence of credit deltas with respect to recovery rate
  - Equity tranche, $R=30\%$

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</tr>
<tr>
<td>6</td>
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<td>0.000</td>
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- Equity tranche, $R=40\%$

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<td>2</td>
<td>2.04%</td>
<td>0.000</td>
</tr>
<tr>
<td>3</td>
<td>1.56%</td>
<td>0.000</td>
</tr>
<tr>
<td>4</td>
<td>1.08%</td>
<td>0.000</td>
</tr>
<tr>
<td>5</td>
<td>0.60%</td>
<td>0.000</td>
</tr>
<tr>
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</tr>
<tr>
<td>7</td>
<td>0.00%</td>
<td>0.000</td>
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Small dependence of credit deltas with respect to recovery rate

- Initial delta with respect to the credit default swap index
  - Only a small dependence of credit deltas with respect to recovery rates

First conclusion:

- Thanks to stringent assumptions
  - credit spreads driven by defaults + homogeneity + Markovian
  - It is possible to compute a dynamic hedging strategy
    - Based on the CDS index
  - That fully replicates the CDO tranche payoffs
When dealing with the risk management of CDOs, traders concentrate upon credit spread and correlation risk – Neglect default risk.

What about default risk?
- For large indices, default of one name has only a small direct effect on the aggregate loss.

Is it possible to build a framework where hedging default risk can be neglected?

And where one could only consider the hedging of credit spread risk?
- See paper “A Note on the risk management of CDOs”
Main and critical assumption
– Default times follow a multivariate Cox process
  ➢ For instance, affine intensities
  ➢ Duffie & Garleanu, Mortensen, Feldhütter, Merrill Lynch

2. the default times follow a multivariate Cox process:

\[
\tau_i = \inf \left\{ t \in \mathbb{R}^+; U_i \geq \exp \left( - \int_0^t \lambda_{i,u} du \right) \right\}, \quad i = 1, \ldots, n
\]

where \( \lambda_1, \ldots, \lambda_n \) are strictly positive, \( \mathcal{F} \) - progressively measurable processes, \( U_1, \ldots, U_n \) are independent random variables uniformly distributed on \([0,1]\) under \( Q \) and \( \mathcal{F} \) and \( \sigma(U_1, \ldots, U_n) \) are independent under \( Q \).

No contagion effects
III - Hedging credit spread risks for large portfolios

• No contagion effects
  – credit spreads drive defaults but defaults do not drive credit spreads
  – For a large portfolio, default risk is perfectly diversified
  – Only remains credit spread risks: parallel & idiosyncratic

• Main result
  – With respect to dynamic hedging, default risk can be neglected
  – Only need to focus on dynamic hedging of credit spread risks
  ➢ With CDS
  – Similar to interest rate derivatives markets
III - Hedging credit spread risks for large portfolios

• Formal setup
  - \( \tau_1, \ldots, \tau_n \) default times
  - \( N_i(t) = 1_{\{\tau_i \leq t\}}, i = 1, \ldots, n \) default indicators
  - \( H_t = \bigvee_{i=1, \ldots, n} \sigma(N_i(s), s \leq t) \) natural filtration of default times
  - \( F_t \) background (credit spread filtration)
  - \( G_t = H_t \lor F_t \) enlarged filtration, \( P \) historical measure
  - \( l_i(t, T), i = 1, \ldots, n \) time \( t \) price of an asset paying \( N_i(T) \) at time \( T \)
Sketch of the proof

Step 1: consider some smooth shadow risky bonds
  – Only subject to credit spread risk
  – Do not jump at default times

Projection of the risky bond prices on the credit spread filtration

**Definition 3.2** The default free $T$ forward loss process associated with name $i \in \{0, \ldots, n\}$, denoted by $p^i(\cdot, T)$ is such that for $0 \leq t \leq T$:

$$
p^i(t, T) \triangleq E^Q \left[ p^i(T) \mid \mathcal{F}_t \right] = E^Q \left[ N_i(T) \mid \mathcal{F}_t \right] = Q(\tau_i \leq T \mid \mathcal{F}_t).
$$

(3.2)

**Lemma 3.1** $p^i(t, T)$, $i = 1, \ldots, n$ are projections of the forward price processes $l^i(t, T)$ on $\mathcal{F}_t$:

$$
p^i(t, T) = E^Q \left[ l_i(t, T) \mid \mathcal{F}_t \right],
$$

(3.3)

for $i = 1, \ldots, n$ and $0 \leq t \leq T$. 

III - Hedging credit spread risks for large portfolios
• Step 2: Smooth the aggregate loss process
• … and thus the tranche payoffs
  – Remove default risk and only consider credit spread risk
  – Projection of aggregate loss on credit spread filtration

**Definition 3.1** We denote by \( p^i(\cdot) \), the default-free running loss process associated with name \( i \in \{0, \ldots, n\} \), which is such that for \( 0 \leq t \leq T \):

\[
p^i(t) \triangleq E^Q[N_i(t) \mid \mathcal{F}_t] = Q(\tau_i \leq t \mid \mathcal{F}_t) = 1 - \exp(-\Lambda_{i,t}).
\] (3.1)

**Definition 3.5** default-free aggregate running loss process The default free aggregate running loss at time \( t \) is such that for \( 0 \leq t \leq T \):

\[
p_n(t) \triangleq \frac{1}{n} \sum_{i=1}^{n} p^i(t).
\] (3.7)
III - Hedging credit spread risks for large portfolios

- Step 3: compute perfect hedge ratios of the smoothed payoff
  - With respect to the smoothed risky bonds
    - Smoothed payoff and risky bonds only depend upon credit spread dynamics
    - Both idiosyncratic and parallel credit spread risks
    - Similar to a multivariate interest rate framework
    - Perfect hedging in the smooth market

Assumption 2 There exists some bounded $\mathcal{F}$-predictable processes $\theta_1(\cdot), \ldots, \theta_n(\cdot)$ such that:

$$ (p_n(T) - K)^+ = E^Q \left[ (p_n(T) - K)^+ \right] + \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{T} \theta_i(t) dp_i(t, T) + z_n, $$

(4.2)

where $z_n$ is $\mathcal{F}_T$-measurable, of $Q$-mean zero and $Q$-strongly orthogonal to $p^1(\cdot, T), \ldots, p^n(\cdot, T)$. 
Step 4: apply the hedging strategy to the true defaultable bonds

Main result

- Bound on the hedging error following the previous hedging strategy
- When hedging an actual CDO tranche with actual defaultable bonds
- Hedging error decreases with the number of names

Default risk diversification

**Proposition 1** Under Assumptions (1) and (2), the hedging error $\varepsilon_n$ defined as:

$$
\varepsilon_n = (l_n(T) - K)^+ - E^Q [(l_n(T) - K)^+] - \frac{1}{n} \sum_{i=1}^{n} \int_0^T \theta_i(t) dI^i(t, T),
$$

is such that $E^P[||\varepsilon_n||]$ is bounded by:

$$
\frac{1}{\sqrt{2n}} \left( 1 + \left( E^Q \left[ \left( \frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \right) + \frac{1}{n} \left( E^Q \left[ \left( \frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \left( \sum_{i=1}^{n} (Q(\tau_i \leq T) + E^Q [B_i|T]) \right)^{1/2}
$$

$$
+ E^P[||z_n||].
$$

(4.5)
III - Hedging credit spread risks for large portfolios

• Provides a hedging technique for CDO tranches
  – Known theoretical properties
  – Takes into account idiosyncratic and parallel gamma risks
  – Good theoretical properties rely on no simultaneous defaults, no contagion effects assumptions
  – Empirical work remains to be done

• Thought provocative
  – To construct a practical hedging strategy, do not forget default risk
  – Equity tranche [0,3%]
  – iTraxx or CDX first losses cannot be considered as smooth
Hedging credit spread risk for large portfolios

- Linking pricing and hedging?
- The black hole in CDO modeling?
- Standard valuation approach in derivatives markets
  - Complete markets
  - Price = cost of the hedging/replicating portfolio
- Mixing of dynamic hedging strategies
  - for credit spread risk
- And diversification/insurance techniques
  - For default risk
• Two different models have been investigated
• Contagion homogeneous Markovian models
  – Perfect hedge of default risks
  – Easy implementation
  – Poor dynamics of credit spreads
  – No individual name effects
• Multivariate Cox processes
  – Rich dynamics of credit spreads
  – But no contagion effects
  – Thus, default risk can be diversified at the index level
  – Replication of CDO tranches is feasible by hedging only credit spread risks.