Advanced Pricing of CDOs

Global Derivatives
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Advanced Pricing of CDOs

- CDO pricing and product development
- Interpolation of base correlations
- Parametric factor copulas
- Non parametric factor approaches
- Loss dynamics
- Purpose of the talk
  - *To give an overview of current issues in the pricing of CDOs and related products*
  - *Try to show some possible directions regarding future modelling*

CDO pricing and product development

- CDO pricing and product development
  - *Models drive product development*
  - *New products call for new pricing techniques*
- Examples :
  - *CDO, CDO squared do not involve the dynamics of the loss distribution*
  - *Which is required for forward starting CDOs*
  - *Options on tranches moreover require the joint dynamics of losses and premiums : dynamic loss models*
  - *Interest hybrids : rather call for intensity models*
- The case of EDS, EDOs
  - *Three or four years ago, was expected to be a booming market*
  - *Calls for structural, barrier type models, possibly with Levy processes well-suited for equity-credit hybrids*
  - *Market did not go in that direction yet*

CDO pricing and product development

- The next markets shifts are not easy to forecast
  - *More liquidity on iTraxx, CDX related products ?*
  - *More attachment points, more traded maturities*
  - *FTD on indices ?*
- Depends on the bank strategy
  - *Willingness to carry default, credit spread, correlation risk*
  - *Focus either on exotics, custom-made or rather promote standardized liquid plain vanilla product*

CDO pricing and product development

- Global vs local pricing approaches
  - *Global : aim is to provide a consistent framework for pricing and hedging a large range of products*
  - *Local : focuses on a specific product*
  - *Example : forward starting CDOs involve the joint distribution of aggregate loss over two time horizons*
    - Local approach: extract the marginal loss distributions over the two maturities involved, then couple those marginals through some suitable copula
    - Global approach: construct a dynamic model of the loss, or of conditional copulas.
- Implementation constraints
  - *Numerical efficiency*
    - Large dimensional problems when dealing with a name per name approach
    - Factor approaches, large portfolio approximations
  - *Calibration issues, risk management of parameters*

Interpolation of base correlations

- Given base correlations of [0,3%] and [0,6%] (say)
  - *Compute by linear interpolation (or other smoothing technique) the base correlation of a [0,5%] tranche (say)*
- Pros
  - *Easy to think of pricing procedure*
- Cons
  - *Does not always produce arbitrage free prices*
    - Possibly some negative loss probabilities
  - *Extrapolation for out of range attachment points is hazardous*
    - Super-senior, [0,1%] tranches
  - *Prices for a [6%,7%] depends on the pricing of the [0,3%] which is counterintuitive*

Interpolation of base correlations

- Cons (following)
  - *The market does not usually quote "zero-coupon CDO"*
    - Base correlations are associated with loss distributions of different maturities
  - *Smoothing in two dimensions (maturity + attachment/detachment points) and arbitrage-free constraints?*
    - Loss must increase through time
    - First order stochastic dominance of loss distributions
  - *Rescaling techniques for the pricing of bespoke and CDO squared often lead to arbitrary prices*
  - *Forward-starting CDOs ?*

Parametric factor copulas

- Stochastic correlation, Random factor loadings
- Pros
  - *Guarantee arbitrage-free prices of CDOs*
  - *Easy pricing of CDOs through semi-analytical methods*
  - *Sparse number of parameters*
- Cons
  - *not a perfect fit to market quotes*
  - *matching of both 5Y and 10Y tranches?*
  - *Need of some procedure for the pricing of bespoke*
    - Though less arbitrary than base correlation approaches
  - *False sense of security*
    - example of stochastic correlation vs RFL, see below

Parametric factor copulas

- Modelling approaches
  - *Direct modelling of* $L(t)$*: collective model*
    - Dealing with heterogeneous portfolios
    - non stationary, non Markovian
    - Aggregation of portfolios, bespoke portfolios?
    - Risk management of correlation risk?
  - *Modelling of default indicators of names: individual model*

$$L(t) = \sum_{i=1}^{n} LGD_i 1_{\tau_i \le t}$$

  - *Numerical approaches*
    - e.g. smoothing of base correlation of liquid tranches

Parametric factor copulas

- Individual model / factor based copulas
  - *Allows to deal with non homogeneous portfolios*
  - *Arbitrage free prices*
    - non standard attachment –detachment points
    - Non standard maturities
  - *Consistent pricing of bespoke, CDO^2, zero-coupon CDOs*
  - *Computations*
    - Semi-explicit pricing, computation of Greeks, LHP
  - *But…*
    - Poor dynamics of aggregate losses (forward starting CDOs)
    - Risk management, credit deltas, theta effects
    - Calibration onto liquid tranches (matching the skew)

Parametric factor copulas

- Factor approaches to joint default times distributions:
  - *V: low dimensional factor*
  - *Conditionally on V, default times are independent.*
  - *Conditional default and survival probabilities:*

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right), \quad q_t^{i|V} = Q\left(\tau_i > t \mid V\right).$$

- Why factor models ?
  - *Tackle with large dimensions (i-Traxx, CDX)*
- Need of tractable dependence between defaults:
  - *Parsimonious modelling*
  - *Semi-explicit computations for CDO tranches*
  - *Large portfolio approximations* $L(t) \simeq p_t^{i|V}$

Parametric factor copulas

- Stochastic correlation
  - *Latent variables* $V_i = \tilde{\rho}_i V + \sqrt{1-\tilde{\rho}_i^2}\,\overline{V}_i, \quad i = 1,\ldots,n$

$$\tilde{\rho}_i = (1-B_s)(1-B_i)\rho + B_s$$

$\tilde{\rho}_i$, stochastic correlation,

$Q(B_s = 1) = q_s)$, systemic state,

$Q(B_i = 1) = q$, idiosyncratic state

  - *Conditional default probabilities*

$$p_t^{\cdot|V,B_s=0} = (1-q)\Phi\left(\frac{\Phi^{-1}\left(F(t)\right) - \rho V}{\sqrt{1-\rho^2}}\right) + qF(t),$$ $F(t)$ default probability

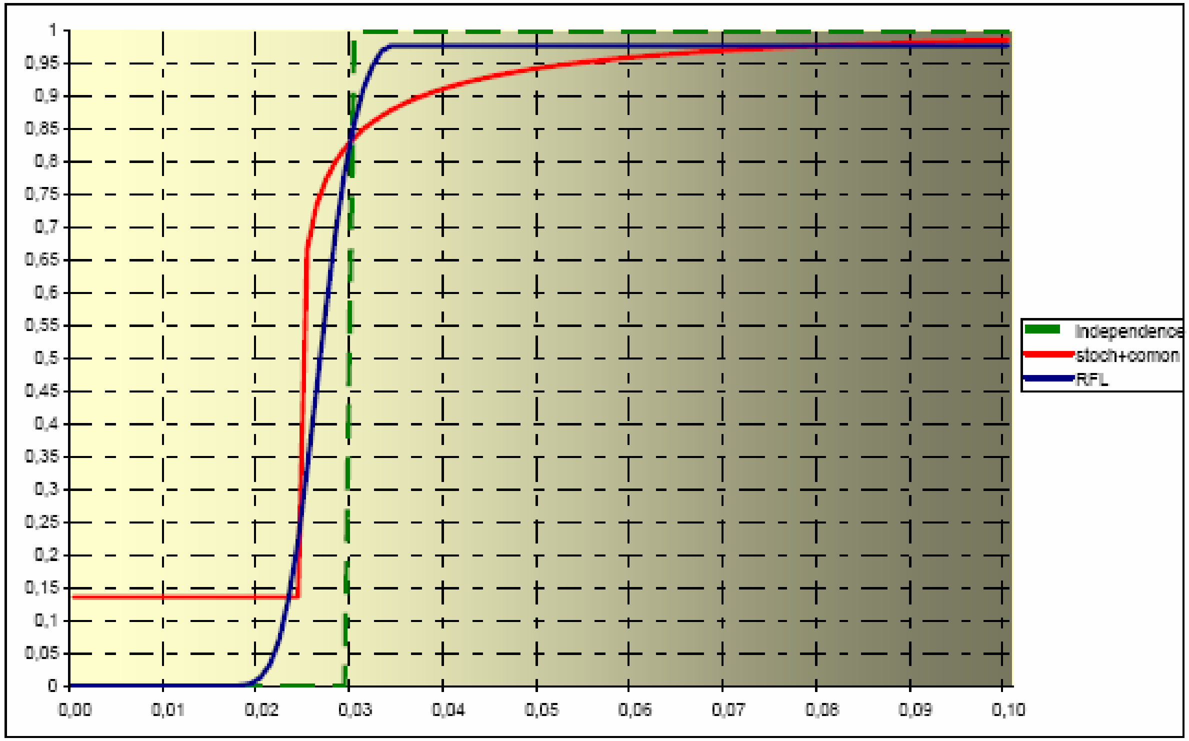
$$p_t^{\cdot|V,B_s=1} = 1_{V \le \Phi^{-1}\left(F(t)\right)},$$ comonotonic

Parametric factor copula

- Stochastic correlation $\tilde{\rho}_i = (1 - B_s)(1 - B_i)\rho + B_s$
  - *Semi-analytical techniques for pricing CDOs available*
  - *Large portfolio approximation can be derived*
  - *Allows for Monte Carlo*
  - $\nearrow \rho, \searrow q_s, \searrow q$ *leads to increase senior tranche premiums*
- State dependent correlation $V_i = m_i(V)V + \sigma_i(V)\bar{V}_i, \quad i = 1, \ldots, n$
  - *Local correlation* $V_i = -\rho(V)V + \sqrt{1 - \rho^2(V)}\bar{V}_i$
    - Turc et al
  - *Random factor loadings* $V_i = m + \left(l 1_{V<e} + h 1_{V \geq e}\right)V + \nu \bar{V}_i$
    - Andersen & Sidenius

Parametric factor copulas

- Distribution functions of conditional default probabilities
  - *stochastic correlation vs RFL*



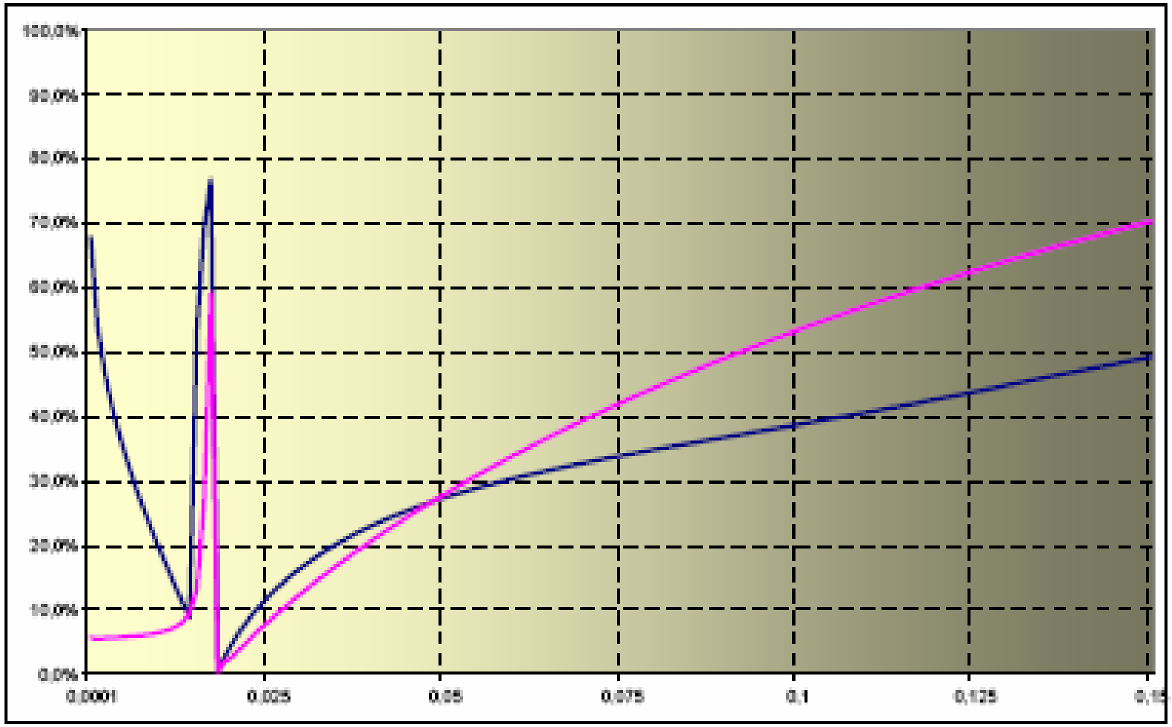
  - *With respect to level of aggregate losses*
  - *Also correspond to loss distributions on large portfolios*

Parametric factor copulas

- Marginal compound correlation
  - *Compound correlation of a* $[\alpha,\alpha]$ *tranche*
    - Digital call on aggregate loss
  - *obtained from conditional default probability distribution*
  - *Need to solve a second order equation*
  - *zero, one or two marginal compound correlations*

Parametric factor copulas

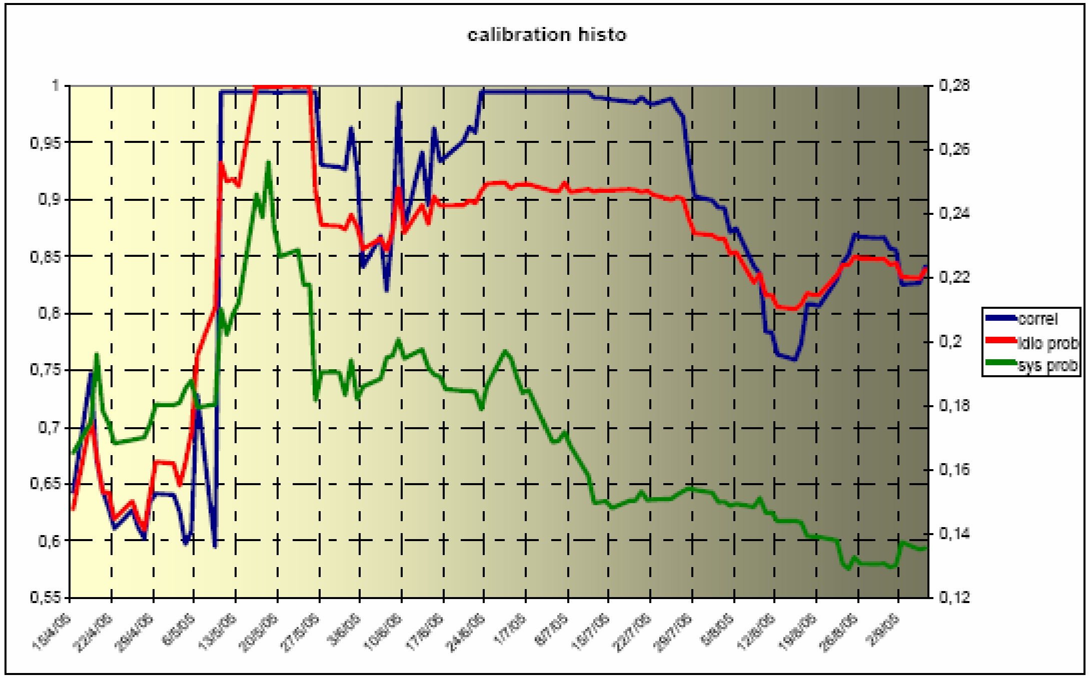
- Marginal compound correlations:
  - *With respect to attachment – detachment point*



  - *Stochastic correlation vs RFL*
  - *zero marginal compound correlation at the expected loss*

Parametric factor copulas

- Calibration history (from 15 April 2005)
  - *Implied correlation, implied idiosyncratic and systemic probabilities*



  - *Trouble in fitting during the crisis*
  - *Since then, decrease in systemic probability*

Non parametric factor approaches

- Still remains in the factor copula framework
  - *Semi-analytical pricing techniques for CDOs*
  - *Taking into account heterogeneity across names*
- Non parametric specification of conditional default probabilities $p_t^{i|V}$
  - *Under some constraints* $t \le t' \Rightarrow p_t^{i|V} \le p_{t'}^{i|V}$
  - *Consistency with marginal credit curves* $E^Q\left[p_t^{i|V} \right] = Q\left(\tau_i \le t\right)$
  - *Consistency with quotes of liquid tranches* $E\left[\left(p_t^{i|V} - k_i\right)^+\right] = \pi_{i,t}$
  - *Local correlation, implied copulas, entropic calibration*

Non parametric factor approaches

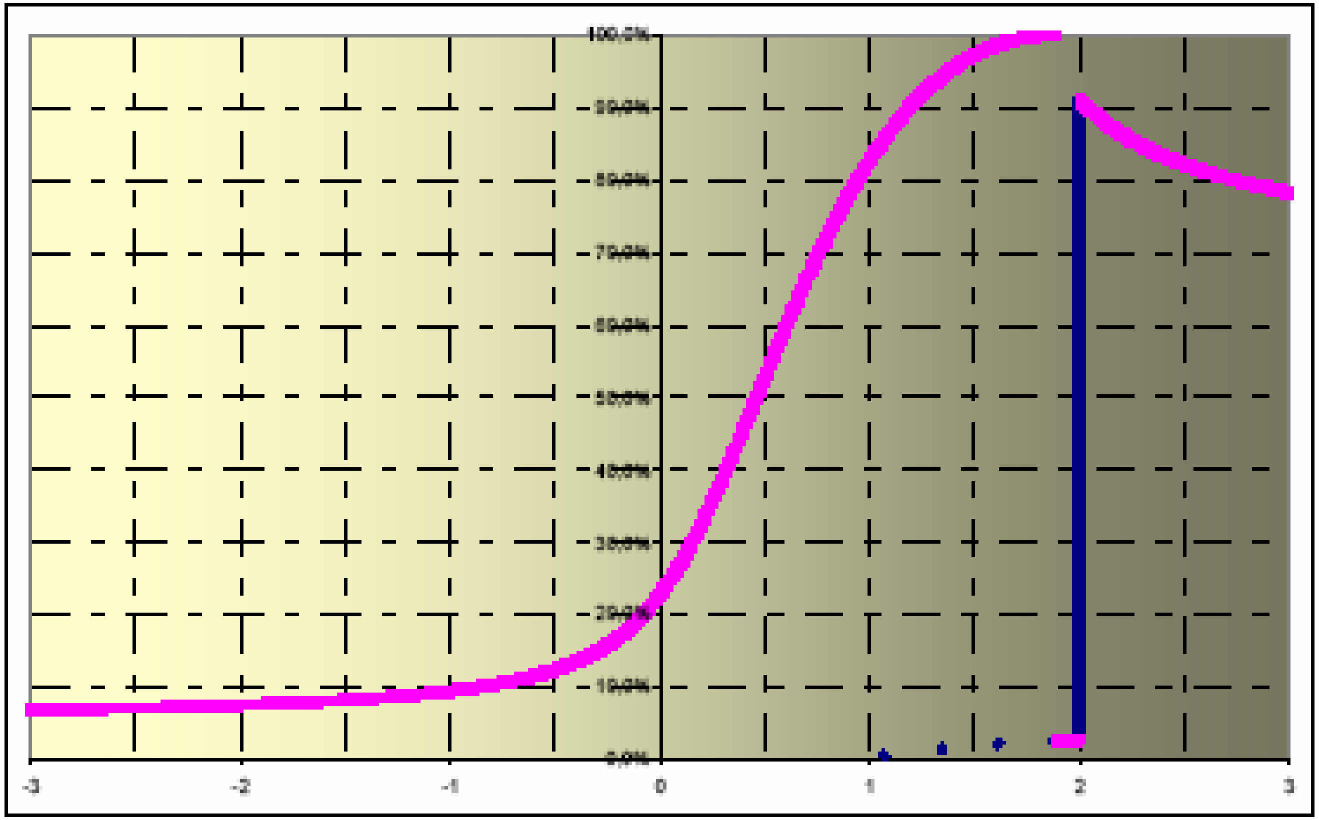
- Implied copula (Hull & White)
  - *discrete distribution of conditional default probabilities* $p_t^{i|V}$
- Local correlation

$$V_i = -\rho(V)V + \sqrt{1-\rho^2(V)}\bar{V}_i$$

  - *Can be computed from the distribution of* $p_t^{i|V}$
    - Through some fixed point algorithm
  - *Local correlation at step one: rescaled marginal compound correlation*
    - Same issues of uniqueness and existence as marginal compound correlation

Non parametric factor approaches

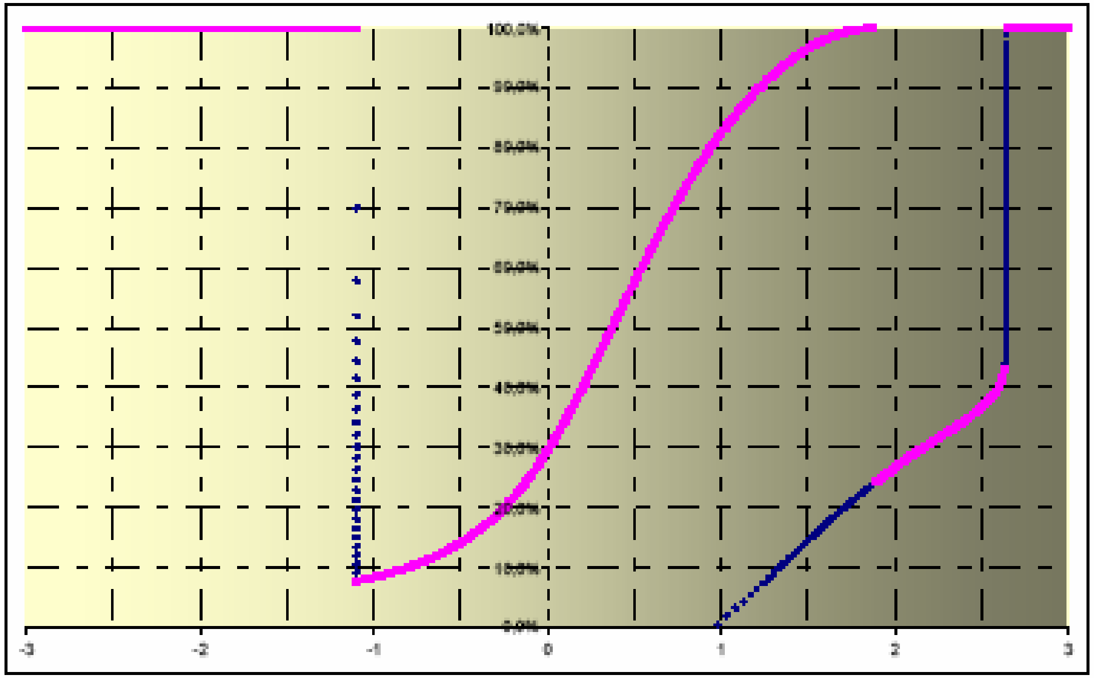
- Local correlation associated with RFL (as a function of the factor)



  - *Jump at threshold 2, low correlation level 5%, high correlation level 85%*
  - *Possibly two local correlations*

Non parametric factor approaches

- Local correlation associated with stochastic correlation model
  - *With respect to factor V*



  - *Correlations of 1 for high-low values of V (comonotonic state)*
  - *Possibly two local correlations leading to the same prices*
  - *As for RFL, rather irregular pattern*

Non parametric factor approaches

- Entropic calibration
  - *Start from some base parametric factor model*
  - g_0 *a priori density function of* $p_t^{i|V}$
  - *Look for some a posteriori density function g of* $p_t^{i|V}$

$$\min_g \int g(p) \ln \frac{g(p)}{g_0(p)} dp$$

  - *under consistency constraints* $\int_0^1 (p - k_i)^+ g(p) dp = \pi_i$

$$g(p) = g_0(p) \exp\left(\lambda + \sum_{i=0}^{I} \lambda_i (p - k_i)^+ \right)$$

  - *Semi-analytical form of the distribution of default probabilities*
    - Guarantees positivity

Loss dynamics

- Loss dynamics for factor models
- Consider the large portfolio approximation:

$$L(t) \simeq p_t^{i|V}$$

- As a consequence $L(t)$ and $L(t')$ are "perfectly correlated" (comonotonic)
  - Forward starting CDOs ?
- Dynamic factor approach $p_t^{i|V(t)}$
  - *Under the constraint that* $p_t^{i|V(t)}$ *is stochastically increasing in t*

Loss dynamics

- Dynamic models of the loss
  - *Intensity models*
    - Cox (doubly stochastic) models
    - Default times are independent upon some pre-specified default intensities
      - No jumps in credit spreads at default time arrivals
    - Factor approach still applicable (see Mortensen $V(t) = \int_0^t \lambda(s)ds$)
    - Allows to deal with name heterogeneity
    - Most likely to be well suited for analyzing hedging issues
  - *Contagion models*
    - Jumps in credit spreads at default time arrivals
    - Taking into account large portfolios
      - Numerical issues

Loss dynamics

- SPA, Schönbucher
  - The term structure of the aggregate loss distributions is the starting point
- Collective model of the loss
  - *Dealing with heterogeneity across names?*
  - *Needs the set of loss distributions over all horizons as a starting point*
    - Rather demanding
    - Small losses region
  - *Well suited for standard indices*
    - Bespoke ?
  - *Well suited for path dependent loss payoffs*
  - *Specification of the volatility inputs ?*
  - *Hedging issues ?*

Advanced Pricing of CDOs

- Linking pricing and hedging ?
- The black hole in CDO modelling ?
- Standard valuation approach in derivatives markets
    - Complete markets
    - Price = cost of the hedging/replicating portfolio
- Hedging CDOs with underlying CDS and indices
    - *Local risk minimization ?*
- Hedging non standard CDOs with liquid tranches