Pricing CDOs with state dependent stochastic recovery rates

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Pricing CDOs with state dependent stochastic recovery rates

- Main practical issue
  - Better understanding of large credit portfolio losses
  - After the credit and liquidity crisis
  - By introducing stochastic recovery rates
  - « correlated » together
  - And « correlated » with default dates
  - Through dependence upon common factor(s)
  - Study the properties of such (bottom-up) models

- Results of interest for market risk assessment
  - And not only portfolio credit risk
Pricing CDOs with state dependent stochastic recovery rates

- Need to distinguish CDOs of subprimes
  - Overestimated ratings for AAA senior tranches
    - Comonotonic losses
      - Related to real estate market in the US
      - Overestimation of diversification effects amongst assets
    - Underestimation of marginal default probabilities
      - Huge adverse selection problems with originate and distribute system especially in the low-quality
  - Huge losses borne by so-called “sophisticated investors”
    - … such as regional banks in Europe
    - “Because of the dispersion of financial risks to those more willing and able to bear them, the economy and the financial system are more resilient,”
      - Ben Bernanke keynote address, Federal Reserve Bank of Chicago’s annual conference on bank structure and competition on May 18, 2006
Pricing CDOs with state dependent stochastic recovery rates

- Need to distinguish between CDOs of subprimes and corporate CDOs
  - CDO of subprimes are CDO squared
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- Need to distinguish between CDOs of subprimes and corporate CDOs

ABX 7-1 Prices

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Pricing CDOs with state dependent stochastic recovery rates

- Huge losses to “sponsors” of SIV
  - Mainly US banks actively operating in private securitization of subprime mortgages
  - A SIV being a shadow bank, with highly illiquid low rated MBS on the asset side and on the liability side, no core equity, funding itself issuing short-term CP
    - Obvious solvency and liquidity issues for such SIV
  - How did it infect the sponsor banks?
  - through “accounting engineering” such as 365 days lines of credit
Pricing CDOs with state dependent stochastic recovery rates

- Huge losses to “sponsors” of SIV
  - Credit and liquidity exposures unconsolidated?
  - poor regulation (Basel I) and banking supervision
  - “Citigroup has agreed to pay $75m to settle civil charges that it misled investors over potential losses from high-risk mortgages”
  - Citigroup had said in 2007 that its exposure was $13bn or less. The SEC said it exceeded $50bn.
  - SEC Enforcement Director Robert Khuzami said Citigroup had misled analysts and the market of its ability to reduce its subprime exposure.
State dependent recovery rates

Practical context

Calibration of super senior tranches during the liquidity and credit crisis

- Insurance against very large credit losses
- [30-100] tranche on CDX starts to pay when (approximately) 50% of the 125 major companies in North America are in default
  - Contributed to the collapse of AIG
- AIG reinsurer of major banks
  - Sold protection through AIG Financial Products (London) and Banque AIG (Paris)
  - Between 440 and 500 billion “CDS” outstanding
  - Issues with accounting, counterparty risk, collateral management and liquidity.
    - Large MTM losses
    - Though no insurance payments were to be made
State dependent recovery rates

- Asymmetric CSA and downgrading of AIG triggered huge collateral posting
  - 30 billion USD of collateral to be posted for super senior tranches
  - Not corresponding to actual credit losses on tranches but to «mark to market» of highly illiquid insurance policies
- What occurred when US Treasury took over AIG?
Pricing CDOs with state dependent stochastic recovery rates

- Practical context: high spreads on senior tranches
- Increase of risk for individual losses leads to increase of risk in aggregate losses
  - For proper positive dependence
  - General results likely to be useful for market risk analysis
- Comparing risks when claim frequency increase and claim amount decrease (with equal mean)
  - Analysis of changing recovery rate assumptions on convex measures of risk
- Comparing risks for granular portfolios sharing the same large portfolio limit
  - Stochastic recovery rate versus recovery markdown
- Numerical issues
  - Expansion techniques vs recursion techniques
**State dependent recovery rates**

- High spreads on super senior tranches
  - *Could not be calibrated with a standard 40% recovery rate*

### Average Sr. Unsecured Bond Recovery Rates by Year Prior to Default, 1982-2008

<table>
<thead>
<tr>
<th></th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>n.a.</td>
<td>3.33%²</td>
<td>n.a.</td>
<td>97.00%</td>
<td>85.55%</td>
</tr>
<tr>
<td>Aa</td>
<td>43.60%</td>
<td>40.15%</td>
<td>43.45%</td>
<td>57.61%</td>
<td>43.40%</td>
</tr>
<tr>
<td>A</td>
<td>42.48%</td>
<td>45.45%</td>
<td>44.50%</td>
<td>38.28%</td>
<td>40.95%</td>
</tr>
<tr>
<td>Baa</td>
<td>41.85%</td>
<td>44.56%</td>
<td>44.09%</td>
<td>45.44%</td>
<td>42.68%</td>
</tr>
<tr>
<td>Ba</td>
<td>48.00%</td>
<td>42.68%</td>
<td>41.58%</td>
<td>41.15%</td>
<td>41.12%</td>
</tr>
<tr>
<td>B</td>
<td>36.98%</td>
<td>35.41%</td>
<td>35.88%</td>
<td>36.91%</td>
<td>40.68%</td>
</tr>
<tr>
<td>Caa-C</td>
<td>33.96%</td>
<td>33.25%</td>
<td>33.11%</td>
<td>39.59%</td>
<td>41.94%</td>
</tr>
<tr>
<td><strong>Investment-Grade</strong></td>
<td><strong>42.05%</strong></td>
<td><strong>44.23%</strong></td>
<td><strong>44.24%</strong></td>
<td><strong>44.57%</strong></td>
<td><strong>43.37%</strong></td>
</tr>
<tr>
<td>Speculative-Grade</td>
<td>36.26%</td>
<td>35.71%</td>
<td>36.30%</td>
<td>38.26%</td>
<td>40.90%</td>
</tr>
<tr>
<td>All Rated</td>
<td>36.56%</td>
<td>36.65%</td>
<td>37.50%</td>
<td>39.52%</td>
<td>41.51%</td>
</tr>
</tbody>
</table>

1. Issuer-weighted, based on 30-day post default market prices.
2. Based on three Icelandic bank defaults.
State dependent recovery rates

- High spreads on super senior tranches
  - *Could not be calibrated with a standard 40% recovery rate*
State dependent recovery rates

- High spreads on super senior tranches
  - Could not be calibrated with a standard 40% recovery rate

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Spread (bps)</th>
<th>Base Correlation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-3%</td>
<td>500, Upfront 68.51 points</td>
<td>39.45</td>
</tr>
<tr>
<td>3-7%</td>
<td>773.99</td>
<td>67.12</td>
</tr>
<tr>
<td>7-10%</td>
<td>435.52</td>
<td>72.58</td>
</tr>
<tr>
<td>10-15%</td>
<td>240.05</td>
<td>85.18</td>
</tr>
<tr>
<td>15-30%</td>
<td>126.50</td>
<td>-</td>
</tr>
<tr>
<td>30-100%</td>
<td>69.57</td>
<td>-</td>
</tr>
</tbody>
</table>

Source: Data from March 12, supplied by Markit
State dependent recovery rates

Practical context

- Steep “base correlations”
- Implied dependence as measured by implied Gaussian copula correlation
- Increases strongly with respect to attachment point
  - Reflecting “fat tails” in aggregate loss distributions
  - A bunch of issues of trading desks
    - Negative tranchelet prices
    - Delta discriminance
    - Weird Idiosyncratic gamma
- These issues are (partly) solved in a stochastic recovery rate approach
- Main issue during 2008 and 2009 for investment banks
State dependent recovery rates

Exhibit 10 – Correlation between Recovery Rates and Annual Default Rates, 1983-2004

Recovery Rate = 0.52 - 6.9 \times \text{Default Rate}

R^2 = 0.6521
Credit models often focus on the dependence between default dates

- **Bottom-up models**
  - Well-suited to analyze changes of portfolio allocation

- **Top-down models**
  - Markov models for aggregate losses
  - Dependence through contagion effects: jumps in aggregate loss intensity at default times
  - It is not obvious to relate risks to portfolio structure
  - Unit losses are capped by credit nominal, aggregate loss is also capped

- Our approach is (currently) related to bottom-up approach
  - *When clustering comes (only) through simultaneous defaults*
  - *It can actually create huge dependence effects (common shocks)*
  - *For example, possibility of an Armageddon risk*
    - Is this building really safe regarding earthquakes?
State dependent recovery rates and credit modelling

- Competing approaches for modelling default date dependencies
  - **Joint defaults: common shock models**
  - **Multivariate structural models**
    - CreditMetrics, Basel II, Moody’s KMV
  - **Correlated intensities**
    - Multivariate Cox processes
  - **Frailty models (Archimedean copulas)**
    - Hierarchical Archimedean copulas (partially nested)
  - **Gaussian copula**
    - Li (2000)
  - **Factor copulas**
    - Associated with a wide range of dependence structures
State dependent recovery rates and credit modelling

- **Markov Copulae**
  - Bielecki and co-authors
  - In between top-down and bottom-up
  - Small homogeneous portfolios may be considered as Markov
  - Dependence comes from simultaneous defaults (related with paper?)

- **GPL: Brigo et al.**

- No embedding framework

- Large credit losses can also come from stochastic recovery rates
  - “collateral damage”
  - Consider a model with factor dependence
  - Large homogeneous approximation with factor dependent recovery rate
  - Change of mixing distribution for defaults or change recovery rates?

Identification issue $\rightarrow (1 - \delta(V)) p^V$
State dependent recovery rates and credit modelling

- Dependence in large dimension
  - The puzzling issue of parametrization
    - Take the Gaussian copula case as the simplest example

- Homogeneous portfolios (static case)
  - De Finetti theorem
  - One factor
    - Partially exchangeable portfolios
    - A number of ways to introduce sector-based effects
      - Homogeneous sub-portfolios

- Common shock model is rather well-known
  - Multivariate exponential distributions
  - Marshall Olkin copulas
  - Within the factor copula framework
  - This eases CDO computations and model analysis
State dependent recovery rates and credit modelling

- Common shock models developed for CDOs by Elouerkhaoui
- The model can be associated with very large dependence
  - Much higher than Cox process models and even that frailty models
  - Allows to control for loss distributions (here small mezzanine tranches)

### Implied Compound Correlation

- Market
- Gaussian
- Double t 4/4
- Clayton
- Exponential
- t-Student 12
- t-Student 6
State dependent recovery rates and credit modelling

- Properties of the common shock model
- Specifying the dependence structure
  - Huge overfitting
  - $n$ names can lead to $2^n$ intensities!
  - Checking model restrictions?
- Dynamics of credit spreads
  - No contagion effects
- Dependence only due to simultaneous defaults
- Due to the large number of states, incomplete markets
  - Requires more involved techniques to construct risk-mitigating dynamic strategies
What are we looking at?

- **Risk measurement**
  - At which time horizon?
  - Need to account for rating migration, changes in credit spreads
  - (not only defaults)
  - Possible changes in the (local) correlation structure.
  - Static versus dynamic

- **CDO pricing**
  - *Investment grade names (100 names), medium size corporate portfolios, mortgages*

Not the same inputs

- *historical default data, recovery rates, definition of a default, credit spreads, ratings, bond prices, etc.*
State dependent recovery rates and credit modelling

- Coping with Basel 2 “++”
  - Capital requirements for CDS and CDO trading books
  - CRM : Comprehensive Risk Measure
  - Incremental Risk Capital Charge (IRC)
  - Stressed VaR : 99.9%, 1 Year time horizon
  - Must take into account dynamic hedging with CDO tranches, credit migration, credit spread volatility, stochastic correlation, stochastic recovery rates,...
  - Urgent action required (completion by end of year 2010)

- Moody’s KMV, CreditMetrics and related packages are frontrunners
State dependent recovery rates and credit modeling

Timing of defaults and default date definition

- *Not that clear in the corporate world*
- *Costly non-defaults, costless defaults*
- *For example, is a bail-out a default?*
  - What has occurred to Merrill Lynch counterparties after BofA stepped-in?
    - Then, it is associated with a joint default event, together with Lehman

Credit migration?

- Prior to Bear Stearns bail out by JP Morgan, many counterparties transferred their OTC exposures to thirds parties
- Novation: transfer rights and obligations to a third party
- “In the three weeks preceding Bear Stearns's collapse, GS, Citadel and Paulson exited about 400 trades where Bear Stearns was the trading partner, more than any other firms did.”
- GS unloaded a number of swap contracts. Positions were transferred to a variety of players, including Lehman Brothers and Morgan Stanley.
State dependent recovery rates and credit modelling

- (Almost) costless defaults: Fannie Mae Subordinated,
  - Final price, 6th October CDS auction: 99.9

- Jarrow et al. (2008)
  - Distressed Debt Prices and Recovery Rate Estimation

- Large discrepancies between economic and recorded default dates
  - Likely to be a major issue when dealing with the estimation of a model with simultaneous defaults
  - More problematic then in the case of no simultaneous defaults

- Recovery rates also contribute to dependence between individual default dates
State dependent recovery rates

- Theoretical context
  - Aggregate loss = sum of individual losses
  - Individual loss = default indicator times loss given default
  - Recovery rate = 1 – loss given default / credit notional
  - Recovery rates are stochastic

- Cross dependencies
  - Amongst default events (copula models, etc.)
  - Between default events and recovery rates
  - Amongst recovery rates

- Dependence through common latent factors
  - For convenience
State dependent recovery rates

- When does an increase in individual risk leads to an increase in the risk on the aggregate portfolio (sum of individual risks)?
  - *(Multivariate)* Gaussian risks
    - Individual risks with same expectation
    - Increase in risk = increase in variance
    - Increase in aggregate portfolio risk occurs if and only if pairwise correlations are non-negative
  - *What about the general case?*
    - Stochastic orders
      - Univariate case: convex order (close to second order stochastic dominance)
      - Positive dependence between individual risks
State dependent recovery rates

- Positive dependence
  - **MTP2**: *Multivariate Total Positivity of Order 2* (Karlin & Rinott (1980))
    - Log-density is supermodular
  - **Conditionally Increasing**
    - \( X = (X_1, \ldots, X_n) \) is CI if and only if \( E \left[ \phi(X_i) \mid (X_j)_{j \in J} \right] \) is increasing in \((X_j)_{j \in J}\) for increasing \( \phi \)
  - **Positive association** (Esary, Proschan & Walkup (1967))
  - **PSMD**: *positive supermodular dependent*

- Gaussian copula
  - **Positive association = PSMD = positive pairwise correlations**
  - **MTP2 = CI** (Müller & Scarsini (2001))
State dependent recovery rates

Theoretical context

- **Non Gaussian framework**
  - Individual risks have a probability mass at 0

- **Increase of risk of individual risks: convex order**

- **Theorem (Müller & Scarsini (2001))**
  - $X$ and $Y$ random vectors with common conditionally increasing copula
  - $X_i$ smaller than $Y_i$ for all $i$
  - Then $X$ smaller than $Y$ with respect to dcx (directionally convex) order
    - Then $X$ smaller than $Y$ with respect to stop-loss order

- **Gaussian copula dependence**
  - Conditionally increasing if and only if the inverse of covariance matrix is a $M$-matrix
  - $\Sigma$ non singular, entrywise non negative, $\Sigma^{-1}$ has positive non diagonal entries
State dependent recovery rates

- Dependence in large dimension
- Well known to finance people
- Factor models
  - Arbitrage pricing theory, asymptotic portfolios
    - Chamberlain & Rothschild (1983)
  - Large portfolio approximations (infinite granular portfolios)
    - Conditional law of large numbers
  - Qualitative data with spatial dependence
    - CreditRisk + (Binomial mixtures), Creditmetrics, Basel II (Gaussian copula)
  - Factor models may not be related to a causal view upon dependence
    - De Finetti, exchangeable sequences of Bernoulli variables are Binomial mixtures
    - Mixing random variable latent factor
State dependent recovery rates

- Spatial dependence with qualitative data
  - *Factor models have been used for long in other fields*
    - IQ tests (differential psychology), Bock & Lieberman (1970), Holland (1981)
    - Item Response Models

- Stochastic recovery rates
  - *Modeling of cross dependencies*
State dependent recovery rates

- Stochastic recovery rates
  - **Modeling of cross dependencies**
    - Individual loss = default indicator times loss given default
    - What is important for the computation of tranche premiums (or risk measures) is the joint distribution of individual losses
    - Direct approach: (discretized) individual loss seen as a polychotomous (or multinomial) variable
      - Multivariate Probit model (Krekel (2008))
      - Dual view of Creditmetrics (default side versus ratings)
  - Sequential models
    - Probit or logit models for default events (dichotomous model)
State dependent recovery rates

- **Gaussian copula**
  - *When is it conditionally increasing?*
  - *One factor case (positive betas)*
    - Gaussian copula is Conditionally Increasing (proof based on Holland & Rosenbaum (1986))
  - *Multifactor case: more intricate, even if all betas are positive, Gaussian copula may not be Conditionally Increasing*
    - Counterexamples
      - Gaussian copula with positive pairwise correlation
      - Increase of marginal risk (convex order)
      - May lead to a decrease of convex risk measures on aggregate portfolio
      - Constraints on conditional covariance matrix
  - *Hierarchical Gaussian copulas*
    - Conditionally Increasing copula (proof based upon Karlin & Rinott (1980))
State dependent recovery rates

- Consequences of previous analysis
  - Other examples of Conditionally Increasing copulas
  - Archimedean copulas, Müller & Scarsini (2005)
  - Dichotomous models with monotone unidimensional representation
    - Default indicators conditionally independent upon scalar $V$
    - Conditional default probabilities are non decreasing in $V$
    - Most known and used models
      - Includes additive factor copula models (Cousin & Laurent (2008)), such as generic one factor Levy model of Albrecher et al. (2007).
State dependent recovery rates

- Consequences of previous analysis
  - Non stochastic recovery rates
  - Analysis of a “recovery markdown”
  - Change recovery rate assumption from 40% to 30% (say)
  - Change marginal default probability so that expected loss unit is unchanged
  - Lemma: increase of marginal risk with respect to convex order

- Then, given a CI copula, increase of risk of the aggregate portfolio with respect to convex order
  - Increase in senior tranche premiums
  - Or CDO senior tranche spreads
State dependent recovery rates

- Consequences of previous analysis
  - Stochastic recovery rate of Amraoui and Hitier (2008)
  - Depends only upon latent factor
    - As in Altman et al (JoB 2005)
  - Specification of recovery rate is such that conditional upon latent factor is the same as in a recovery mark-down case
  - Same conditional expected losses
    - Same large portfolio approximations
    - Same “infinitely granular” portfolios
    - When number of names tends to infinity, strong convergence of aggregate losses to large portfolio limits

- Stochastic recovery rate (AH) versus recovery markdown
  - Same infinitely granular portfolios
  - But finitely granular portfolios behave (slightly) differently
**State dependent recovery rates**

- **Stochastic recovery rate (AH) vs recovery markdown**
  - Main comparison result
  - Aggregate losses are ordered with respect to convex order
  - Smaller risks in stochastic recovery rate specification
  - Smaller spreads on senior tranches
  - Small numerical discrepancies

- **Numerical issues**
  - Computation of aggregate loss distributions in individual loss model with spatial dependence (factor models)
  - Actuarial methods (recursions, etc.)
  - FFT, inverse of Laplace transforms
  - Expansions (Stein’s method, Gram-Charlier expansions)
State dependent recovery rates

Numerical issues

- Lots of smuggling around
- Key issues for implementation
  - Computation of prices
  - Much quicker than Monte Carlo
    - Issues for the use of Hierarchical Archimedean Copulas
  - More importantly computations of Greeks
  - Risk Management
  - Maximum Likelihood methods
- Needs to be reassessed in case of stochastic recovery models

Maximum Likelihood methods

Needs to be reassessed in case of stochastic recovery models