Interest Rate Models: from Parametric Statistics to Infinite Dimensional Stochastic Analysis

December 27, 2000
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Preface

These lecture notes, and especially the last four chapters, grew out of a graduate seminar given during the summer of 2000 at Princeton University. The crash course on the mechanics of the bond market was prepared in December of 2000 for the tutorial presented by the author in Los Angeles at the IPAM the 3rd, 4th and 5th of January 2001. The chapter on data analysis grew out of lecture notes prepared for the ORF 405 class in Data Analysis at Princeton University. I would like to thank Mike Tehranchi for typing a preliminary version of the chapter on integration in a Banach space, and Mike and Manuel Sales for proofreading an early version of the set of notes.

The level of complexity of the bond market is higher than for stocks: one simple reason is contained in the fact that the underlying instruments on which the derivatives are written are more sophisticated than mere stock shares. As a consequence, the mathematical models needed to describe their time evolution will have to be more involved. Indeed on each given day $t$, instead of being given by a single number $S_t$ as the price of one share of a common stock, the term structure of interest rates is given by a curve determined by a finite discrete set of values. This curve is interpreted as the sampling of the graph of a function $T \mapsto P(t,T)$ of the date of maturity of the instrument. In particular, whenever we had to deal with stock models involving ordinary or stochastic differential equations or finite dimensional dynamical systems, we will have to deal with stochastic partial differential equations or infinite dimensional systems!

The main goal of these lectures is to present in a self contained manner, the empirical facts needed to understand the sophisticated mathematical models developed by the financial mathematics community over the last decade. So after a very elementary introduction to the mechanics of the bond market, and a thorough statistical analysis of the data available to any curious spectator without any special inside track information, we gradually introduce the mathematical tools needed to analyze the stochastic models most widely used in the industry. Our point of view has been strongly influenced by recent works of Cont and his collaborators and the PhD of Fillipovic. They merge the original proposal of Musiela inviting us to rewrite the HJM model as a stochastic partial differential equation, together with Bjork’s proposal to recast the HJM model in the framework of stochastic differential equations in a Banach space. The main thrust of these lectures is to present this approach from scratch, in a rigorous and self-contained manner.

The Heath-Jarrow-Morton [56] framework for modeling the interest rate term structure is to take the (instantaneous) forward rate as the underlying state variable. The field $f(t,T)$, interpreted as the forward rate prevailing at time $t$ for maturity $T$, is assumed to
evolve according to the stochastic differential equation

\[ df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t. \]

In the absence of arbitrage, there is the famous HJM condition expliciting constraining the drift to a function of volatility for a risk neutral probability measure.

In the classical \( d \)-factor HJM model, we take \( W_t = [W^{(1)}_t, \ldots, W^{(d)}_t] \) to be standard \( \mathbb{R}^d \)-valued Brownian motion, and \( \sigma(t, T) = [\sigma^{(1)}(t, T), \ldots, \sigma^{(d)}(t, T)] \) to be valued in the dual space of \( \mathbb{R}^d \), which is again \( \mathbb{R}^d \). Note that there are an infinite number of equations, on for each value of \( T > t \), but only finite sources of randomness.

As noted by various authors, a finite factor forward rate model has several shortcomings, one of which is that every interest rate contingent claim can be hedged by \( d + 1 \) bonds of any maturity, contrary to actual market practice. In order to address this issue, these notes explore the proposed generalization to an infinite-factor case where \( W_t \) is a Brownian motion valued in a Banach space.
Part I

The Term Structure of Interest Rates
Chapter 1

The Term Structure of Interest Rates: A Crash Course

The size and the level of sophistication of the market of fixed income securities increased dramatically over the last 10 years and it became a prime test bed for financial institutions and academic research. The fundamental object to model is the term structure of interest rates and we shall approach it via the prices of treasury bond issues. Models for these prices are crucial to price derivatives such as swaps, quantify and manage financial risk, and set monetary policy. We mostly restrict ourselves to Treasury issues to avoid to have to deal with the possibility of default. The highly publicized defaults of counties (such as the bankruptcy of Orange County in 1994), of sovereigns (like Russia defaulting on its bonds in 1998) and the ensuing ripple effects on worldwide markets have brought the issue of credit risk to the forefront. Unfortunately, because of time and space limitations, we will not be able to address these issues in much detail. The reader interested in credit risk issues and further financial engineering applications of the mathematical models of the fixed income markets is referred to Part III of [27].

Before we tackle some of the fundamental statistical issues of the bond markets, we need a crash course on the mechanics of interest rates and the fixed income securities. This gives us a chance to introduce the notation and the terminology used throughout these lectures.

1.1 The Time Value of Money

We first introduce the time value of money by valuing the simplest possible fixed income instrument. It is a financial instrument providing a cash flow with a single payment of a fixed amount (the principal $X$) at a given date in the future. This date is called the maturity date. If the time to maturity is exactly $n$ years, the present value of this instrument is:

$$P(X, n) = \frac{1}{(1+r)^n} X$$  \hspace{1cm} (1.1)
This formula gives the present value of a nominal amount $X$ due in $n$ years time. Such an instrument is called a discount bond or a zero coupon bond because the only cash exchange takes place at the end of the life of the instrument, i.e. at the date of maturity. The positive number $r$ is referred to as the (yearly) discount rate or spot interest rate for time to maturity $n$ since it is the interest rate which is applicable today (hence the terminology spot) on an $n$-year loan. Formula 1.1 is the simplest way to quantify the adage: one dollar is worth more today than later!

**Treasury Bills**

Treasury bills are securities issued by the US government with a time to maturity of one year or less. A noticeable difference with the other securities discussed later is the fact that they do not carry coupon payments.

Let us consider for example the case of an investor who buys a $100,000 13-week T-bill at a 6% yield (rate.) The investor pays (approximately) $98,500 at the inception of the contract, and receives the nominal value $100,000 at maturity 13 weeks later. Since $13 = 52/4$ weeks represent one quarter, and since 6% is understood as an annual rate, the discount is computed as $100,000 \times 0.06/4 = 1,500$.

So in order to price a 5.1% rate T-bill which matures in 122 days, we first compute the discount rate:

$$\delta = 5.1 \times (122/360) = 1.728$$

which says that the investor receives a discount of $1.728 per $100 of nominal value. Consequently, the price of a T-bill with this (annual) rate and time to maturity should be:

$$\$ 100 - \delta = 100 - 1.728 = 98.272$$

per $100 of nominal value.

Rates, yields, spreads, . . . are usually quoted in basis points. There are 100 basis points in one percentage point. The Treasury issues bills with time to maturity of 13 weeks, 26 weeks and 52 weeks. These bills are called "three-month bills", "six-month bills" and "one-year bills" although these names are accurate only at their inception. Thirteen-week bills and twenty-six week bills are auctioned off every Monday while the fifty-two week bills are auctioned off once a month.

The T-bill market is high volume, and liquidity is not a major issue. The left pane of Figure 1.1 reproduces a short article from the Wall Street Journal of December 22, 2000 in which an offering was announced. Despite a very high transaction volume, there is always a slight difference between the bid and ask prices. The right pane of Figure 1.1 shows how T-bills are quoted daily in the Wall Street Journal.

The first column give the date of maturity of the bill, the second column giving the number of days to maturity. The third and fourth columns give the bid and asked prices in decimal form. Compare with the bid and asked columns of Figure 1.2 and the discussion of the quotes of Treasury notes and bonds as discussed below.
1.2 The Discount Factors

Since the nominal value $X$ appears merely as a plain multiplicative factor in formula (1.1), it is convenient to assume that this value is equal to 1, and effectively drop it from the notation. This leads to the notion of discount factor. Discount factors are the quantities used at a given point in time to obtain the present value of future cash flows. At a given time $t$, the discount factor $d_{t,m}$ with time to maturity $m$, or maturity date $T = t + m$, is given by the formula:

$$d_{t,m} = \delta(t, T) = \frac{1}{(1 + r_{t,m})^m} \quad (1.2)$$

where $r_{t,m}$ is the yearly spot interest rate in force at time $t$ for this time to maturity. We assumed implicitly that the time to maturity $T - t$ is a whole number $m$ of years. Definition (1.2) can be rewritten in the form:

$$\log(1 + r_{t,m}) = -\frac{1}{m} \log d_{t,m}$$

and considering the fact that $\log(1 + x) \sim x$ when $x$ is small, the same definition gives the approximate identity:

$$r_{t,m} \sim -\frac{1}{m} \log d_{t,m}$$

which becomes an exact equality if we use continuous compounding. This formula justifies the terminology discount rate for $r$. Considering payments occurring in $m$ years time, the
Chapter 1 The Term Structure of Interest Rates: A Crash Course

spot rate \( r_{t,m} \) is the single rate of return used to discount all the cash flows for the discrete periods from time \( t \) to time \( t + m \). As such, it appears as some sort of composite of interest rates applicable over a shorter period. Moreover, this formula offers a natural generalization to continuous time models with continuous compounding of the interest. In this case, it reads:

\[
d(t, T) = e^{-(T-t)r(t,T)}.
\]

(1.3)

The discount function is very useful. Indeed, according to the above discussion, the present value of any future cashflow can be computed by multiplying its nominal value by the appropriate value of the discount factor. Nevertheless, the information contained in the discount function is often re-packaged into quantities which better quantify the returns associated with purchasing future cashflows at their present value. These quantities go under the names of spot interest rate curve, par yield curve, and implied forward rate curve. This section is devoted to the introduction of these quantities in the discrete time setting, and to the definition of their analogs in the continuous time limit. The latter is a mathematical convenience which makes it possible to use the rules of the differential and integral calculus. It is somehow unrealistic because money is lent for discrete periods of time, but when these periods are short, the continuous time limit models become reasonable. Also, one of the goals of this subsection is to show how to go from discrete data to continuous time models and vice versa.

1.3 Coupon Bearing Bonds

Now that we know what a zero coupon bond is, it is time to introduce the notion of coupon bearing bond. If a zero coupon bond was involving only one cash flow, what is called a bond (or a coupon bearing bond), is a regular stream of future cash flows. To be more specific, a coupon bond is a series of payments amounting to \( C_1, C_2, \ldots, C_m \), at times \( T_1, T_2, \ldots, T_m \), and a terminal payment \( X \) at the maturity date \( T_m \). \( X \) is called the nominal value, or the face value, or the principal value of the bond. According to the above discussion of the discount factors, the bond price at time \( t \) should be given by the formula:

\[
B(t) = \sum_{t \leq T_j} C_j d(t, T_j).
\]

(1.4)

This all purpose formula can be specialized advantageously for in most cases, the payments \( C_j \)’s are made at regular time intervals. These coupons payments \( C_j \)’s are most often quoted as a percentage \( c \) of the face value \( X \) of the bond. In other words, \( C_j = cX \). This percentage is given as an annual rate, even though payments are sometimes done every six months, or according to another periodicity. It is found convenient to introduce a special notation, say \( n_y \), for the number of coupon payments per year. For example, \( n_y = 2 \) for coupons paid semi-annually. If we denote by \( r_1, r_2, \ldots, r_m \) the interest rates for the \( m \) periods ending with the coupon payments \( T_1, T_2, \ldots, T_m \), then the present value of the
1.3 Coupon Bearing Bonds

Bond cash flow is given by the formula:

\[ B(t) = \frac{C_1}{1 + r_1/n_y} + \frac{C_2}{(1 + r_2/n_y)^2} + \cdots + \frac{C_m}{(1 + r_m/n_y)^m} \]

\[ = \frac{cX}{n_y(1 + r_1/n_y)} + \frac{cX}{n_y(1 + r_2/n_y)^2} + \cdots + \frac{cX + X}{n_y(1 + r_m/n_y)^m} \]  \hspace{1cm} (1.5)

Note that we divided the rates \( r_n \) by the frequency \( n_y \) because the rates are usually quoted as yearly rates. Formulae (1.4) and (1.5) are often referred to as the bond price equations. An important consequence of these formulae is the fact that on any given day, the value of a bond is entirely determined by the discount curve (i.e. the sequence of the \( d(t, n) \) for \( n = 1, \cdots, m \)) on that day.

Remarks:
1. Reference to the present date \( t \) will often be dropped from the notation when no confusion is possible. Moreover, instead of working with the absolute dates \( T_1, T_2, \cdots, T_m \), which can represent coupon payment dates as well as maturity dates of various bonds, it will be often more convenient to work with the times to maturities which we denote by \( x_1 = T_1 - t, x_2 = T_2 - t, \cdots, x_m = T_m - t \). We will use whatever notation is more convenient for the discussion at hand.
2. Unfortunately for us, bond prices are not quoted as a single number. Instead, they are given by a bid-ask interval. We shall ignore the existence of this bid-ask spread for most of the following discussion, collapsing this interval to a single value by considering its midpoint only. We shall re-instate the bid-ask spread in the next section when we discuss the actual statistical estimation procedures.

Treasury Notes and Treasury Bonds

Treasury notes, are Treasury securities with time to maturity ranging from 1 to 10 years at the time of sale. Unlike bills, they have coupons: they pay interest every six months. Notes are auctioned on a regular cycle. The Fed acts as agent for the Treasury, awarding competitive bids in decreasing order of price, highest prices first. The smallest nominal amount is $5,000 for notes with two to three years to maturity at the time of issue, and it is $1,000 for notes with four or more years to maturity at the time of issue. Both types are available in multiples of $1,000 above the minimum nominal amount.

Treasury bonds or T-bonds, are Treasury securities with more than 10 years to maturity at the time of sale. Like Treasury notes, they are sold at auctions, they are traded on a dollar price basis, they bare coupons and they accrue interest. If not for their different life spans, the differences between Treasury notes and bonds are few. For example, bonds have a minimal amount of $1,000 with multiples of $1,000 over that amount. Also, some bonds can have a call feature (see Section 1.7 below.) So we shall give the same treatment to notes and bonds, and we shall only talk about T-bills and T-bonds in these lectures.

On any given day, there is a great variety of notes and bonds outstanding, with maturity ranging from a few days to 30 years, and coupon rates as low as 3 or 4% and as high as...
Chapter 1 The Term Structure of Interest Rates: A Crash Course

9%. Figure 1.2 shows how Treasury notes and bonds are quoted daily in the Wall Street Journal.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>ATE</th>
<th>Maturity</th>
<th>BID</th>
<th>ASKED</th>
<th>CHG</th>
<th>YLD</th>
</tr>
</thead>
<tbody>
<tr>
<td>4%</td>
<td>Dec 01n</td>
<td>99.50</td>
<td>99.55</td>
<td>+ 1</td>
<td>3.75</td>
<td></td>
</tr>
<tr>
<td>6%</td>
<td>Dec 01n</td>
<td>99.20</td>
<td>99.30</td>
<td>+ 2</td>
<td>4.98</td>
<td></td>
</tr>
<tr>
<td>4%</td>
<td>Jan 01n</td>
<td>99.26</td>
<td>99.30</td>
<td>+ 2</td>
<td>5.14</td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>Feb 01n</td>
<td>99.31</td>
<td>100.00</td>
<td>+ 3</td>
<td>5.07</td>
<td></td>
</tr>
<tr>
<td>7%</td>
<td>Apr 01n</td>
<td>101.70</td>
<td>101.12</td>
<td>+ 3</td>
<td>5.88</td>
<td></td>
</tr>
<tr>
<td>12%</td>
<td>Jun 01n</td>
<td>100.31</td>
<td>100.31</td>
<td>+ 3</td>
<td>6.01</td>
<td></td>
</tr>
<tr>
<td>6%</td>
<td>Aug 01n</td>
<td>99.92</td>
<td>99.92</td>
<td>+ 3</td>
<td>5.25</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1.2. Wall Street Journal Treasury notes and bond quotes on December 22, 2000.

The first column gives the rate while the second column gives the month and year of maturity, with a lower case letter "n" in case the instrument is a note. The third and fourth columns give the bid and asked prices. Notice that none of the decimal parts happen to be greater than 31. Compare with the T-bill quotes given earlier in Figure 1.1. The reason is that the prices of Treasury notes and bonds are quoted in percentage points and 32nds of a percentage point. These are percentages of the nominal amount. But even though the figures contain a decimal point, the numbers to the right of the decimal point give the number of 32nds. So the third bid price which reads 99.28 is actually 99 + 28/32 = 99.875 which represents $998,750 per million of dollars of nominal amount. The fifth column gives the change in asked price with the asked price of the last trading day while the last column gives the yield computed on the asked price. More needs to be said on the way this yield is computed for some of the bonds (for callable bonds for example) but this level of detail is beyond the scope of these introductory lectures.

Because of their large volume, Treasury notes and bonds can easily be bought and sold at low transaction cost. They pay interest semi-annually, most often on the anniversary date of the date of issue. This income is exempt from state income taxes.

**STRIPS**

Formula (1.4) shows that a coupon bearing bond can be viewed as a composite instrument comprising a zero coupon bond with the same maturity $T_m$ and face value $(1 + c)X/n_y$. 
1.4 Clean Prices

and a set of zero coupon bonds whose maturity dates are the coupon payment dates $T_j$ for $1 \leq j < m$ and face value $cX/nw$. This remark is much more than a mere mathematical curiosity. Indeed, the principal and the interest components of US Treasury bonds have been traded separately under the Treasury STRIPS (Separate Trading of Registered Interest and Principal Securities) program since 1986. These instruments were created to meet the demand for zero coupon obligations. They are not special issues: the Treasury merely declares that specific notes and bonds (and no others) are eligible for the STRIPS program, and the stripping of these issues is done by government bond dealers who give a special security identification number (CUSIP in the jargon of financial data) for these issues. Figure 1.3 shows how STRIPS are quoted daily in the Wall Street Journal.

<table>
<thead>
<tr>
<th>U.S. TREASURY STRIPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
</tr>
<tr>
<td>Feb. 01</td>
</tr>
<tr>
<td>Feb. 01</td>
</tr>
<tr>
<td>May</td>
</tr>
<tr>
<td>Aug.</td>
</tr>
<tr>
<td>Nov.</td>
</tr>
<tr>
<td>Nov.</td>
</tr>
<tr>
<td>Nov.</td>
</tr>
<tr>
<td>Feb. 02</td>
</tr>
<tr>
<td>May</td>
</tr>
<tr>
<td>Aug.</td>
</tr>
<tr>
<td>Nov.</td>
</tr>
<tr>
<td>Jun.</td>
</tr>
</tbody>
</table>

Figure 1.3. Wall Street Journal STRIPS quotes on December 22, 2000.

1.4 Clean Prices

Formulae (1.4) and (1.5) implicitly assumed that $t$ was the time of a coupon payment, and consequently, that the time to maturity was an integer multiple of the time separating two successive coupon payments. Because of the very nature of the coupon payments occurring at specific dates, the bond prices given by the bond pricing formula (1.4) are discontinuous, in the sense that they jump at the times the coupons are paid. This is regarded as an undesirable feature, and systematic price corrections are routinely implemented to remedy the jumps. The technicalities behind these price corrections increase the level of complexity of the formulae, but since most bond quotes (whether they are from the US Treasury, or from international or corporate markets) are given in terms of these corrected prices, we thought
that it would be worth our time to look into a standard way to adjust the bond prices for these jumps.

The most natural way to smooth the discontinuities is to adjust the bond price for the accrued interest earned by the bond holder since the time of the last coupon payment. This notion of accrued interest is quantified in the following way. Since the bond price jumps by the amount \( cX/n \), at the times \( T_j \) of the coupon payments, if the last coupon payment (before the present time \( t \)) was made on date \( T_n \), then the accrued interest is defined as the quantity:

\[
AI(T_n, t) = \frac{t - T_n}{T_{n+1} - T_n} \cdot \frac{cX}{n_y},
\]

and the clean price of the bond is defined by the requirement that the transaction price be equal to the clean price plus the accrued interest. In other words, if \( T_n \leq t < T_{n+1} \), the clean price \( CP(t, T_m) \) is defined as:

\[
CP(t, T_m) = PX,C(t, T_m) - AI(t, T_n)
\]

where \( B(t) \) is the transaction price given by (1.4) with the summation starting with \( j = n + 1 \).

### 1.5 The Spot (Zero Coupon) Yield Curve

The spot rate \( r_{t,m} \) or \( r(t, T) \) defined from the discount factor via formulae (1.2) or (1.3) is called the zero coupon yield because it represents the yield to maturity on a zero coupon bond (also called a discount bond.) Given observed values \( d_j \)'s of the discount factor, these zero coupon yields can be computed by inverting formula (1.2). Dropping the date \( t \) from the notation, we get:

\[
r_j = \left( \frac{1}{d_j} \right) - 1
\]

for the zero coupon yield. The sequence of spot rates \( \{ r_j; j = 1, \cdots, m \} \) where \( m \) is a distant maturity is what is called the term structure of (spot) interest rate or the zero coupon yield curve. It is usually plotted against the time to maturity \( T_j - t \) in years. See next chapter for examples. Figure 1.4 shows how the Treasury yield curves are pictured daily in the Wall Street Journal. Notice the non uniform time scale on the horizontal axis.

#### 1.5.1 The Par Yield Curve

The par yield curve has been introduced to give an account of the term structure of interest rate when information about coupon paying bonds are the only data available. Indeed, yield computed from coupon paying bonds can be quite different from the zero coupon yields computed as above.
1.5 The Spot (Zero Coupon) Yield Curve

Figure 1.4. Wall Street Journal Treasury yield curves on December 22, 2000.

A coupon paying bond is said to be priced at par if its current market price equals its face (or par) value. If a bond price is less than its face value, we say the bond trades at a discount. In this case, its yield is higher than the coupon rate. On the other hand, if its price is higher than its face value, it is said to trade at a premium. In this case the yield is lower than the coupon rate. These price/yield qualitative features are quite general: everything else being fixed, higher yields correspond to lower prices, and vice versa. Also note that, if the prices of two bonds are equal, then the one with the larger coupon has the highest yield.

The par yield is defined as the yield of a bond priced at par. In other words, if the bond is \( n \) time periods away from maturity, and if we denote by \( P \) this par value, the par yield is the value of \( y \) for which we have the following equality:

\[
P = \sum_{j=1}^{m} \frac{y}{(1 + y/n)^j} + \frac{P}{(1 + y/n)^m}.
\]

### 1.6 The Forward Rate Curve

To restate this last property of the spot interest rate, we introduce the notation \( f_{t,m} \) for the rate applicable from the end of the \((m - 1)\)-th period to the end of the \( m \)-th period. With this notation at hand we have:

\[
\begin{align*}
1/d_{t,1} &= 1 + r_{t,1} = 1 + f_{t,1} \\
1/d_{t,2} &= (1 + r_{t,2})^2 = (1 + f_{t,1})(1 + f_{t,2}) \\
& \quad \cdots \quad \cdots \\
1/d_{t,j-1} &= (1 + r_{t,j-1})^{j-1} = (1 + f_{t,1})(1 + f_{t,2}) \cdots (1 + f_{t,j-1}) \\
1/d_{t,j} &= (1 + r_{t,j})^j = (1 + f_{t,1})(1 + f_{t,2}) \cdots (1 + f_{t,j-1})(1 + f_{t,j})
\end{align*}
\]

Computing the ratio of the last two equations gives:

\[
\frac{d_{t,j-1}}{d_{t,j}} = 1 + f_{t,j}
\]

or equivalently:

\[
f_{t,j} = \frac{d_{t,j-1} - d_{t,j}}{d_{t,j}} = -\frac{\Delta d_{t,j}}{d_{t,j}} \tag{1.8}
\]

if we use the standard notation \( \Delta d_{t,j} = d_{t,j} - d_{t,j-1} \) for the first difference of a sequence (i.e. the discrete time analog of the first derivative of a function.) The rates \( f_{t,1}, f_{t,2}, \cdots, f_{t,j} \) implied by the discount factors \( d_{t,1}, d_{t,2}, \cdots, d_{t,j} \) are called the implied forward interest rates. The essential difference between the spot rate \( r_{t,j} \) and the forward rate \( f_{t,j} \) can be best restated by saying that \( r_{t,j} \) gives the average rate of return of the next \( j \) periods while the forward rate \( f_{t,j} \) gives the marginal rate of return over the \( j \)-th period, for example the one year rate of return in 10 years time instead of today’s 10 year rate.
1.7 Extensions

We shall denote by \( x \mapsto d(t, x) \) the discount factor as a function of the time to maturity, whether the latter is an integer (giving the number of years to maturity) or more generally a fraction or even a nonnegative real number. With this generalization in mind, formula (1.8) gives:

\[
 f(t, x) = -\frac{d'(t, x)}{d(t, x)}
\]

where \( d'(t, x) \) denotes the derivative (partial derivative to be correct) of \( d \) with respect to the continuous variable \( x \) giving the time to maturity. Integrating both sides and taking exponentials of both sides we get the following expressions for the discount factor:

\[
 d(t, x) = e^{-\int_0^x f(t, s)ds} \tag{1.9}
\]

and the spot rate:

\[
 r(t, x) = -\frac{1}{x} \int_0^x f(t, s)ds. \tag{1.10}
\]

This relation can be inverted to express the forward rates as function of the spot rate:

\[
 f(t, x) = r(t, x) + x r'(t, x).
\]

1.7 Extensions

Many important issues will not be addressed in these lectures. This should not reflect on the lack of challenges with these issues. It is only a matter of time and space (and may be of taste and competence of the lecturer!) we list some of them for the sake of completeness. References are given in the Notes & Complements at the end of the chapter.

Tax Issues

Tax considerations may very much change the attractiveness of some issues. Here is a simple example. As we already mentioned, the income from coupon interest payments on Treasury notes and bonds are exempt from state income taxes. Continuous compounding is a reasonable model for zero coupon bonds because they automatically reinvest the interest earnings at the rate the bond was originally bought. This feature is very attractive to some investors, except for the fact that the IRS (our friendly Internal Revenue Service) requires some bond owners to report these earned interests (though not paid.) This explains in part why zero coupons bonds and STRIPS are often held by institutional investors and accounts exempt from federal income tax. They include pension funds and individual retirement accounts such as IRA’s and Koegh plans.
Chapter 1 The Term Structure of Interest Rates: A Crash Course

Municipal Bonds

Municipal bond is a general terminology to cover debt securities issued by states, cities, townships, counties, US Territories and their agencies. The interest income of these securities were exempt of federal taxes up until the Tax Reform Act (RFA for short) of 1986. This tax advantage was one of the attractive features which made the municipal bonds (munis for short) very popular. If the interest income of all the securities issued before 1986 remain tax exempt, the situation is more complex for the securities issued after that date. The primary offerings of municipal issues are usually underwritten by specialized brokerage firms. Even though instances of default have not been plentiful, several high profile events have given publicity to the credit risk associated with the municipal securities: we shall mention only the 1975 default of the city of New York defaulting on a note issue, and the highly publicized bankruptcy of Orange County in 1994. As part of the information given to the potential security buyers, municipal bond issuers hire a rating agency or even two (the most popular are S&P and Moody’s) to rate each bond issue. On the top of that, some issuers enter in a contract with an insurance company which will pay interest and principal in case of default of the issuer. Although they are regarded as generally safe, municipal bonds carry a risk, and the buyer of these securities are rewarded by a yield which is higher than the yield of a Treasury security with the same features. This difference in yield is called the yield spread over treasury. It is expressed in basis points, and prices of municipal bonds are most often quoted by their spread over Treasury.

Index Linked Bonds

Index linked bonds were created in an attempt to guarantee real returns and protect the cash flows from inflation. They are bonds with coupon payments and/or principal amounts which are tied to a particular price index. There are four types of index linked bonds.

- **indexed principal bonds** for which both coupons and principal are adjusted for inflation
- **indexed coupon bonds** for which only the coupons are adjusted for inflation
- **zero coupon bonds** which pay no coupon but for which the principal is adjusted for inflation
- **indexed annuity bonds** which pay inflation adjusted coupons and no principal on redemption.

The most common index used is the Consumer Price Index (CPI for short.) These issues seem to be more popular in Europe than in the US. Figure 1.5 shows how inflation indexed Treasury securities are quoted daily in the Wall Street Journal.

The first column gives the rate while the second column gives the month and year of maturity. The third column gives the bid and asked prices, while the fourth column gives
1.7 Extensions

Figure 1.5. Wall Street Journal inflation indexed Treasury securities as quoted on December 22, 2000.

the change since the quote of the last trading day. The fifth column gives the yield to maturity on the accrued principal as given in the last column.

**Corporate Bonds**

Corporations raise funds in a number of ways. Short term debts (typically less than five years) are handled via bank loans. For longer periods, commercial banks are reluctant to be the source of funds and corporations usually use bond offerings to gain access to capital. As for municipal bonds, each issue is rated by S&P and/or Moody’s, and the initial rating is a determining factor in the success of the offering. These ratings are updated periodically, usually every six months. They are the main source of information buyers and potential buyers use to quantify the credit risk associated with these bonds. For this reason, they are determining factors in the values of the bonds, and a change in rating is usually accompanied by a change in the spread over Treasury. Bond issues with poor ratings are called non-investment grade bonds or junk bonds. Their spread over Treasury is usually relatively high, and for this reason they are also called high yield bonds. Bond issues with the best ratings are safer; they are called investment grade bonds and their spread over Treasury is smaller.

Figure ?? shows a comparison of several yield curves, and so doing, gives an indication of the yield spread due to credit risk. Our source is again the Wall Street Journal of December 22, 2000.

The indenture of a corporate bond can be extremely involved. Indeed, some corporate
bonds are callable (some Treasury issues do have this feature too), other are convertible. Callable bonds give the issuer the option to recall the bond under some conditions. Convertible bonds give the buyer the option to convert under some conditions, the debt into a specific number of shares of the company stock. All these features make the fair pricing of the issues the more difficult.

**Asset Backed Securities**

Once mortgage loans are made, individual mortgages are pooled together into large bundles which are securitized in the form of bond issues backed by the interest income of the mortgages. Prepayments and default risks are the main factors in pricing of these securities. But the success of this market has encouraged the securitization of many other risky future incomes, ranging from catastrophic risk due to natural disasters such as earthquakes and hurricanes, to intellectual property such as the famous example of the Bowie bond issued by the rock star borrowing on the future cash flow expected from its rights and record sales. As before, the complexity of the indenture of these bonds is a challenge for the modellers trying to price these issues.
Chapter 2
Statistical Estimation of the Term Structure

The previous chapter illustrated clearly the duality between continuous time and discrete time modeling. Data comprise real numbers corresponding to measurements taking place at discrete times, and the choice of a continuous time model is a modeling decision. In other words, we may want to imagine that the data are observations taking place at specific times of continuously evolving quantities. In this way, the mathematics of functional analysis can be used to define and study the models, while the tools of parametric and nonparametric statistics can be brought to bear to fit the models and validate or invalidate them with the tools of statistical inference. This chapter is a first step in this direction. It introduces some of the function spaces in which the mathematical models could be set, and it presents some of the statistical techniques used in practice to infer continuous time yield and forward curves from discrete time observations.

2.1 Function Spaces for Yield and Forward Curves

In this first section, we give a quick review of some of the function spaces used recently in the analysis of the forward curve. See the Notes & Complements at the end of the chapter for bibliographic references. At this stage of the lectures, the choice of such a space can only be made on the discussion of the previous chapter and the numerical examples described therein. We shall see later that there are serious restrictions imposed by the (stochastic) dynamics of the curves. Indeed, as time evolves (for example from one day to the next) the curve is subject to a combination of a backward shift in time, and (random) perturbations. Mathematically speaking, this means that the elements of a reasonable function space should be essentially differentiable, or in other words, that such a space needs to support a reasonable extension of the first derivation operator. By this we mean the existence of a densely defined closed operator $A$ whose domain contains enough smooth functions (if they could form a core that would be great) and which coincides with the first derivative (i.e. $Af = f'$) on these smooth functions. We will see that a convenient way to define such an unbounded operator is to define first a strongly continuous semigroup of
operators implementing the shift operators, as defined for example as in (??), and then to
identify its infinitesimal generator to the desired operator $A$.

This discussion is presumably premature at this stage. It is only purpose at this stage is
to justify some of the choices made below. The need for these choices will become clear
when we discuss the details of the stochastic models in Chapter ??.

Example 2.1

The first example we give is the real separable Hilbert space $F = L^2([0, 1])$ which was
proposed as a model for the fluctuations of the yield curve around a (straight) baseline
joining the short and the long interest rates. Even though this Hilbert space is very well
understood from the point of view of functional analysis, the possible lack of smoothness
of most of its elements will be one of the reasons we shall not use it in these lectures.

Example 2.2

The analysis of the invariant measures for a finitely many factor HJM model in Musiela’s
notation was attempted in [87] when the state space $F$ is one of the Sobolev’s spaces:

$$H^1_\gamma = \{ f \in L^2(\mathbb{R}_+, e^{-\gamma x} dx); f' \in L^2(\mathbb{R}_+, e^{-\gamma x} dx) \}$$

(2.1)
equipped with the norm:

$$\| f \|_{\gamma}^2 = \int_0^\infty |f(x)|^2 e^{-\gamma x} dx + \int_0^\infty |f'(x)|^2 e^{-\gamma x} dx$$

(2.2)

where $\gamma \geq 0$ is a specific real parameter. One can use formula (??) to define the shift
operators $S_t$ on these spaces $F$, and it is not difficult to see that they form a strongly con-
tinuous semigroup of operators. The infinitesimal generator $A$ can be used as a reasonable
extension of the first derivative operator.

$F = H^1_{\gamma}$ is presumably too small a state space for the evolution of forward curves
when $\gamma = 0$. Indeed, it does not contain any non zero constant function, or any function
converging toward a non-zero limit at infinity. Indeed, its elements are functions which
converge in an average sense to 0 at infinity.

On the other hand, $H^1_{\gamma}$ is presumably too large a state space when $\gamma > 0$. Indeed, its
elements and their derivatives can be very large at infinity (as long as they can be tamed by
the negative exponential weight.) A sign of this shortcoming is given by the fact that there
exist probability measures on this space which are not concentrated on constant functions
and which are nevertheless invariant under the action of the shifts $S_t$. Take for example a
continuous function with period $x_p$, and let us consider the uniform probability on the set
of translates by $x \in [0, x_p]$ of the original function. This probability measure is invariant
under the shifts $S_t$.

Example 2.3
2.1 Function Spaces for Yield and Forward Curves

The following space \( F = H_w \) was used in [45] as a state space for the forward curve in a generalized HJM model in the Musiela’s notation.

\[
H_w = \{ f \in L^1_{\text{loc}}(\mathbb{R}_+, dx); \text{f is absolutely continuous and } \int_0^\infty f'(x)^2 w(x)dx < +\infty \}
\]

(2.3)

where the weight function \( w \) is any nondecreasing continuously differentiable function from \( \mathbb{R}_+ \) onto \([1, \infty)\) such that \( w^{-1/3} \) is integrable. Notice that fundamental theorem of calculus applies to the elements of \( H_w \) and we have:

\[
f(x) = f(0) + \int_0^x f'(y)dy.
\]

From this one easily sees that the elements of \( H_w \) are continuous functions (hence defined everywhere) and the evaluation functionals:

\[
\delta_x : H_w \ni f \mapsto \delta_x(f) = f(x)
\]

make sense. The space \( H_w \) is a Hilbert space for the norm:

\[
\|f\|^2 = |f(0)|^2 + \int_0^\infty |f'(x)|^2 w(x)dx
\]

\{S_t; t \geq 0\} as defined by (??) for all \( f \in H_w \), and \( t \geq 0 \) and \( x \in [0, \infty) \), is a strongly continuous semigroup of bounded operators on \( H_w \). Let us see first that each \( S_t \) so defined is a bounded operator on \( H_w \). Using again the fact \( f(x) = f(0) + \int_0^x f'(y)dy \), the Cauchy-Schwartz inequality, and the monotonocity of \( w \) we get:

\[
\|S_t f\|^2 = \|[S_t f](0)\|^2 + \int_0^\infty |[S_t f]'(x)|^2 w(x)dx
\]

\[
= |f(t)|^2 + \int_0^\infty |f'(x + t)|^2 w(x)dx
\]

\[
= \left| f(0) + \int_0^t f'(y)dy \right|^2 + \int_0^\infty |f'(x + t)|^2 w(x)dx
\]

\[
\leq 2|f(0)|^2 + 2 \left( \int_0^t |f'(y)|dy \right)^2 + \int_0^\infty |f'(x)|^2 w(x)dx
\]

\[
\leq 2|f(0)|^2 + 2 \left( \int_0^t |f'(y)|^2 w(y)dy \right)^2 \left( \int_0^t w(y)^{-1}dy \right)
\]

\[
+ \int_0^\infty |f'(x)|^2 w(x)dx
\]

\[
\leq c\|f\|^2
\]

for some constant \( c > 0 \).
Example 2.4

Our last example is an attempt to recast the main ideas behind the introduction of the space described in the previous example, in a framework which includes spaces of functions defined on a fixed bounded time interval. Indeed, the previous example, cannot accommodate a model where we assume that there is an overall upper bound for the times to maturities of the bonds we consider. This assumption is in fact quite realistic if we restrict ourselves to Treasury issues. After all, we rarely encounter bonds with time to maturity greater than 30 years.

2.2 The Effective Dimension of the Space of Yield Curves

Our first application concerns the markets of fixed income securities introduced in Section ?? . We shall use freely the definitions and notation from this section. As we have seen, the term structure of interest rates is conveniently captured by the daily changes in the yield curve. The dimension of the space of all the possible yield curves is presumably very high, potentially infinite. But it is quite sensible to try to approximate these curves by functions from a class chosen in a parsimonious way. We already did just that when we used the Nelson-Siegel and the Nelson-Siegel-Svensson families in Subsection ?? . Without any a priori choice of the type of functions to be used to approximate the yield curve, PCA can be used to extract one by one, the components responsible for the variations in the data.

As in Section ?? and Subsection ?? , we assume that the bonds used are default free, that there are no embedded options, no call or convertibility features, and we ignore the effects of taxes and transaction costs.

2.2.1 PCA of the Yield Curve

For the purpose of illustration, we use data on the US yield curve as provided by BIS. As we already discussed, these data are the result of a nonparametric processing (smoothing spline regression to be specific) of the raw data. The details will be given in Subsection ?? of Chapter ?? , but for the time being, we shall ignore the possible effects of this pre-processing of the raw data. The data are imported into an S-object named us.bis.yield which gives, for each of the 1352 successive trading days following January 3rd 1995, the yields on the US Treasury bonds for times to maturity

\[ x = 0, 1, 2, 3, 4, 5, 5.5, 6.5, 7.5, 8.5, 9.5 \text{ months.} \]

```r
> dim(us.bis.yield)
[1] 1352 11
> us.bis.yield.pca <- princomp(us.bis.yield)
> plot(us.bis.yield.pca)
```
2.2 The Effective Dimension of the Space of Yield Curves

Figure 2.1. Proportions of the variance explained by the components of the PCA of the daily changes in the US yield.

```
[1] 0.700000 1.900000 3.100000 4.300000 5.500000
> title("Proportions of the Variance Explained by the Components")
> X <- c(0,1,2,3,4,5,5.5,6.5,7.5,8.5,9.5)
> par(mfrow=c(2,2))
> plot(X,us.bis.yield.pca$loadings[,1],ylim=c(-.7,.7))
> lines(X,us.bis.yield.pca$loadings[,1])
> plot(X,us.bis.yield.pca$loadings[,2],ylim=c(-.7,.7))
> lines(X,us.bis.yield.pca$loadings[,2])
> plot(X,us.bis.yield.pca$loadings[,3],ylim=c(-.7,.7))
> lines(X,us.bis.yield.pca$loadings[,3])
> plot(X,us.bis.yield.pca$loadings[,4],ylim=c(-.7,.7))
> lines(X,us.bis.yield.pca$loadings[,4])
> par(mfrow=c(1,1))
> title("First Four Loadings of the US Yield Curves")
```

Figure 2.1 gives the proportions of the variation explained by the various components. The first three eigenvectors of the covariance matrix (the so-called loadings) explain 99.9% of the total variation in the data. This suggests strongly that the effective dimension of the space of yield curves could be three. In other words, any of the yield curves from this period can be approximated by a linear combination of the first three loadings, the relative error being very small. Figure 2.2 gives the plots of the first four loadings.
Chapter 2  Statistical Estimation of the Term Structure

Figure 2.2. From left to right and top to bottom, sequential plots of the first four US yield loadings.
2.3 The Example of the Swap Rate Curves

The first loading is essentially flat, so a component on this loading will essentially represent the average yield over the maturities. Because of the monotone and increasing nature of the second loading, the second component measures the upward trend (if the component is positive and the downward trend otherwise) in the yield. The shape of the third loading suggests that the third component captures the curvature of the yield curve. Finally, the shape of the fourth loading does not seem to have an obvious interpretation. It is mostly noise (remember that most of the variations in the yield curve are explained by the first three components.) These features are very typical, and they should be expected in most PCA’s of the term structure of interest rates.

The fact that the first three components capture so much of the yield curve may seem strange when compared to the fact that some estimation methods which we discussed use parametric families with more than three parameters! There is no contradiction there. Indeed, for the sake of illustration, we limited the analysis of this section to the first part of the yield curve. Restricting ourselves to short maturities makes it easier to capture all the features of the yield curve in a small number of functions with a clear interpretation.

2.3 The Example of the Swap Rate Curves

Swap contracts have been traded publicly since 1981. As of today, they are the most popular fixed income derivatives. Because of this popularity, the swap markets are extremely liquid, and as a consequence, they can be used to hedge interest rate risk of fixed income portfolios at a low cost.

2.3.1 Swap Contracts

As implied by its name, a swap contract obligates two parties to exchange (or swap) some specified cash flows at agreed upon times. The most common swap contracts are interest rate swaps. In such a contract, one party, say counterparty A, agrees to make interest payments determined by an instrument $P_A$ (say, a 30 year US Treasury bond rate), while the other party, say counterparty B, agrees to make interest payments determined by another instrument $P_B$ (say, the London Interbank Offer Rate – LIBOR for short) Even though there are many variants of swap contracts, in a typical contract, the principal on which counterparty A makes interest payments is equal to the principal on which counterparty B makes interest payments. Also, the payment schedules are identical and periodic, the payment frequency being quarterly, semi-annualy, . . . .

It should be clear from the above discussion that a swap contract is equivalent to a portfolio of forward contracts, but we shall not use this feature here. In this section, we shall restrict ourselves to the so-called plain vanilla contracts involving a fixed interest rate and the 3 or 6 months LIBOR rate.
2.3.2 A Price Formula for a Plain Vanilla Swap

Let us denote by \( X \) the common principal, by \( R \) the fixed interest rate on which the swap is written, and by \( T_1, T_2, \ldots, T_m \) the dates after the current date \( t \), at which the interest rate payments are scheduled. On each payment date \( T_j \), the variable interest rate used to compute the payment at this time is taken from the period \([T_{j-1}, T_j)\), so that the floating interest payment for this period will be \( X(T_j - T_{j-1})r(t, T_{j-1}) \), while the fixed interest payment for the same period will by \( X(T_j-T_{j-1})R \). Using the discount factors to compute the present value of the cashflows, we get:

\[
P_{\text{swap}} = X \sum_{j=1}^{m} (T_j - T_{j-1}) (r(t, T_{j-1}) - R) d(t, T_j)
\]

where we used the convention \( T_0 = t \). Notice that, if we were to add a payment of the principal \( X \) at time \( T_m \), then the cashflows of the swap would be identical to the cashflows generated by a portfolio long a (fixed rate) coupon bearing bond and short a floating rate bond with the same face value.

The valuation problem for a swap is solved by computing the difference of the floating rate bond and the fixed coupon bond. After simple algebraic manipulations we get:

\[
P_{\text{swap}}(t, T_m) = X \left( 1 - \left[ P(t, T_m) + R \sum_{j=1}^{m} (T_j - T_{j-1}) P(t, T_j) \right] \right)
\]

where we used the standard notation \( P(t, T) \) for the price at time \( t \) of a riskless zero coupon bond with maturity date \( T \) and nominal value 1.

2.3.3 The Swap Rate Curve

On any given day \( t \), the swap rate \( R_{\text{swap}}(t, T) \) with maturity \( T = T_m \) is the unique value of the fixed rate \( r \), which, once injected in formula (2.5) makes the swap price equal to 0. In other words, the swap rate is the value of the fixed interest rate for which the counterparties will agree to enter the swap contract without paying or receiving a premium. This swap rate is obtained by solving for \( r \) the equation obtained by setting \( P_{\text{swap}}(t, T) = 0 \) in formula (2.5). This gives:

\[
R_{\text{swap}}(t, T_m) = \frac{1 - P(t, T_m)}{\sum_{j=1}^{m} (T_j - T_{j-1}) P(t, T_j)}.
\]

Notice that in practice, the interest payments are regularly distributed over time (in other words all the time intervals \( T_j - T_{j-1} \) are all equal), and for this reason, one uses a parameter \( \omega \) for the frequency of the payments.
2.3 The Example of the Swap Rate Curves

2.3.4 PCA of the Swap Rates

Our second application of principal component analysis concerns the features of the swap rates curves described above. As before, we denote by $M$ the dimension of the vectors. We use data downloaded from Data Stream. It is quite likely that the raw data have been processed, but we are not quite sure what kind of manipulation is performed by Data Stream so for the purpose of this illustration, we shall ignore the possible effects of the pre-processing of the data. In this example, the day $t$ labels the rows of the data matrix. The latter has $M = 15$ columns, containing the swap rates with maturities $T$ conveniently labeled by the times to maturity $x = T - t$ which have the values $1, 2, \cdots, 10, 12, 15, 20, 25, 30$ years in the present situation. We collected these data for each day $t$ of the period from May 1998 to March 2000, and we computed the covariance matrix of the daily changes. We rearranged these numerical values in a matrix $R = [r_{i,j}]_{i=1,\cdots,N, j=1,\cdots,M}$. Here, the index $j$ stands for the time to maturity while the index $i$ codes the day the curve is observed.

If the data $R$ on the swap rates is contained in the $S$ object `swap`, the PCA can be performed in Splus with the command:

```r
> dim(swap)
[1] 496 15
> swap.pca <- princomp(swap)
> plot(swap.pca)
```

```
> title("Proportions of the Variance Explained by the Components")
> YEARS <- c(1,2,3,4,5,6,7,8,9,10,12,15,20,25,30)
> par(mfrow=c(2,2))
> plot(YEARS, swap.pca$loadings[,1], ylim=c(-.6,.6))
> lines(YEARS, swap.pca$loadings[,1])
> plot(YEARS, swap.pca$loadings[,2], ylim=c(-.6,.6))
> lines(YEARS, swap.pca$loadings[,2])
> plot(YEARS, swap.pca$loadings[,3], ylim=c(-.6,.6))
> lines(YEARS, swap.pca$loadings[,3])
> plot(YEARS, swap.pca$loadings[,4], ylim=c(-.6,.6))
> lines(YEARS, swap.pca$loadings[,4])
> par(mfrow=c(1,1))
> title("First Four Loadings of the Swap Rates")
```

Figure 2.3 gives the proportions of the variation explained by the various components while Figure 2.4 gives the plots of the first four eigenvectors.

It is clear from Figure 2.4 that the first component is capturing the overall level of the swap rate, while the second component measures an upward trend in the rate. The third component captures the curvature of the rate curve, while the fourth component does not
Figure 2.3. Proportions of the variance explained by the components of the PCA of the daily changes in the swap rates for the period from May 1998 to March 2000.

Figure 2.4. From left to right and top to bottom, sequential plots of the eigenvectors (loadings) corresponding to the 4 largest eigenvalues. Notice that we changed the scale of the horizontal axis to reflect the actual times to maturity.
2.4 Yield Curve Estimation

This section reviews some of the methods of yield curve estimation used by the market makers and the central banks which report to the Bank for International Settlements (BIS for short.) Except for the U.S. and Japan which use nonparametric smoothing techniques based on splines, most central banks use parametric estimation methods to infer smooth curves from finitely many discrete values. Parametric estimation is appropriate if the set of yield curves can be parametrized by a (small) finite number of parameters. Mathematically speaking this means that, despite the fact that we recast the set of all possible curves in an infinite dimensional space, the set of all the actual curves which do occur in real life is a finite dimensional (possibly nonlinear) manifold in this infinite dimensional function space. The use of parametric estimation methods is justified by the principal components analysis performed earlier. Indeed, we showed that the effective dimension of the space of yield curves is low, and consequently, a small number of parameters should be enough to describe the elements of this space. Moreover, another advantage of the parametric approach is the fact that one can estimate the term structure of interest rates by choosing to estimate first the par yield curves, or the spot rate curves, or the forward rate curves, or even the discount factor curves as functions of the maturity. Indeed, which one of these quantities is estimated first is irrelevant: once the choice of a set of curves and of their parametrization are made, the parameters estimated from the observations, together with the functional form of the curves, can be used to derive estimates of the other sets of curves. We shall most often parametrize the set of forward rate curves, and derive formulae for the other curves (yields, spot rates, discount factors, . . .) by means of the relationships made explicit in the crash course of the previous section.

On each given day, say \( t \), one uses the available values of the discount factors to produce a curve \( x \mapsto f(t, x) \) for the instantaneous forward rates as functions of the time to maturity \( x \). For the sake of notation convenience, we shall drop the reference to the present \( t \) in most of our discussions below. In this section we introduce some of the most popular methods used to solve this problem. They are based on the fitting of a parametric family of curves to the data.

The estimation of the discount factor curve would be an easy problem, if we had observations of zero-coupon bond prices. Unfortunately, these instruments have maturities of less than two years: fixed income securities of longer maturities have coupons. As a
Chapter 2 Statistical Estimation of the Term Structure

consequence, the calibration procedures will be based on observations of coupon bearing bond prices.

2.4.1 Parametric Estimation Procedures

The Nelson-Siegel Family

This family is parametrized by a 4-dimensional parameter $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$. It is defined by:

$$f_{NS}(x, \theta) = \theta_1 + (\theta_2 + \theta_3 x) e^{-x/\theta_4}$$

(2.7)

where $\theta_4$ is assumed to be strictly positive, and as a consequence, the parameter $\theta_1$, which is also assumed to be strictly positive, gives the asymptotic value of the forward rate. The value $\theta_1 + \theta_2$ gives the forward rate today, i.e. the starting value of the forward curve. Since this value $f(t, 0)$ has the interpretation of the short interest rate $r_t$, it is also required to be positive. The remaining parameters $\theta_3$ and $\theta_4$ are responsible for the so-called hump. This hump does exist when $\theta_3 > 0$ but it is in fact a dip when $\theta_3 < 0$. The magnitude of this hump/dip is a function of the size of the absolute value of $\theta_3$ while $\theta_3$ and $\theta_4$ conspire to force the location along the maturity axis of this hump/dip. Once the four parameters have been estimated, formulae for the discount factor and the zero-coupon yield (or spot interest rate) can be obtained by plain integration from formulae (1.9) and (1.10) respectively. We get:

$$d_{NS}(x, \theta) = \exp \left[ -\theta_1 x + \left( \theta_4 (\theta_2 + \theta_3 x) + \theta_3 \theta_4 x \right) e^{-x/\theta_4} \right]$$

(2.8)

and

$$r_{NS}(x, \theta) = -\theta_1 + \left( \frac{\theta_4 (\theta_2 + \theta_3 x)}{x} + \theta_3 \theta_4 \right) e^{-x/\theta_4}$$

(2.9)

These formulae are used in countries such as Finland and Italy to produce yield curves.

The Svensson Family

To improve the flexibility of the curves and the fit, Svensson proposed a natural extension to the Nelson-Siegel’s family by adding an extra exponential term which can produce a second hump/dip. This extra flexibility comes at the cost of two extra parameters which have to be estimated. The Svensson family is generated by mixtures of exponential functions of the Nelson-Siegel type. To be specific, the Svensson family is parametrized by a 6-dimensional parameter $\theta$, and defined by:

$$f_S(x, \theta) = \theta_1 + (\theta_2 + \theta_3 x) e^{-\theta_4 x} + \theta_5 x e^{-\theta_6 x}$$

(2.10)
2.4 Yield Curve Estimation

As before, once the parameters are estimated, the zero-coupon yield curve can be estimated by plain integration of (2.10). We get:

\[
r_S(x, \theta) = \theta_1 - \frac{\theta_1 \theta_4}{x} (1 - e^{-\theta_4 x}) + \frac{\theta_2}{\theta_4} \left[ \frac{1}{\theta_4 x} (1 - e^{-\theta_4 x}) - e^{-\theta_4 x} \right] + \frac{\theta_5}{\theta_6} \left[ \frac{1}{\theta_6 x} (1 - e^{-\theta_6 x}) - e^{-\theta_6 x} \right].
\] (2.11)

The Svensson family is used in many countries including Canada, Germany, France and the UK.

2.4.2 Practical Implementation

BIS provides information on the methodologies used by the major central banks to produce yield curves. It also provides samples of curve estimates. We shall use several of these estimates as illustrations and points of comparison for the experiments we run in the next subsection.

Description of the Available Data

On any given day \( t \), financial data services provide, for a certain number of bond issues, the times to maturity \( x_j = T_j - t \), the coupon payments and their frequencies, and various precomputed quantities. We used data from Data Stream. For the purpose of illustration, we chose to collect data on German bonds. These instruments are very liquid and according to BIS, the Deutsche Bundesbank uses the Svensson’s extension of the Nelson-Siegel family to produce yield curves. As an added bonus, the coupons on the instruments we chose are paid annually.

The Actual Fitting Procedure

Denoting by \( B_j \) the bond prices available on a given day, say \( t \), and denoting by \( B_j(\theta) \) the prices one would obtain using formula (1.4) with discount factors (or equivalently zero coupon yields) given by formula (2.8) or (2.11) above, the term structure estimation procedure boils down to finding the vector \( \theta \) of parameters which minimizes the quadratic loss function:

\[
L(\theta) = \sum_j w_j |B_j - B_j(\theta)|^2
\] (2.12)

where the weights \( w_j \)'s are chosen as a function of the duration (??) and the yields to maturity of the \( j \)-th bond. The dependence of the loss function upon the parameters \( \theta \) appears to be complex and extremely nonlinear. Fitting the parameters, i.e. finding the \( \theta \)'s minimizing \( L(\theta) \), depends upon delicate optimization procedures which can be very unstable and computer intensive. We give sample results below.
Chapter 2 Statistical Estimation of the Term Structure

Remarks

⋄ Many central banks do not use the full spectrum of available times to maturity. Indeed, the prices of many short term bonds are very often influenced by liquidity problems. For this reason, they are often excluded from the computation of the parameters. For example both the Bank of Canada, the Bank of England and the Deutsche Bundesbank consider only bonds with a remaining time-to-maturity above three months. The French central bank also filter out the short term instruments.

⋄ Even though it appears less general, the Nelson-Siegel family is often preferred to its Svensson relative. Being of a smaller dimension, the model is more robust and less unstable. This is especially true for countries with a relatively small number of issues. Finland is one of them. Spain and Italy are other countries using the original Nelson-Siegel family for stability reasons.

⋄ The bid-ask spread is another form of illiquidity. Most central banks choose the midpoint of the bid-ask interval for the value of $B_j$. The Banque de France does just that for most of the bonds, but it also uses the last quote for some of them. Suspicious that the influence of the bid-ask spread could overwhelm the estimation procedure, the Finnish central bank uses a loss function which is equal to the sum of squares of errors where the individual errors are defined as the distance from $B(j, \theta)$ to the the bid-ask interval (this error being obviously 0 when $B(j, \theta)$ is inside this interval.)

⋄ It is fair to assume that most central banks use accrued interests and clean prices to fit a curve to the bond prices. This practice is advocated in official documents of the Bank of England and the US Treasury.

⋄ Some of the countries relying on the Svensson family, fit first a Nelson-Siegel family to their data. Once this 4-dimensional optimization problem is solved, they use the argument they found, together with two other values for $\theta_5$ and $\theta_6$ (often 0 and 1), as initial values for the minimization of the loss function for the Svensson family. And even then, these banks do opt for the Svensson family, only when the final $\theta_5$ is significantly different from 0 and $\theta_6$ is not too large! This procedure is used by Belgium, Canada and France.

2.4.3 Splus Experiments

For the sake of definiteness, we chose to work with German bond quotes available on May 17, 2000. After some minor reformatting and editing to remove the incomplete records, we imported the data into the data frame GermanB041700. Figure 2.5 shows the variables we kept for our analysis.

We then write an Splus function to compute the price $B_{NS}(j, \theta)$ given to a bond by the Nelson-Siegel parametric formula.

We can now compute and minimize the loss (2.12) in the case of the Nelson-Siegel
2.4 Yield Curve Estimation

Figure 2.5. German bond quotes imported into Splus for the purpose of testing the nonlinear fit of the Nelson-Siegel family.

parametrization. We shall denote it by $L_{NS}(\theta)$. It is given by the formula:

$$L_{NS}(\theta) = \sum_{j} w_j |B_j - B_{NS}(j, \theta)|^2$$  \hspace{1cm} (2.13)

Here is the Splus code of a short function to compute the price of a bond from its coupon rate COUPON, the accrued interest AI, the time to maturity LIFE given in years and the parameters THETA of the Nelson-Siegel family.

```splus
bns <- function(COUPON, AI, LIFE, X = 100, THETA = c(0.06,0,0,1)) {
  LL <- floor(1 + LIFE)
  DISCOUNT <- seq(to = LIFE, by = 1, length = LL)
  DISCOUNT <- exp(-THETA[1] * DISCOUNT + (TTT + TT * DISCOUNT) * exp(-DISCOUNT/THETA[4]))
  CF <- rep((COUPON * X)/100, LL)
  CF[LL] <- CF[LL] + X
  PRICE <- sum(CF * DISCOUNT) - AI
  PRICE
}
```

The parameters THETA[j] are obtained by minimizing the sum of square deviations (2.13). Since we do not know what kind of weights (if any) are used by the German central bank, we set $w_j = 1$. We use the Splus function ms to perform the minimization of the sum of square errors.

> GB.fit <-
ms( (Price-bns(Coupon,Accrud.Intrst,Life,THETA))^2, data=GermanB041700) > GB.fit
value: 8222.603 parameters:
   THETA1   THETA2   THETA3   THETA4
  0.06175748  6.127430  7.636384  1.452340
formula:  (Price - bns(Coupon, Accrud.Intrst, Life, T HETA))^2
47 observations
call:  ms(formula =  (Price - bns(Coupon, Accrud.Intrst, Life, THETA))^2, data = GermanB041700)

Remarks
The results reported above show an extreme variability of the estimates at the short end of the curve. This confirms the widely admitted fact that the term structure of interest rates is more difficult to estimate for short maturities. This is one of the reasons why many central banks do not provide estimates of the term structure for the left hand of the maturity spectrum.

All in all it seems clear that the various estimates are stable and reliable in the maturity range from one to ten years.

2.4.4 Nonparametric Estimation Procedures

given our newly acquired knowledge of nonparametric curve estimation, we revisit the problem of the estimation of the instantaneous forward interest rates and the yield curve, which was tackled in Section 2.4 by means of parametric methods.

A First Estimation of the Instantaneous Forward-Rate Curve

The first procedure we present was called iterative extraction by its inventors, but it is known on the street as the bootstrapping method. We warn the reader that this use of the word bootstrapping is more in tune with the everyday use of the word bootstrapping than with the statistical terminology.

We assume that the data at hand comprise coupon bearing bonds with maturity dates $T_1 < T_2 < \cdots < T_m$, and we search for a forward curve which is constant on the intervals $[T_j, T_{j+1})$. For the sake of simplicity we shall assume that $t = 0$. In other words, we postulate that:

$$f(0, T) = f_j \quad \text{for} \quad T_j < T \leq T_{j+1}$$

for a sequence $\{f_j\}$ to be determined recursively by calibration to the observed prices. Let us assume momentarily that $f_1, \cdots, f_j$ have already been determined and let us describe the procedure to identify $f_{j+1}$. If we denote by $X_{j+1}$ the principal of the $(j+1)$-th bond, by $\{t_{j+1,i}\}_i$ the sequence of coupon payment times, and by $C_{j+1,i} = c_{j+1}/n_y$ the
corresponding payment amounts (recall that we use the notation $c_j$ for the annual coupon rate, and $n_y$ for the number of coupon payments per year), then its price at time $t = 0$ can be obtained by discounting all the future cash flows associated with this bond:

$$B_{j+1} = \sum_{t_{j+1,i} \leq T_j} d(0, t_i) \frac{c_{j+1}X_{j+1}}{n_y} + d(0, T_j) \left( \sum_{T_j < t_{j+1,i} \leq T_{j+1}} e^{-\left( (T_{j+1}-T_j) f_{j+1} \right)} \frac{c_{j+1}X_{j+1}}{n_y} + e^{-\left( (t_{j+1,i}-T_j) f_{j+1} \right)} \right)$$

(2.14)

Notice that all the discount factors appearing in this formula are known since, for $T_k \leq t < T_{k+1}$ we have:

$$d(0, t) = \exp \left[ \sum_{h=1}^{k} (T_h - T_{h-1}) f_h + (t - T_k) f_{k+1} \right]$$

and all the forward rates are known if $k < j$. Consequently, rewriting (2.14) as:

$$\frac{B_{j+1} - \sum_{t_{j+1,i} \leq T_j} d(0, t_i) \frac{c_{j+1}X_{j+1}}{n_y}}{d(0, T_j)} = \frac{c_{j+1}X_{j+1}}{n_y} \sum_{T_j < t_{j+1,i} \leq T_{j+1}} e^{-\left( (T_{j+1}-T_j) f_{j+1} \right)} + e^{-\left( (t_{j+1,i}-T_j) f_{j+1} \right)}$$

(2.15)

and since the left hand side can be computed and the unknown forward rate $f_{j+1}$ appears only in the right hand side, this equation can be used to determine $f_{j+1}$ from the previously evaluated values $f_k$ for $k \leq j$.

**Remark**

Obviously, the forward curve produced by the bootstrapping method is discontinuous, since by construction, it jumps at all the input maturities. These jumps are the source of an artificial volatility which is only due to the method of estimation of the forward curve. This is the main shortcoming of this method of estimation. Several remedies have been proposed to alleviate this problem. The simplest one is to increase artificially the number of maturity dates $T_j$ to interpolate between the observed bond (or swap) prices. Another proposal is to add a smoothness penalty which will force the estimated curve to avoid jumps. This last method is in the spirit of the smoothing spline estimation method which we will discuss in Chapter ??.

**2.4.4.1 A Direct Application of Smoothing Splines**

Let us consider the following data contained in the Splus data frame USBN041700. These data comprise the quotes on April 17, 2000 of the outstanding US Treasury Notes...
and Bonds. Figure 2.6 gives the plot of the redemption yield as a function of the time to maturity, together with the plot of the smoothing spline. This plot was created with the following commands.

```r
> attach(USBN041700)
> plot(LIFE, INT.YIELD, main="Smoothing Spline for 04/17/00 Yields")
> lines(smoothing.spline(LIFE, INT.YIELD))
```

The (smooth) yield curve plotted in Figure 2.6 is unusual because it has two humps, but it is still regarded as a reasonable yield curve.

### 2.4.4.2 US and Japan Instantaneous Forward Rates

Even though we stated in Section ?? that the yield curves and the forward rate curves published by the US Federal Reserve and the Bank of Japan were computed using smoothing (cubic) splines, they are not produced in the simplistic approach described above. The instantaneous forward rate curves produced on a given day $t$ is the function $t \mapsto \varphi(t)$ which
minimizes the loss function:

\[ L_{JUS}(\varphi) = \sum_{i=1}^{n} w_{i} |P_{i} - P_{i}(\varphi)|^2 + \lambda \int |\varphi''(t)|^2 dt \]  \hspace{1cm} (2.16)

where \( \varphi''(t) \) stands for the second derivative of \( \varphi(t) \), where the \( P_{i} \)'s are the prices of the outstanding bonds and notes available on day \( t \), and the \( P_{i}(\varphi) \)'s are the prices one would get pricing the bonds and notes on the forward curve given by \( \varphi \). Recall that these prices are computed according to the theory presented in Section ??.

2.5 Notes & Complements

The space \( F \) described in Example 2.1 was used by Cont in [?] for the analysis of the fluctuations of the forward curve around a random line determined by the (jointly Markovian) dynamics of the short and long interest rates. The spaces \( F \) described in Example 2.2 were used by Vargiolu in [87] for the analysis of the invariant measures of the finitely many factor HJM dynamics in the Musiela notation. As far as we know, the space \( F \) of Example 2.3 was introduced by Filipovic in his Ph.D. [45]. We introduced Example 2.4 for the sake of having a space of forward curves defined on a bounded interval on which the action of the shift semigroup could still be defined.
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