Markovian Projection, Heston Model and Pricing of European Basket Options with Smile

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References

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European Options on geometric baskets

- Let us assume we want to find the value of an European call option on the basket $S = S_1 \cdot S_2$ where S_1, S_2 are the prices of two currencies in our domestic currency.
- We assume that each currency is driven by geometric Brownian motion.

$$dS_i = S_i(\mu_i dt + \sigma_i dW_i(t))$$

  with the correlation $dW_1 dW_2 = \rho dt$.
- Using Ito's product rule $dS = S_1 dS_2 + S_2 dS_1 + dS_1 dS_2$ it is easy to see that

$$dS = S\left((\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)dt + \sqrt{(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)}dW(t)\right)$$

- Hence we can price our European call option on S using the standard Black Scholes formula for European options.

Actually, the above argument generalizes to a (geometric basket) of n currencies given by $S = \prod_i S_i^{a_i}$ when $a_i \in \mathbb{R}_{>0}$.

European Options on arithmetic baskets

- However, often we are interested in arithmetic baskets defined by

$$S = S_1 + S_2.$$

- Assuming again geometric Brownian motion for S_1, S_2, and even setting $W_1(t) = W_2(t)$ we find

$$\begin{aligned} dS &= dS_1 + dS_2 = (S_1 + S_2)(\sigma_1 + \sigma_2)dW(t) \\ &+ S_1\mu_1 dt + S_2\mu_2 dt \end{aligned}$$

- Hence only in the special case of $\mu_1 = \mu_2 \equiv \mu$ we find

$$dS = S(\mu dt + (\sigma_1 + \sigma_2)dW(t))$$

  and can use the standard Black Scholes formula to price call options on S.
- We have just shown that in general the sum of lognormal random variables is not a lognormal random variable. Hence we need to find "good" analytic approximations or use numeric techniques to price our European call option.

Pricing using Moment Matching

- A classical method to find analytic solutions is moment matching.
- We observe that the price of a European call option $C(S_0, K, T) = E_0(S_T - K)$ depends only on the distribution of S at T.
- Our approximation is in the choice of distribution, we assume it is lognormal. Hence we need to find its first and second moment.
- For any lognormal random variable $X = \exp Y, \; Y \sim N(\mu, \sigma^2)$ the higher moments are given by $E[X^n] = \exp\left(n\mu + \frac{n}{2}\sigma^2\right)$.
- Using the fact that

$$\begin{aligned} E(S_T) &= E(S_{1T}) + E(S_{2T}) \\ E(S_T^2) &= E(S_{1T}^2) + E(S_{2T}^2) + 2E(S_{1T} \cdot S_{2T}) \end{aligned}$$

and that the process S_{1T}, S_{2T} and $S_{1T} \cdot S_{2T}$ are lognormal distributed, we can solve for σ and μ corresponding to S_T and again use Black Scholes formula to price the European call option.

Pricing via Markovian Projection

- Let us consider the general case of n currencies S_i, driven by geometric Brownian motion

$$dS_i = S_i(\mu_i dt + \sigma_i dW_i(t))$$

with correlation matrix $dW_i dW_j = \rho_{ij} dt$ and a European option on the basket with constant weights w_i

$$S = \sum_i w_i S_i$$

- We define a drift μ and a volatility σ

$$\mu(S_1, \ldots, S_n) = \frac{1}{S} \sum_i \mu_i S_i, \quad \sigma^2(S_1, \ldots, S_n) = \frac{1}{S^2} \sum_{ij} S_i S_j \sigma_i \sigma_j \rho_{ij}.$$

Using Lévy theorem, it is easy to see that

$$dW(t) = (S\sigma)^{-1} \sum_i S_i \sigma_i dW_i(t).$$

defines a Brownian motion ($dW(t)^2 = 1$).

Pricing via Markovian Projection

- Hence we can write the process for the basket as

$$dS = \mu S dt + \sigma S dW(t).$$

- However, despite its innocent appearance, this is not quite geometric Brownian motion. The drift and the volatility μ and σ are not constant, but stochastic processes which are not even adapted (they are not measurable with respect to the filtration generated by $W(t)$).
- We are looking for a process $S^*(t)$ given

$$dS^*(t) = \mu^*(t, S^*)S^* dt + \sigma^*(t, S^*)S^* dW(t)$$

which would give the same prices on European options as $S(t)$.
- Since the price of a European option on S with expiry T and strike K depends only on the one dimensional distribution of S at time T, it would be sufficient for S^* to have the same one dimensional distribution as S.
- Exactly this "Markovian projection" is the context of Gyongy's Lemma.

Gyongy Lemma 1986

- Let the process $X(t)$ be given by

$$dX(t) = \alpha(t)dt + \beta(t)dW(t), \tag{1}$$

where $\alpha(t)$, $\beta(t)$ are adapted bounded stochastic processes such that the SDE admits a unique solution.

- Define $a(t,x)$ and $b(t,x)$ by

$$a(t,x) = E(\alpha(t)|X(t)=x) \tag{2}$$

$$b(t,x) = E(\beta(t)^2|X(t)=x) \tag{3}$$

- Then the SDE

$$dY(t) = a(t, Y(t))dt + b(t, Y(t))dW(t) \tag{4}$$

with $Y(0) = X(0)$ admits a weak solution $Y(t)$ that has the same one-dimensional distributions as $X(t)$ for all t.

- Hence we can use $Y(t)$ to price our basket. Because of its importance we will give an outline of the proof of Lemma in the case $\alpha(t) = 0$.

Tanaka's formula

- Consider the function $c(x, K) = (x - K)^+$. We can take the derivative in the distributional sense and find

$$\begin{aligned}\partial_x c(x, K) &= 1_{(x>K)} \\ \partial_x^2 c(x, K) &= \delta(x - K) \\ \partial_k^2 c(x, K) &= \delta(x - K)\end{aligned}$$

- Tanaka's formula (a generalized Ito rule applicable to distributions) states that the differential of $(Z(t) - K)^+$ for a stochastic process $Z(t)$ is given by

$$d(Z(t) - K)^+ = 1_{(Z(t)>K)} dZ(t) + \frac{1}{2}\delta(Z - K) dZ^2(t)$$

- We assume that the process $Z(t)$ has no drift. Hence we find for the price of a European option

$$C(t, K) = E_0(Z(t) - K)^+) = (Z(0) - K)^+ + \frac{1}{2}\int_0^t E_0(\delta(Z(s) - K) dZ^2(s)).$$

Dupire's Formula

- Lets apply Tanaka's formula to the process $dY(t) = b(t, Y)dW(t)$.

$$\begin{aligned} C(t,K) &= E_0(Y(t)-K)^+) = (Y(0)-K)^+ + \frac{1}{2}\int_0^t E_0(\delta(Y(s)-K)dZ^2(s)) \\ &= (Y(0)-K)^+ + \frac{1}{2}\int_0^t \int \phi_s(y)\delta(Y(s)-K)b^2(s,y)ds \\ &= (Y(0)-K)^+ + \frac{1}{2}\int_0^t \phi_s(K)b^2(s,K)ds \end{aligned}$$

Here $\phi_s(y)$ denotes the density of $Y(s)$ which obeys

$$\begin{aligned} \partial_K^2 C(t,K) &= E_0(\partial_K^2(Y-K)^+) = \int \phi_t(y)\partial_K^2(y-K)^+ dy \\ &= \int \phi_t(y)\delta(y-K)dy \\ &= E_0(\delta(Y-K)) = \phi_t(K) \end{aligned}$$

Dupire's Formula

- Hence we find as the differential form of Tanaka's formula the Dupire's formula
$$\partial_t C(t,K) = \frac{1}{2}\partial_K^2 C(t,K) b^2(t,K).$$
- This formula shows that the local volatility $b(t,y)$ is determined by the European call prices for all strikes K.
- It also shows that the if we know the local volatility function $b(s,K)$ for all $s \in [0,t]$ we can determine the prices of European call options with expiry t uniquely (up to boundary conditions).

Proof: Final steps

- To finish the proof lets apply Tanaka's formula to the process $dX(t) = \beta(t)dW(t)$. We find

$$\begin{aligned}\partial_t C(t,K) &= \frac{1}{2}E_0(\delta(X-K)dX^2(t)) \\ &= E_0(\delta(X-K))E_0(dX^2(t)|X(t)=K) \\ &= \partial_K^2 C(t,K)E_0(dX^2(t)|X(t)=K).\end{aligned}$$

- Choosing the local volatility function

$$b(t,K) = E_0(dX^2(t)|X(t)=K),$$

implies that the process $X(t)$ and $Y(t)$ have the same prices for all European call options, and hence the same one dimensional distributions for all t.

- The hard work using the approach of "Markovian projection" to price European options one basket lays in the challenge of the explicit computation of the conditional expectation values.

Conditional expectation values

- Let start simple: Assume two normally distributed random variables $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$. It is easy to see that

$$E(X|Y) = EX + \frac{Covar(X,Y)}{Var(Y)}(Y - EY).$$

- This can be extended to a Gaussian approximation. Let's assume that the dynamics of $(S(t), \Sigma^2(t))$ can be written in the following form

$$dS(t) = S(t)dW(t), \quad d\Sigma^2(t) = \eta(t)dt + \epsilon(t)dB(t),$$

where $S(t), \eta(t)$ and $\epsilon(t)$ are adapted stochastic processes and $W(t), B(t)$ are both Brownian motions. Then the conditional expectation value can be approximated by

$$E(\Sigma^2(t)|S(t) = x) = \bar{\Sigma}^2(t) + r(t)(x - S_0).$$

with the corresponding moments, e.g. $\bar{\Sigma}^2(t) = \int_0^t (E\eta(s))ds$

Conditional expectation values

- The Gaussian approximation can be applied to $S(t) = \sum w_n S_n(t)$ where each asset $S_n(t)$ follows the process

$$dS_n(t) = \phi_n(S_n(t))dW_n(t).$$

The N Brownian motions are correlated via $dW_i \cdot dW_j = \rho_{ij} dt$. We assume that the volatility functions are linear,

$$\phi_n = p_n + q_n(S_n(t) - S_n(0)).$$

Using Gaussian approximation in computing $E(S_n(t) - S_n(0)|S(t) = x)$, the process $S(t)$ can be approximated via

$$dS(t) = \phi(S(t))dW(t)$$

where $\phi(x)$ is is such that

$$\phi(S(0)) = p \quad \phi(S(0))' = q$$

with appropriate constants.

- Restricting to the case $\phi_n(x) = x$, this provides a solution to our original problem of an arithmetic basket driven by n geometric Brownian motions.

Conditional expectation values

- In the case of non linear volatility functions $\phi(x)$, other approximations can be made. In particular, Avellaneda et al (2002) develops a heat-kernel approximation and saddle point method for the expectation value.
- Finally one could try to exploit the variance minimizing property of the conditional expectation value. Clearly, $E[X|Y]$ is Y measurable function. Actually, $E[X|Y]$ is the best Y measurable function, in the sense that it minimizes the functional

$$\chi = E((X - E[X|Y])^2).$$

by varying over all Y measurable functions. Choosing an appropriate ansatz for $E[X|Y]$, this could be solvable.

Heston model and its parameters

- We started forming baskets from processes following geometric Brownian motion.
- But currencies are not described by geometric Brownian motion, they admit "Smile", That is simply the fact that the implied volatility of European options depends on the strike K. A natural candidate to explain "Smile" is the Heston model. It is driven by the following SDE:

$$\begin{aligned} dS(t) &= \mu S dt + \sqrt{v(t)} S dz_1 \\ dv(t) &= \kappa(\theta - v(t))dt + \sigma \sqrt{v(t)} dz_2 \end{aligned}$$

  where the Brownian motions z_1 and z_2 are correlated via ρ.
- We note in particular, that the variance is driven by its own Brownian motion. That implies that it does not follow the spot process. This feature is driven by the volatility of volatility σ. When σ is zero, the volatility is deterministic and spot returns have normal distribution. Otherwise it creates fat tails in the spot return (raising far in and out of the money option prices and lowering near the money prices.

Heston model and its parameters

- The variance drifts towards a long run mean θ with the mean reversion speed κ. Hence an increase in θ increases the price of the option. The mean reversion speed determines how fast the variance process approaches this mean.
- The correlation parameter ρ positively affects the skewness of the spot returns. Intuitively, positive correlation results in high variance when the spot asset raises, hence this spreads the right tail of the probability density for the return. Conversely, the left tail is associated with low variance. In particular, it rises prices for out of the money call options. Negative correlation has the inverse effect.

Solutions to Heston model

- Following standard arguments, any price for a tradable asset $U(S, v, t)$ must obey the partial differential equation (short rate $r = 0$)

$$\frac{1}{2} v S^2 \frac{\partial^2 U}{\partial^2 S} + \rho \sigma v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2} v \sigma^2 \frac{\partial^2 U}{\partial^2 v} + \frac{\partial U}{\partial t} - \kappa(\theta - v(t)) \frac{\partial U}{\partial v} = 0$$

- A solution for European call option can be found using following strategy (Heston):
- Make an ansatz $C(S, v, t) = SP_1(S, v, t) - KP_2(S, v, t)$ as in standard Black Scholes (P_1 conditional expected value of spot given that option is in the money, P_2 probability of exercise of option)
- Obtain PDE for $P_i(S, v, t)$, and hence a PDE on its Fourier transform. $\tilde{P}_i(u, v, t)$.
- Make an ansatz $P_i(S, v, t) = \exp\left(C(u, t)\theta + D(u, t)v\right)$.
- Obtain ODE for $\tilde{P}_i(u, v, t)$ which can be solved explicitly.
- Obtain $P_i(S, v, t)$ via inverse Fourier transform.

Simulation of Heston process

- Recall the Heston process

$$\begin{aligned} dS(t) &= \mu S dt + \sqrt{v(t)} S dz_1 \\ dv(t) &= \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dz_2 \end{aligned}$$

- A simple Euler discretization of variance process

$$v_{i+1} = v_i - \kappa(\theta - v_i)\Delta t + \sigma\sqrt{v_i}\sqrt{\Delta t}Z$$

  with Z standard normal random variable may give raise to negative variance. Practical solution are absorbing assumption (if $v < 0$ then $v = 0$) or reflecting assumption (if $v < 0$ then $v = -v$). This requires huge numbers of time step for convergence.

- Feller condition: $\frac{2\lambda\theta}{\sigma^2} > 1$ then theoretically the variance stays positive (in $\Delta t \to 0$ limit). However, Feller condition with real market data often violated.
- Sampling from exact transition laws, since marginal distribution of v is known. This methods are very time consuming (Broadie-Kaya, Andersen).

Introduction shifted Heston model

- Instead of using the Heston model to describe the dynamics of our currency, we will use a shifted Heston model. There exists a analytic transformation between these models, hence the analytic solutions of the Heston model can be used.

$$\begin{aligned} dS(t) &= (1+(S(t)-S(0))\beta)\sqrt{z(t)}\lambda \cdot dW(t) \\ dz(t) &= a(1-z(t))dt + \sqrt{z(t)}\gamma dW(t),\ z(0)=1 \end{aligned}$$

- We represent N dimensional Brownian motion by $W(t)$, hence

$$\lambda \cdot dW(t) = \sum_N \lambda^i dW^i.$$

Note that the shifted Heston model has two natural limits, $\beta = 0$, the stock process is "normal", while $\beta = \frac{S(t)-1}{S(t)-S(0)}$, it is "lognormal".

Setup

- The initial process driven by $2n$ Brownian motions representing the basket is the weighted sum

$$S(t) = \sum_i w_i S_i(t)$$

of n shifted Heston models.

$$\begin{aligned} dS_i(t) &= (1 + \Delta S_i(t)\beta_i)\sqrt{z_i(t)}\lambda_i \cdot dW(t) \\ dz_i(t) &= a_i(1 - z_i(t))dt + \sqrt{z_i(t)}\gamma_i \cdot dW(t),\ z_i(0) = 1 \end{aligned}$$

- Our goal is to find an effective shifted Heston model

$$\begin{aligned} dS^*(t) &= (1 + \Delta S^*(t)\beta))\sqrt{z(t)}\sigma_H \cdot dW(t) \\ dz(t) &= \theta(t)(1 - z(t))dt + \sqrt{z(t)}\gamma_z \cdot dW(t),\ S^*(0) = 1,\ z_i(0) = 1 \end{aligned}$$

which represents the dynamic of our basket, such that European options on S^* have the same price as on S.

Generalized Gyongy Lemma

- Consider an N-dimensional (non-Markovian) process $x(t) = x_1(t), \ldots, x_N(t)$ with an SDE

$$dx_n(t) = \mu_n(t)dt + \sigma_n(t) \cdot dW(t)$$

The process x(t) can be mimicked with a Markovian N-dimensional process $x^*(t)$ with the same joint distributions for all components at fixed t. The process $x^*(t)$ satisfies the SDE

$$dx_n^*(t) = \mu_n^*(t, x^*(t))dt + \sigma_n^*(t, x^*) \cdot dW(t)$$

with

$$\begin{aligned} \mu_n^*(t, y) &= E[\mu_n(t)|x(t) = y] \\ \sigma_n^*(t, y) \cdot \sigma_m^*(t, y) &= E[\sigma_n(t) \cdot \sigma_m(t)|x(t) = y] \end{aligned}$$

Choice of process

- Having in mind the shifted Heston as the projected process, we write the SDE for the rate $S(t) = \lambda(t) \cdot dW(t)$ in the following form

$$dS(t) = (1 + \beta(t)\Delta S(t))\Lambda(t) \cdot dW(t)$$

Here $\Delta S(t) = S(t) - S(0)$, $\beta(t)$ is a deterministic function (determined later) and

$$\Lambda(t) = \frac{\lambda(t)}{(1 + \beta(t)\Delta S(t))}.$$

The second equation for the variance $V(t) = |\Lambda(t)|^2$,

$$dV(t) = \mu_V(t)dt + \sigma_V(t) \cdot dW(t)$$

This completes the SDEs for the non-Markovian pair $(S(t), V(t))$.

Projection to a Markovian process

- Applying the extension of Gyongy's Lemma to the process pair $(S(t), V(t))$

$$\begin{aligned} dS(t) &= (1+\beta(t)\Delta S(t))\Lambda(t)\cdot dW(t) \\ dV(t) &= \mu_V(t)dt + \sigma_V(t)\cdot dW(t) \end{aligned}$$

we find the Markovian pair $(S^*(t), V^*(t))$

$$\begin{aligned} dS^*(t) &= (1+\beta(t)\Delta S(t))\sigma_S^*(t; S^*, V^*)\cdot dW(t) \\ dV^*(t) &= \mu_V(t; S^*, V^*)dt + \sigma_V(t; S^*, V^*)\cdot dW(t) \end{aligned}$$

where

$$\begin{aligned} |\sigma_S^*(t; s, u)|^2 &= E[|\Lambda(t)|^2 | S(t) = s, V(t) = u] = u \\ |\sigma_V^*(t; s, u)|^2 &= E[|\sigma_V(t)|^2 | S(t) = s, V(t) = u] \\ \sigma_S^*(t; s, u)\cdot\sigma_V^*(t; s, u) &= E[\Lambda(t)\cdot\sigma_V(t) | S(t) = s, V(t) = u] \\ \mu_V(t; s, u) &= E[\mu_V(t) | S(t) = s, V(t) = u] \end{aligned}$$

Fixing the Markovian process

- To ensure that the Markovian process is closely related to the Heston process, we define the variance $V^*(t) = z(t)|\sigma_H(t)|^2$. Using this ansatz we find

$$
\begin{aligned}
dS^*(t) &= (1+\beta(t)\Delta S^*(t))\frac{\sqrt{V^*(t)}}{|\sigma_H(t)|}\sigma_H(t)\cdot dW(t) \\
dV^*(t) &= \left(V^*(t)\left((log|\sigma_H(t)|^2)' - \theta(t)\right) + \theta(t)|\sigma_H(t)|^2\right) dt \\
&+ |\sigma_H(t)|\sqrt{V^*(t)}\sigma_z(t)\cdot dW(t)
\end{aligned}
$$

In particular, the coefficients are given by

$$
\begin{aligned}
\mu_V(t;s,v) &= v\left((log|\sigma_H(t)|^2)' - \theta(t)\right) + \theta(t)|\sigma_H(t)|^2 \\
|\sigma_V^*(t;s,v)|^2 &= |\sigma_H(t)|^2 v|\sigma_z(t)|^2 \\
\sigma_S^*(t;s,v)\cdot\sigma_V^*(t;s,u) &= v\sigma_z(t)\cdot\sigma_H(t)
\end{aligned}
$$

Computing the coefficients of the Heston process

- Simultaneously minimizing the three regression functionals

$$\chi_1^2(t) = E\left[\left(\mu_V(t) - V(t)\left((log|\sigma_H(t)|^2)' - \theta(t)\right) + \theta(t)|\sigma_H(t)|^2\right)^2\right]$$

$$\chi_2^2(t) = E\left[\left(|\sigma_V(t)|^2 - |\sigma_H(t)|^2 V(t)|\sigma_z(t)|^2\right)^2\right]$$

$$\chi_3^2(t) = E\left[(\Lambda(t)\cdot\sigma_V(t;s,u) - V(t)\sigma_z(t)\cdot\sigma_H(t))^2\right]$$

determines the parameters for the shifted Heston (choose $\beta(t)$ to minimize projection defects).

$$|\sigma_H(t)|^2 = E[V(t)], \quad \rho(t) = \frac{E[V(t)\Lambda(t)\cdot\sigma_V(t)]}{\sqrt{E[V^2(t)]E[V(t)|\sigma_V(t)|^2]}}$$

$$|\theta(t)|^2 = (logE[V(t)])' - \frac{1}{2}(logE[\delta V^2(t)])' + \frac{E[|\sigma_V(t)|^2]}{2E[\delta V^2(t)]}$$

$$|\sigma_z(t)|^2 = \frac{E[V(t)|\sigma_V(t)|^2]}{E[V^2(t)]E[V(t)]}, \quad \delta V(t) = V(t) - E[V(t)]$$

Closed form solutions

- To obtain closed form solutions for the parameters for the shifted Heston one can assume that $S(t)$ follows a separable process, that is, its volatility function $\lambda(t)$ can be represented by a linear combination of several processes X_n which together form an n dimensional Markovian process.

$$dS(t) = \lambda(t) \cdot dW(t) = \sum_n X_n(t) a_n(t) \cdot dW(t),$$

where $a_n(t)$ are deterministic vector functions and $X_n(t)$ obey

$$dX_n(t) = \mu_n(t, X_k(t))dt + \sigma_n(t, X_k(t)) \cdot dW(t).$$

where the drift terms μ_n are of the second order in volatilities. Then closed form expressions $|\sigma_H(t)|, |\sigma_z(t)|, \theta(t)$ and $\rho(t)$ in the leading order in volatilities can be found. $\beta(t)$ must be found a solution to a linear ODE.

Explicit formula

- Recalling our setup, that we wanted project n Heston models

$$\begin{aligned} dS_i(t) &= (1 + \Delta S_i(t)\beta_i)\sqrt{z_i(t)}\lambda_i dW(t) \\ dz_i(t) &= a_i(1 - z_i(t))dt + \sqrt{z_i(t)}\gamma_i dW(t),\ z_i(0) = 1 \end{aligned}$$

driving our basket $S = \sum_i w_i S_i$ to one effective Heston model

$$\begin{aligned} dS^*(t) &= (1 + \Delta S^*(t)\beta))\sqrt{z(t)}\sigma_H dW(t) \\ dz(t) &= \theta(t)(1 - z(t))dt + \sqrt{z(t)}\gamma_z dW(t),\ S^*(0) = 1,\ z_i(0) = 1 \end{aligned}$$

and after defining a drift less processes $y_i = y_i(z_i)$ our basket can be approximated via a separable process and we can give the explicit formulas for the coefficients of the projected Heston model.

Explicit formula

$$
\begin{aligned}
\sigma_H &= \sum_i w_i \lambda_i \\
\sigma_z &= 2\frac{\sum_i w_i d_i(\beta_i \lambda_i + \frac{1}{2})}{|\sigma_H|^2} - 2\beta\sigma_H \\
\theta(t) &= -\frac{\int_0^t \partial_t |\Omega(t,\tau)|^2 d\tau}{\int_0^t |\Omega(t,\tau)|^2 d\tau}
\end{aligned}
$$

where $d_i = \lambda_i \cdot \sigma_H$ and

$$
\begin{aligned}
\Phi(t,\tau) &= \sum_i w_i d_i \left(\beta_i \lambda_i + \frac{1}{2} \exp(-t a_i) \gamma_i \exp(\tau a_i) \right) \\
\Omega(t,\tau) &= 2(\Phi(t,\tau) - \beta(t)|\sigma_H|^2 \sigma_H)
\end{aligned}
$$

with $\beta(t)$ solving linear ODE and initial value $\beta(0) = \frac{\sum_i \beta_i d_i^2}{|\sigma|^4}$

Numerical results

- Consider a European call $C(S,t)$ on the spread $S = S_1 - S_2$
- S_1, S_1 two currencies calibrated to the market (GBP, EUR)
- Price the call option for an expiry of 10 years, ATM and compare prices generated by $4d$ Monte Carlo on S with prices generated by $2d$ Monte Carlo on projected process S^*: Error up to 20%.
- Consider Black Scholes limit: OK
- Recall problems by modeling Heston process: Negative variance: After using analytic solution for S^* error reduced to 10%
- Outlook: What would happen if Broadie-Kaya or Anderson method is used for $4d$ Monte Carlo?

Calibration

- Lets assume we have calibrated n Heston models the the market data of n currencies. Our original basket is driven by $2n$ Brownian motions. What are the correlations?
- Clearly, the correlation between the spot processes and the variance processes are given by the calibration procedure. However, the remaining $2(n^2 - n)$ correlations still need to be determined.
- To get an idea, recall the situation for 2 currencies in the Black Scholes limit. We assume there are three currencies, S_1, S_2, S_3 driven by geometric Brownian Motion. If we assume that $S_3 = \frac{S_1}{S_2}$ then it is easy to show that the correlation $dW_1 dW_2 = \rho dt$ is given by

$$\sigma_3^2 = \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2.$$

Calibration

- Assume now that S_i are "lognormal" shifted Heston processes. It follows that the processes

$$x_i = \ln S_i$$

are "normal" shifted Heston processes. In addition, they are related by

$$x_3 = x_1 - x_2.$$

In particular, we can compare the process x_3^* with the calibrated process x_3. This procedure indeed fixes all 6 correlation up to one scaling degree of freedom.