Double Impact: Credit Risk Assessment and Collateral Value

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Abstract

This papers deals with credit portfolio risk analysis. The benchmark Basel II IRB approach relies on the independence between losses given defaults and default events. Nevertheless, empirical evidence shows that recovered values are likely to be lower when the number of defaults increases, such as in recession periods. We consider a model embedding Basel II that allows to deal with dependence between recovery rates and default events. We then study loss distributions for large credit portfolios. We show that both expected credit losses and standard risk measures such as credit VaR or Expected Shortfall tend to increase compared with the Basel II approach.

1. Introduction

In the reduced-form approach to credit initiated by Jarrow and Turnbull [1995], default dates are considered as unpredictable stopping times and the recovery rates are usually independent of default dates. The recovery rate may be constant (see Canabarro et al. [2003], Jarrow and Turnbull [1995] or CreditRisk+ [1997]), or stochastic as in Moody's KMV (see Crosby and Bohn [2002]) or CreditMetrics (see Gupton, Finger and Bahia [1997]). The Basel II quantitative IRB approach to risk capital (see Basel Committee on Banking Supervision [2001a,b,c]) provides a benchmark framework for credit risk assessment that follows the same lines. This modelling choice is rather a matter of mathematical simplification, leading to analytically tractable expressions for credit portfolio losses.

The previous approaches rely on the independence between recovery rates and default events. Nevertheless, Altman and Kishore [1996], Altman et al. [2003], Hu and Perraudin [2002] find that recovered values are likely to be lower when the number of default increases, such as in recession periods. This can be highlighted for instance in the corporate bond market, where default rates increase as recovery rates decreases, or for bank secured loans, for which default affects the recovery rate through fluctuating collateral market value. This stylised fact can be explained by negative correlation between recovery rates and default

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rates. As a result, expected loss as well as unexpected loss seem to be quite underestimated (Altman [1996], Frye [2000a]). Schuermann [2003], Renault and Scaillet [2003] show bimodal distributions for recovery rates. This can be seen as a consequence of the dependence between recovery rates and default rates, these two quantities depending upon common factors related to the business cycle.

The dependence between default events and losses given default is often introduced through a single factor that drives both default events and recovery rates (see Jokivuolle and Peura [2000]). The recovery rate is then modelled by specifying the collateral value distribution: for instance, Frye [2000a] claims for Gaussian collateral value, Pykhtin [2003] for log-normal. This single factor approach is related to the structural modelling initiated by Merton [1974], and further generalised by Black and Cox [1976], Longstaff and Schwartz [1995], Leland [1994]. This modelling is very similar to the insurance ruin theory. When asset value falls below a predetermined level, default occurs. The asset value is the unique driver of default events and losses given default: However, other effects such as liquidity shortage can drive defaults (see Cetin et al. [2003]). This is consistent with Moody's reporting cases where the asset value of a firm in default exceeds the commitments to the borrowers (see Schuermann [2003]). Moreover, assuming a single risk factor is likely to induce harsh collapse of collateral value, i.e. low recovery rates when default occurs. Disentangling defaults and recovery rates is thus a desirable feature in credit risk modelling.

We thus propose a model consistent with the Basel II framework, which takes into account collateral value and allows a smoother dependence between losses given default and default events. We then study the impact of such dependence on loss distributions, expected losses and credit risk measures such as VaR and Expected Shortfall.

Let us remark that the insurance literature deals with similar issues. For instance, Müller and Pflug [2001] and the references therein look for ruin probabilities when the claims amounts are dependent. The claims amounts correspond here to losses given defaults. In these approaches, there is usually no dependence between claims amounts and the occurrence of claims which is the issue that we want to address. On the other hand, a series of papers (see Denuit et al [2002], Genest et al [2003] and the references therein) deal with dependence between the occurrence of claims. Meanwhile, the claims amounts are still independent of the occurrence of claims. Let us notice that these models share the same flavour as the Basel II approach. Though the mathematical set-up is rather different, Markovian models of the claim surplus process allow to deal with dependencies between the claims amounts and the occurrence of claims (see Asmussen [2000]). However, up to now, the literature regarding these models is not well developed.

The article is organised as follows: in the following section, we describe a model for portfolio credit risk, where default events and losses given default are correlated in different ways. In section 3, we provide some credit risk distributions for large portfolios as such that can be found in retail banking. In section 4, we provide a credit risk assessment for such portfolios, considering the expected loss, the loss distribution and some risk measures such as Value-at-Risk and Expected Shortfall. We study how these risk measures depend upon correlation between asset values and default events.

2. PORTFOLIO CREDIT RISK IN AN EXTENDED BASEL II FRAMEWORK

The Basel II IRB (Internal Ratings Based) provides a benchmark approach to loan loss distributions. Given some probabilistic modelling, the Basel II Committee proposes to compute VaR based measures of risk, though other risk measures such as Expected Shortfall may be considered. In the following, we propose a model that extends the regulatory model in order to better investigate correlation effects between recovery rates (or loss given defaults) and default events. We will firstly decompose credit losses into default events on one hand and losses given defaults (LGD in the Basel II terminology) on the other hand. As for the modelling of default events, we will not depart from the Basel II IRB approach. More details on the Basel II set-up can be found in found in Gordy [2000, 2003], Crouhy, Galai, and Mark [2000] and Chabaane, Chouillou and Laurent [2003].

2.1 Portfolio Credit Loss

Basel II aims at modelling default losses over a given time horizon for a homogeneous portfolio. The aggregated loss of such a homogeneous portfolio of n loans with same unit nominal is given by the sum of individual credit losses:

$$L = \sum_{j=1}^{n} LGD_j \times D_j \tag{1}$$

- D_j is the default indicator for the j^{th} creditor. It is a Bernoulli random variable taking value 1 in case creditor j is in default over the prescribed time horizon and zero otherwise.
- LGD_j stands for the loss severity if default occurs (Loss Given Default) for the j^{th} creditor, i.e. the amount that is non recovered in case of default. LGD_j may be deterministic or stochastic and stands between 0 and 1. Loss given default is equal to one minus the recovery rate on the j^{th} creditor.

Let us remark that using the actuarial terminology, this modelling corresponds to an individual model. The Basel II methodology is also closely related to the credit scoring practices within commercial banks. By using the credit scoring or the qualitative econometrics terminology (see Gouriéroux [2000]), the default indicator is a qualitative random variable while losses given default are quantitative. Let us notice that losses given default can really be assessed only in case of default, while the random variable LGD_j is also defined on the set $D_j = 0$. One may wonder what would be losses given defaults, in case no default has occurred. In a firm-value framework, potential losses given default can be seen as the difference between asset values minus debt commitments minus some liquidation costs.
However, these quantities may be difficult to assess *ex-ante*, and actual losses given defaults are often known some months after default has actually occurred. To deal with these issues, Gordy [2003] proposes a direct modelling of the losses without trying to separate it into its default event and loss given default components.

Let us remark that in this framework, losses occur only in case of default and thus losses due to credit migrations are not taken into account. Whenever market credit spreads or agency ratings are available as for traded bonds, this assumption may be questionable. As far as retail credits are concerned, one may also think to use dynamic internal ratings to cope with credit migrations. Similarly, no interest rate effects are being considered in this framework. As far as standard loans are concerned, the main source of risk comes from defaults. However, when considering interest rate derivatives such as swaps or caps, one needs to carefully deal with the dependence between default events and interest rates in order to assess loss distributions. As mentioned above, it is also difficult to cope consistently with losses over different time horizons, since no dynamic structure is specified.

2.2 Default events modelling

Let us firstly concentrate on the modelling of the default events. In the Basel II framework, D_j are such that $D_j = 1_{\{\Psi_j < z\}}$ where the latent variables $\left(\Psi_j\right)_{1 \le j \le n}$ have a multivariate normal distribution with correlation $\rho = \mathrm{Cov}(\Psi_i, \Psi_j)$.

In the credit scoring terminology, this corresponds to a multivariate probit model. Under the homogeneity assumption, the threshold z and the correlation parameter ρ do not depend on the specific creditor, since the default indicators do share the same distributions.

Under some mild assumptions, thanks to the homogeneity assumption and De Finetti's theorem (see Frey and McNeil [2003]), we can then state a one factor representation of the latent variables $\left(\Psi_j\right)_{1\le j\le n}$:

$$\Psi_j \equiv \sqrt{\rho}\Psi + \sqrt{1-\rho}\ \overline{\Psi}_j \qquad (2)$$

where the random variables Ψ and $\left(\overline{\Psi}_j\right)_{1\le j\le n}$ are assumed to be all independent and have standard Gaussian distribution. $\overline{\Psi}_j$ represents *specific risk* (or *idiosyncratic risk*) to credit j and Ψ a common risk to all credits in the portfolio, or *systematic risk*. Let us point out again that the default indicators are not independent, due to the common factor Ψ .

Thus for homogeneous portfolios, the only modelling assumption is that the latent variables follow a Gaussian distribution. Other approaches, for instance based on the Student *t*-copula can be used (see Frey and McNeil [2003]). Nevertheless, the choice of the right copula for defaults remains an open question and we will still rely on the benchmark Gaussian copula assumption that underlies the Basel II framework. Let us also remark that this corresponds to the individual insurance model of Denuit et al [2002].

Eventually, the threshold z can be calibrated on the common default probability PD:

$$z = \Phi^{-1}\left[PD\right], \qquad (3)$$

where Φ stands for the Gaussian cumulative density function.

2.3 Loss Given Default modelling

2.3.1 Loss given default for a single credit

We now concentrate on the modelling of losses given defaults. We denote by C, the value of the assets at the time horizon. The credit is assumed to have unit nominal. If default occurs, the creditor loses the quantity $LGD = \left[1-C\right]^+ = \max\left(1-C,0\right)$, which is the difference between the credit exposure and the asset's value.

Assets of the creditor are the collateral protection of the loan. If C is seen as the asset's value at default, we do not consider departures from the absolute priority rule. Bankruptcy costs, either fixed or proportional to asset's value can be embedded in our framework without changing the payoff structure. Our analysis can be applied to senior unsecured debt. In order to take into account junior debt, we can simply change the default threshold from 1 to the required level.

Following the standard modelling, we will assume that C is log-normally distributed, i.e. $C = \exp\left(\mu + \sigma\xi\right)$, where μ and σ are some parameters and ξ follows a standard Gaussian distribution.

- σ corresponds to the asset volatility and can be inferred form stock prices.
- the drift term μ can differ from the risk-free rate since we want to compute loss distributions under the historical measure and also due to dividend payments.

In the portfolio credit modelling field, such a log-normal assumption is used by Pykhtin [2003] which ensures the collateral remains positive, while Frye [2000a], Canabarro et al [2003] rather use a Gaussian assumption which seems easier to handle.

Let us remark that for an unsecured loan, the asset's value of the creditor is the only collateral. For secured loans, the specification of the loss given default is more involved since it requires to deal both with the value of the guarantee and of the assets of the creditor. For this reason, when considering secured credits such as mortgages, most authors neglect the guarantee provided by the general assets of the firm (see Frye [2000a,b], Jokivuolle and Peura [2000], Pykhtin [2003]) and thus their model formally collapses into ours.

The credit loss associated with creditor j is then given by:

$$L_j = \left[1 - e^{\mu + \sigma \xi_j}\right]^+ \times 1_{\{\Psi_j < \Phi^{-1}(PD)\}} \tag{4}$$

In the econometrics literature terminology, L_j follows a tobit model (see Gouriéroux [2000]). The loss on a given creditor also corresponds to the payoff of a vulnerable put option. Thus, the expected loss $E\left[L_j\right]$ can be easily using bivariate Gaussian distributions. We refer to Johnson and Stulz [1987], Jarrow and Turnbull [1995], Klein [1996], Klein and Inglis [2001], Augros and Tchapda [2003] for computations and discussion. However, rather than considering the expected loss on the portfolio, $\sum_{j=1}^{n} E\left[L_j\right]$, we thereafter need to compute to loss distribution of $\sum_{j=1}^{n} L_j$. In the option pricing terminology, this corresponds to the valuation of options on portfolios of vulnerable options. In the insurance terminology, we want to be able to evaluate stop loss premiums.

2.3.2 Basel II framework

We still consider homogeneous portfolios. The losses given default $\left(LGD_j\right)_{1\le j\le n}$ are then identically distributed.

In the Basel II framework, it is moreover assumed that the $\left(LGD_j\right)_{1\le j\le n}$ are both independent and independent of the default events $\left(D_j\right)_{1\le j\le n}$. Consequently, for large portfolios, the only quantity to be considered is the expected Loss Given Default while the $\left(LGD_j\right)_{1\le j\le n}$ specific distribution is not required (see below).

2.3.3 Correlated losses given default and asset values.

As discussed above, the independence assumption between default events and losses given defaults does not seem to be realistic. We thereafter propose a simple model where losses given defaults are correlated together and are also correlated with default events. This model is consistent with empirical evidence on losses given defaults (i.e. bimodal distributions) and default events. Moreover, it can be given simple economic interpretation and leads to rather simple interpretations. Not surprisingly, it will lead to changes in the shape of the aggregated loss distribution and thereafter in the expected and unexpected losses.

We now introduce a correlation structure between Loss Given Default $\left(LGD_j\right)_{1\le j\le n}$ and Default Indicators $\left(D_j\right)_{1\le j\le n}$ based upon some extension of the Merton model (i.e. a one period firm value model). In order to deal with credit portfolio distributions, the correlation structure should be emphasised. The $2n$ random variables within an homogeneous credit portfolio, $\left(\Psi_i, \xi_i\right)_{1\le i\le n}$ can be seen as follows:

- homogeneity for default events, and for losses given defaults implies that the cross moments $\mathrm{Cov}\left(\Psi_i, \Psi_j\right)$ as well as $\mathrm{Cov}\left(\xi_i, \xi_j\right)$ do not depend on names $i, j,\ i \ne j$,
- similarly, we must have constant correlation between Default and Loss Given Default: for a single creditor (Ψ_i and ξ_i) and between two creditors $\left(\Psi_i, \xi_j\right)$.

The homogeneity assumption implies that the sequences $(\Psi_i)_{1\le i\le n}$ and $(\xi_i)_{1\le i\le n}$ are exchangeable. From De Finetti's theorem, there exist two sequences of independent standard Gaussian variables $\Psi, \overline{\Psi}_i$, $i=1,\ldots,n,\ldots$ and $\xi, \overline{\xi}_i$, $i=1,\ldots,n,\ldots$ such that:

$$\begin{cases} \Psi_j = \sqrt{\rho}\,\Psi + \sqrt{1-\rho}\,\overline{\Psi}_j \\ \xi_j = \sqrt{\beta}\,\xi + \sqrt{1-\beta}\,\overline{\xi}_j \end{cases} \qquad (5)$$

where ρ,β are some correlation parameters defined on $[0,1]$. Let us denote by $\eta = \operatorname{corr}(\Psi,\xi)$ and $\gamma = \operatorname{corr}(\overline{\Psi}_i,\overline{\xi}_i)$.

Thanks to the homogeneity assumption, $E\left[\xi\overline{\Psi}_i\right]$ is independent of $i=1,\ldots,n,\ldots$ and thus equal to $E\left[\xi\times\frac{1}{n}\sum_{i=1}^{n}\overline{\Psi}_i\right]$. By Cauchy-Schwarz inequality we get: $\left|E\left[\xi\times\frac{1}{n}\sum_{i=1}^{n}\overline{\Psi}_i\right]\right| \le \left\|\frac{1}{n}\sum_{i=1}^{n}\overline{\Psi}_i\right\|_2 = \frac{1}{\sqrt{n}}$.
Thus $\operatorname{corr}(\overline{\Psi}_i,\xi)=0$. Similarly, $\operatorname{corr}(\Psi,\overline{\xi}_i)=0$ for all $i=1,\ldots,n,\ldots$
As a consequence $(\overline{\Psi}_i,\overline{\xi}_i)$ is independent from (Ψ,ξ) and can be seen as a "good" residual.
Let us now consider the correlation terms $\operatorname{corr}(\overline{\Psi}_i,\overline{\xi}_j)$ for $i\neq j$. Let us use the homogeneity assumption again. It can be seen that $E\left[\overline{\Psi}_i\overline{\xi}_j\right]$ does not depend on i,j for $i\neq j$. Thus $E\left[\overline{\Psi}_i\overline{\xi}_j\right] = E\left[\overline{\Psi}_i\times\frac{1}{n}\sum_{j=i+1}^{i+n}\overline{\xi}_j\right]$.
By using again Cauchy-Schwarz inequality and the law of large number in L_2, we obtain that $\operatorname{corr}(\overline{\Psi}_i,\overline{\xi}_j)=0$ for $i\neq j$. As a consequence the random vectors $(\overline{\Psi}_i,\overline{\xi}_i)$ are independent for $i=1,\ldots,n,\ldots$

Thus, equations (5) define a proper factor structure for underlying latent variables $(\Psi_i)_{1\le i\le n}$ and $(\xi_i)_{1\le i\le n}$. Ψ and ξ can be seen as systematic factors (common risk factors to all credits) while $(\overline{\Psi}_j)_{1\le j\le n}$, $(\overline{\xi}_j)_{1\le j\le n}$ are specific risks. Let us remark however that these specific risks can be correlated for a given credit i: we will further denote by $\gamma = \operatorname{corr}(\overline{\Psi}_i,\overline{\xi}_i)$. The correlation structure for $i\neq j$ is summarised in the following table:

	Ψ	ξ	$\overline{\Psi}_i$	$\overline{\Psi}_j$	$\overline{\xi}_i$	$\overline{\xi}_j$
Ψ	1	η	0	0	0	0
ξ	η	1	0	0	0	0
$\overline{\Psi}_i$	0	0	1	0	γ	0
$\overline{\Psi}_j$	0	0	0	1	0	γ
$\overline{\xi}_i$	0	0	γ	0	1	0
$\overline{\xi}_j$	0	0	0	γ	0	1

Table 1 – Factor correlation structure

Let us remark that the correlation between specific risks related to default and recovery rates has to be equal to zero for different credits as a consequence of the homogeneity assumption. Our model only involves four parameters ρ,β,η,γ.

When $\eta = \gamma = 0$ (i.e. no correlation between default events and losses given defaults) and $\beta = 0$ (i.e. no correlation between losses given defaults), the model collapses into the Basel II framework. We emphasise that well-known reduced-form models such as KMV (Crosbie and Bohn [2002]) or CreditMetrics (Gupton et al. [1997]) treat recovery rate as a stochastic variable, independent from the default event and make a similar assumption.

Let us remark that when the correlation between ξ_j and Ψ_j is equal to one, the model turns out to be the Merton model. In that framework, very simple analytical expressions of loss distributions and risk measures can be obtained.

Correlation between default events and losses given default have been introduced in recent works under the assumption of $\eta = 100\%$ (i.e. a one factor model): Frye ([2000a]) assumed no correlation between specific risks ($\gamma = 0$) while Pykhtin [2003] added specific risk correlation ($\gamma > 0$). It should be noticed that these one-factor models induce harsh collapse of collateral value (i.e. strong losses given default) when default occurs. This seems inappropriate for mortgage loans for instance.

Introducing a two-factor model ($\eta < 100\%$) is likely to be associated with smoother loss distributions, which will be shown in the following. Let us notice that Rosen and Sidelnikova [2002] propose a correlation structure similar to ours, but without specific correlation ($\gamma = 0$).

Moreover, the correlation between the latent variables ξ_j and Ψ_j is given by:

$$K = \eta\sqrt{\rho\beta} + \gamma\sqrt{1-\rho}\sqrt{1-\beta}\ .$$

This correlation has two sources: one reflects how systematic risks are correlated to business cycles, while the other describes the idiosyncratic impact on both credit risk and market risk. As a matter of fact, the latter allows also to cope with project finance cases for which the collateral itself drives default events (see Pykhtin [2003]).

3. Loss Distributions for Large Portfolios

For well-diversified credit portfolios, i.e. when the nominal exposure is small compared with total exposure, the aggregated loss can be dramatically simplified thanks to some asymptotic expansion. These large sample approximation techniques are described in Gordy [2003] or Wilde [2001], Martin and Wilde [2002]. Finger [1999] and Vasicek [2002] describe the application of such techniques for the CreditMetrics and KMV portfolio manager respectively. Large sample approximations are well suited for retail portfolio risk analysis and are thus in the core of the Basel II methodology. The following proposition states the limit distribution for a well-diversified credit portfolio.

Proposition 3.1 Portfolio loss limit distribution: *let us consider a homogeneous portfolio of n loans, for which the loss depends on a systematic multivariate risk factor* Ψ *. The individual losses are assumed to be independent under* P^{Ψ}*. With total exposure equal to 1, the aggregated loss converges almost surely:*

$$\frac{1}{n}\sum_{j=1}^{n} LGD_j \times 1_{\{\Psi_j < \Phi^{-1}[PD]\}} \xrightarrow[n\to+\infty]{a.s.} L_{\Psi} = E\left[LGD_j \times 1_{\{\Psi_j < \Phi^{-1}[PD]\}} \middle| \Psi \right] \qquad (6)$$

Proof: see appendix 6.1

Such approximation is done within the Basel II framework and is especially valid for retail portfolios. In Basel II terminology, this is known as the infinite granularity hypothesis.

Corollary 3.1 Basel II case: *in the Basel II framework, the aggregated loss converges almost surely to:*

$$L_{\text{Basel}} = \left(\Phi\left[-\frac{\mu}{\sigma} \right] - e^{\mu+\sigma^2/2} \times \Phi\left[-\frac{\mu}{\sigma} - \sigma \right] \right) \times F_{\Psi}, \qquad (7)$$

with: $F_{\Psi} = \Phi\left[\dfrac{\Phi^{-1}(PD) - \sqrt{\rho}\ \Psi}{\sqrt{1-\rho}} \right]$.

Proof: see appendix 6.3

$E[LGD]$ stands for the non conditional expectation of the random variables $\left(LGD_j\right)_{1\le j\le n}$. In the Basel II framework, the aggregated loss is a decreasing function of a one dimensional Gaussian variable Ψ.

As stated in proposition 3.1, we do not require the independence assumption between default events and losses given default. The infinite granularity approach can still be used:

Corollary 3.2 Correlation between defaults and losses given default: *the aggregated loss involving the systematic factor* (Ψ,ξ) *converges almost surely to:*

$$L = \Phi_2\left[\Phi^{-1}[F_\Psi]; \Phi^{-1}[G_\xi]; \gamma \right] - e^{\mu+\sigma^2/2} \times e^{\sigma\sqrt{\beta}\,\xi - \beta\sigma^2/2} \times \Phi_2\left[\Phi^{-1}[F_\Psi] - \sigma\gamma\sqrt{1-\beta}; \Phi^{-1}[G_\xi] - \sigma\sqrt{1-\beta}; \gamma \right] \qquad (8)$$

with: $F_{\Psi} = \Phi\left[\dfrac{\Phi^{-1}(PD) - \sqrt{\rho}\ \Psi}{\sqrt{1-\rho}} \right]$ *and* $G_\xi = \Phi\left[-\dfrac{\mu/\sigma + \sqrt{\beta}\ \xi}{\sqrt{1-\beta}} \right]$.

Proof: see appendix 6.4

The previous expressions involve the systematic correlated factor (Ψ,ξ) while all specific risks have been diversified. *F* can be seen as the expected default probability while *G* is to be related to the expected loss given default.

4. Risk analysis

As for market risk, in the Basel II IRB approach bank capital charges must match the credit risk exposure through the use of an appropriate credit risk measure, computed from the loss distribution. In the Basel II case, capital charges may be evaluated from:

- the credit losses *L*, as for corporate credits and mortgages ;
- the unexpected losses, namely $L - E^P[L]$ for retail credits apart mortgages[1].

More details may be found in Chabaane et al. [2003].

To investigate the differences between the Basel II approach and the extended one, we firstly address the issue of expected losses.

[1] For retail credits, banks do not have capital charges for expected losses $E^P[L]$, which are covered by the credit margin.

4.1 Comparing expected losses

The Expected Loss EL in the Basel II framework is given by (see appendix 6.3):

$$EL_{\text{Basel}} = PD \times \left(\Phi\left[-\frac{\mu}{\sigma}\right] - e^{\mu+\sigma^2/2} \times \Phi\left[-\frac{\mu}{\sigma}-\sigma\right] \right) \tag{9}$$

In the model with correlated default events and losses given default, a technical but not complicated computation leads to (see appendix 6.4):

$$EL_{\text{correlated}} = \Phi_2\left[\Phi^{-1}(PD)\ ;\ -\frac{\mu}{\sigma}\ ;\ K\right] - e^{\mu+\sigma^2/2} \times \Phi_2\left[\Phi^{-1}(PD) - \sigma K\ ;\ -\frac{\mu}{\sigma}-\sigma\ ;\ K\right] \tag{10}$$

where K denotes the correlation parameter: $K = \eta\sqrt{\rho\beta} + \gamma\sqrt{1-\rho}\sqrt{1-\beta}$.

One should notice that the expected loss is driven by the unique global correlation parameter K, which sums up the dependence between default events and losses given default.

(PD, ρ) and (μ, σ, β) are respectively associated with the marginal distributions of default events and asset values. For this purpose, we claim that the volatility σ and asset correlation β may be appraised from asset values data, separately from default events, while default probability PD and correlation ρ may be estimated from historical default rates within the portfolio. Moreover, μ corresponds to a given expected loss in the Basel II framework and is to be related to the leverage and to the average return of the asset.

In the Basel Committee approach, the default correlation ρ is computed using the following formula:

$$\begin{cases} \rho = \lambda \times \rho_{\min} + (1-\lambda) \times \rho_{\max} \\ \lambda = \dfrac{1-e^{-\alpha \times PD}}{1-e^{-\alpha}} \end{cases}$$

where the parameters $\rho_{\min}, \rho_{\max}, \alpha$ depend on the type of credit[2].

In the following example, the default probability is set to $PD = 1\%$ and the expected loss in the Basel II framework to 0.2 %. Marginal correlations β and ρ are set respectively to 80% and 15%. In figure 1, we represent Expected Loss for the model with correlated default events and losses given default as a function of correlation parameter K. The three curves are associated with different levels of volatility ($\sigma = 0\%$, 20% and 50%).

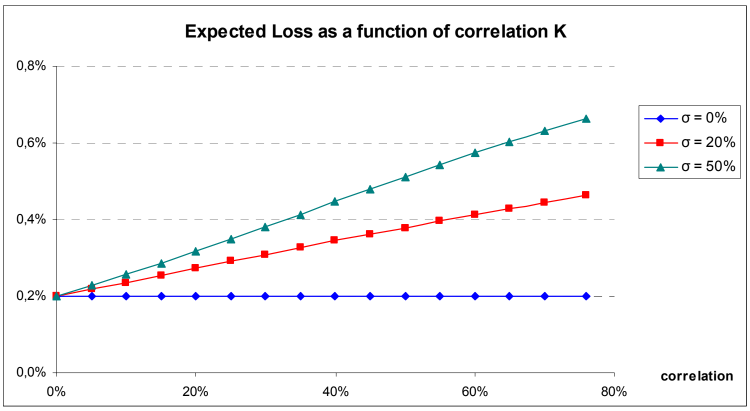

Figure 1 - Expected Loss in a correlated way (EL = 0.2% for Basel II)

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2 The Basel Committee claims that these parameters should be set to $\rho_{\min} = 2\%$, $\rho_{\max} = 12\%$, $\alpha = 35$ for retail portfolios, $\rho_{\min} = 12\%$, $\rho_{\max} = 24\%$, $\alpha = 50\%$ for corporate ones.

As can be seen in figure 1, the correlation effect between losses given default and default events can induce a substantial increase of the expected loss. This should be taken into account in the pricing of such credits. Still, one should keep in mind that for given correlations β and ρ, the correlation parameter K is bounded between 0 and $\sqrt{\rho\beta}+\sqrt{1-\rho}\sqrt{1-\beta}$ (i.e. 76% in our case study).

4.2 Loss distributions

Before evaluating risk measures and reckoning with their relative positions, it seems natural that the loss distribution shapes of Basel II and its extension should be collated. In figure 2, we compare the Basel II loss distribution, as a benchmark, to correlated approach loss distribution where volatility and default/recovery correlation have been taken high concomitantly, in order to enlighten the double impact effect. We find that the distribution is bimodal: when defaults occur, collateral values are likely to nosedive, such that losses are weak, huge but not average.

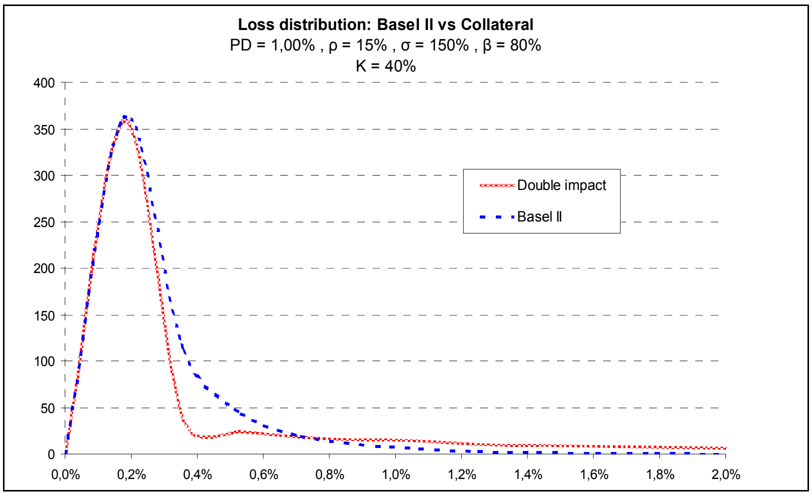

Figure 2 – Loss Distribution: Comparison between Basel II and the extended approach (200,000 Monte-Carlo simulations)

One should keep in mind that this example is quite extreme; still, with more consistent parameters, this phenomenon would be noticeable too, even if less striking. Furthermore, although the expected loss appears hardly changed, the tail distribution is dramatically widened in comparison with Basel II. This trend will be discussed in the following section dealing with risk measures computation.

4.3 Risk measures

Supervisory authorities retained Value-at-Risk as a risk measure to evaluate regulatory capital. For a given random variable *X*, we recall the Value-at-Risk expression for a confidence level α :

$$VaR_\alpha(X)=\inf\ \left(t, P\left[X\le t\right]\ge\alpha\right) \tag{11}$$

In the Basel II framework, the confidence level is $\alpha=99.9\%$. One may notice that this risk measure is not sub-additive and does not take into account the magnitude of large losses. To cope with this harshness, other risk measures may be reckoned with. Expected Shortfall is being considered by large banks a reliable alternative risk measure. For a given random variable *X*, with continuous distribution, the Expected Shortfall stands for the mean of the losses beyond VaR:

$$ES_\alpha(X)=E^P\left[X\middle|X>VaR_\alpha(X)\right] \tag{12}$$

From this expression arises that $VaR_\alpha(X) \le ES_\alpha(X)$ and that Expected Shortfall is a more conservative risk measure than the regulatory one. We underline the fact that the Expected Shortfall is sub-additive while VaR is not (see Artzner et al. [1997]). Throughout this article, the risk measures to be focused on are Value at Risk and the Expected Shortfall.

4.3.1 *Basel II framework*

The monotony property[3] can be applied to the Basel II aggregated loss (see appendix 6.3):

$$VaR_{\text{Basel}}(\alpha) = E\left[LGD\right] \times \Phi\left[\frac{\Phi^{-1}(PD) + \sqrt{\rho}\ \Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right] \tag{13}$$

Moreover, the Expected Shortfall can also be derived analytically (see appendix 6.3):

$$ES_{\text{Basel}}(\alpha) = E\left[LGD\right] \times \frac{\Phi_2\ \left[\Phi^{-1}(PD)\ ;\ \Phi^{-1}(1-\alpha)\ ;\ \sqrt{\rho}\right]}{1-\alpha} \tag{14}$$

4.3.2 *Correlated default events and losses given default*

In the general case of unconstrained correlation, the model involves two distinct systematic factors Ψ and ξ. No analytical expression of Value-at-Risk or Expected Shortfall is available and these risk measures will be thus evaluated numerically. However, in the particular case when credit and market systemic risks are completely correlated (i.e. $\eta = 100\%$), the aggregated loss becomes a decreasing function of a standard normal variable. The Value-at-Risk would then be analytically computable.

4.3.3 *Computation methodology*

Value-at-Risk and Expected Shortfall are computed for a volatility level $\sigma = 20\%$, with current default probability $PD = 1\%$, expected loss $EL = 0.2\%$ and default correlation $\rho = 15\%$. VaR and Expected Shortfall have to be seen as functions of the correlation parameters β and η. More precisely, the quantities we study are the following ratios:

$$\overline{\text{VaR}} = \frac{\text{VaR}_{\text{correlated}}}{\text{VaR}_{\text{Basel}}} \text{ and } \overline{\text{ES}} = \frac{\text{ES}_{\text{correlated}}}{\text{EL}_{\text{Basel}}}$$

$\text{VaR}_{\text{Basel}}$ and EL_{Basel} are known explicitly while risk measures for the extended model are computed by Monte-Carlo simulations.

Tables 2 and 3 show those ratios for different levels of correlation (β, η), for two case study:

- $\gamma = 0$: correlation between default events and collateral values is purely systematic
- $\gamma = 50\%$: correlation between default events and collateral values involves specific risk

[3] The VaR monotony property (see appendix 6.2) states that if the aggregated loss is a decreasing function of a standard Gaussian factor, the VaR can be obtained directly from:

$$\begin{cases} L = f(Z) \\ df / dz < 0 \quad \Rightarrow VaR_\alpha(L) = f\ \ [-\Phi^{-1}(\alpha)] \\ Z \sim N(0,1) \end{cases}$$

η \ β	0%	20%	40%	60%	80%	100%
0%	100,0% *100,0%*	113,1% *115,6%*	122,1% *126,0%*	130,3% *135,7%*	139,1% *144,8%*	145,7% *151,5%*
20%	100,7% *100,8%*	129,9% *134,1%*	143,7% *148,5%*	157,7% *163,2%*	165,4% *172,6%*	175,4% *182,7%*
40%	100,8% *100,8%*	144,3% *149,0%*	165,6% *171,2%*	181,0% *190,2%*	193,3% *201,8%*	204,1% *215,3%*
60%	99,8% *100,0%*	161,2% *168,6%*	186,3% *194,3%*	204,4% *215,7%*	219,8% *230,6%*	232,1% *244,0%*
80%	100,2% *100,4%*	179,5% *188,7%*	209,0% *219,5%*	227,8% *236,9%*	247,2% *258,7%*	261,0% *274,8%*
100%	100,4% *99,7%*	192,6% *201,2%*	225,6% *235,4%*	251,1% *261,8%*	272,9% *287,9%*	281,9% *296,9%*

Table 2 – $\overline{\mathrm{VaR}}$ and $\overline{\mathrm{ES}}$ (in italic) for case study 1 (γ = 0%) as a function of correlation parameters 1,000,000 simulations

η \ β	0%	20%	40%	60%	80%	100%
0%	158,9% *154,8%*	161,0% *160,2%*	164,2% *165,4%*	162,5% *164,7%*	159,3% *162,4%*	145,9% *152,1%*
20%	157,5% *153,9%*	175,4% *175,6%*	182,6% *183,7%*	186,8% *188,6%*	186,0% *192,5%*	172,8% *179,8%*
40%	160,2% *156,0%*	194,1% *196,6%*	207,9% *211,6%*	211,8% *218,7%*	212,6% *219,5%*	205,7% *217,2%*
60%	158,2% *155,2%*	207,4% *210,3%*	227,0% *231,1%*	238,9% *243,0%*	240,8% *249,2%*	234,1% *243,4%*
80%	159,6% *156,0%*	223,1% *229,4%*	244,1% *249,4%*	257,4% *265,1%*	264,5% *271,2%*	260,5% *273,4%*
100%	158,1% *153,9%*	238,9% *246,4%*	262,7% *268,0%*	276,5% *287,3%*	283,3% *296,3%*	286,8% *296,6%*

Table 3 – $\overline{\mathrm{VaR}}$ and $\overline{\mathrm{ES}}$ (in italic) for case study 2 (γ = 50%) as a function of correlation parameters 1,000,000 simulations

5. CONCLUSION

Not surprisingly, taking into account the positive dependence between default events and losses given defaults tend to increase both expected losses and credit risk measures such as VaR and Expected Shortfall. This is associated with a change in the shape of the distribution function. To deal with these effects, we have considered a model that encompasses Basel II and previous models dealing with such dependencies. The model studied can be seen as a bridge between the structural approach and the reduced form approach to credit risk by allowing several levels of dependency between default dates and recovery rates. This model is well suited for large homogeneous portfolios. In fact, the two factors structure that we exhibit is a mere consequence of the homogeneity assumption. Under the infinite granularity assumption, the aggregated losses are computed explicitly as a function of the two factors. Thus, aggregation of homogeneous portfolios remains easy. However, the computed VaR is no more additive with respect to credit exposures.

6. APPENDIX

6.1 Aggregated loss convergence

We denote by $(\Omega, \Im, P)$ a space endowed with a probability measure. Q refers to the regular version of a probability measure knowing a multivariate factor Ψ. Let us consider the losses $(L_k)_{k\leq n}$ characterised by:

- $(L_k)_{k\leq n}$ are all independent knowing Ψ ;
- $(L_k)_{k\leq n}$ have the same expectation under Q : $L_\Psi = E^Q[L_k]$;
- $(L_k)_{k\leq n}$ have finite expectation under P and under Q.

We aim at showing that the average loss converges almost surely to L_Ψ when *n* tends to infinity. Hence, using the law of large numbers:

$$Q\left(\frac{1}{n}\sum_{k=1}^{n} L_k \to L_\Psi\right) = 1 \tag{A1}$$

Moreover, using Fubini's theorem, we obtain:

$$P\left(\frac{1}{n}\sum_{k=1}^{n} L_k \to L_\Psi\right) = \int_\Omega Q\left(\frac{1}{n}\sum_{k=1}^{n} L_k \to L_\Psi\right) dQ = 1 \tag{A2}$$

This last assumption proves the almost surely convergence of the loss to $L_\Psi = E^Q[L_k]$.

6.2 Gaussian VaR

We assume that the loss *L* is a strictly decreasing function of a standard Gaussian variable Ψ : $L = f(\Psi)$. Hence, if g denotes the inverse function of f, still decreasing, we successively get:

$$P[L \geq L_0] = P[f(\Psi) \geq L_0] = P[\Psi \leq g(L_0)] = \Phi[\ g(L_0)].$$

Then $\mathrm{VaR}_\alpha(L) = f\ [\Phi^{-1}(1-\alpha)]$ which finally leads to:

$$\boxed{\mathrm{VaR}_\alpha(L) = f\ [-\Phi^{-1}(\alpha)]} \tag{A3}$$

6.3 Basel II framework

We denote by $(\Omega, \Im, P)$ a space endowed with a probability measure. Q refers to the regular version of a probability measure knowing Ψ .Let us recall that the loss L_j for a single credit is characterised by:

$$\begin{cases} LGD_j \\ D_j = 1_{\left\{\sqrt{\rho}\,\Psi + \sqrt{1-\rho}\,\overline{\Psi}_j < \Phi^{-1}(PD)\right\}} \end{cases} \tag{A4}$$

<u>**Expected Loss**</u>

The Expected Loss is given by $EL = E[LGD] \times PD$. The first quantity is evaluated as follows:

$$E[LGD] = \int_{-\infty}^{+\infty} \left[1 - e^{\mu+\sigma z}\right]^{+} \times \varphi(z)\, dz = \int_{-\infty}^{-\mu/\sigma} \left[1 - e^{\mu+\sigma z}\right] \times \varphi(z)\, dz$$

$$= \Phi[-\mu/\sigma] - \int_{-\infty}^{-\mu/\sigma} e^{\mu+\sigma z} \times \varphi(z)\, dz = \Phi[-\mu/\sigma] - e^{\mu+\sigma^2/2} \times \Phi[-\mu/\sigma - \sigma]$$

We finally get:

$$\boxed{EL_{Basel} = PD \times \left(\Phi\left[-\frac{\mu}{\sigma}\right] - e^{\mu+\sigma^2/2} \times \Phi\left[-\frac{\mu}{\sigma} - \sigma\right]\right)} \tag{A5}$$

Aggregated Loss

The asymptotic property given in appendix 6.1 allows to reduce the aggregated loss L_Ψ to:

$$
\begin{aligned}
E\left[LGD_j \times D_j \mid \Psi\right] &= E\left[LGD_j \mid \Psi\right] \times E\left[D_j \mid \Psi\right] \\
&= E\left[LGD_j\right] \times Q\left[\sqrt{\rho}\,\Psi + \sqrt{1-\rho}\,\overline{\Psi}_j < \Phi^{-1}(PD)\right] \\
&= E\left[LGD\right] \times Q\left[\overline{\Psi}_j < \frac{\Phi^{-1}(PD) - \sqrt{\rho}\,\Psi}{\sqrt{1-\rho}}\right]
\end{aligned}
$$

Thus, the aggregated loss is:

$$\boxed{L_{Basel} = E\left[LGD\right] \times \Phi\left[\frac{\Phi^{-1}(PD) - \sqrt{\rho}\,\Psi}{\sqrt{1-\rho}}\right]} \qquad (A6)$$

Value-at-Risk

Furthermore, the aggregated loss is a decreasing function of the systematic variable Ψ . In fact:

$$\frac{dL}{d\Psi} = -\sqrt{\frac{\rho}{1-\rho}} \times E\left[LGD\right] \times \varphi\left[\frac{\Phi^{-1}(PD) - \sqrt{\rho}\,\Psi}{\sqrt{1-\rho}}\right] \le 0$$

The appendix 6.2 leads then to:

$$\boxed{VaR_{Basel}(\alpha) = E\left[LGD\right] \times \Phi\left[\frac{\Phi^{-1}(PD) + \sqrt{\rho}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right]} \qquad (A7)$$

Expected Shortfall

The Expected Shortfall is given by: $ES_\alpha(X) = E^P\left[X \middle| X > VaR_\alpha(X)\right]$

Since $L(\Psi) > VaR_\alpha(L) \Leftrightarrow \Psi < \Phi^{-1}\left[1-\alpha\right]$, we get:

$$
\begin{aligned}
ES_{Basel}(\alpha) &= \frac{1}{1-\alpha} \times \int_{-\infty}^{\Phi^{-1}(1-\alpha)} L(x) \times \varphi(x)\,dx \\
&= \frac{1}{1-\alpha} \times \int_{-\infty}^{\Phi^{-1}(1-\alpha)} \Phi\left[\frac{\Phi^{-1}(PD) - \sqrt{\rho}\,x}{\sqrt{1-\rho}}\right] \times \varphi(x)\,dx \\
&= \frac{1}{1-\alpha} \times \int_{-\infty}^{\Phi^{-1}(1-\alpha)} \int_{-\infty}^{\frac{\Phi^{-1}(PD) - \sqrt{\rho}\,x}{\sqrt{1-\rho}}} \varphi(x) \times \varphi(y)\,dx\,dy
\end{aligned}
$$

We perform the following linear change of variables: $\begin{cases} x_1 = x \\ y_1 = \sqrt{\rho}\,x + \sqrt{1-\rho}\,y \end{cases}$

which lead to: $ES_{Basel}(\alpha) = \dfrac{1}{1-\alpha} \times \displaystyle\int_{-\infty}^{\Phi^{-1}(1-\alpha)} \int_{-\infty}^{\Phi^{-1}(PD)} \varphi_2(x_1\,;y_1\,;\sqrt{\rho})\,dx_1\,dy_1$

The analytical regulatory expression for Expected Shortfall finally falls to:

$$\boxed{ES_{Basel}(\alpha) = \frac{1}{1-\alpha} \times \Phi_2\left[\Phi^{-1}(PD)\,; \Phi^{-1}(1-\alpha)\,; \sqrt{\rho}\right]} \qquad (A8)$$

6.4 Correlation between defaults and losses given default

We denote by $(\Omega, \Im, P)$ a space endowed with a probability measure. Q refers to the regular version of a probability measure knowing (Ψ, ξ). The credit loss is characterised by:

$$\begin{aligned} LGD_j &= \left[1 - e^{\mu+\sigma\sqrt{\beta}\,\xi+\sigma\sqrt{1-\beta}\,\bar{\xi}_j} \right]^+ \\ D_j &= 1_{\left\{\sqrt{\rho}\,\Psi+\sqrt{1-\rho}\,\bar{\Psi}_j < \Phi^{-1}(PD)\right\}} \end{aligned} \qquad (A9)$$

<u>**Expected Loss**</u>

Let us recall that the expression of a credit loss is $L_j = \left[1 - e^{\mu+\sigma\,\xi_j} \right]^+ \times 1_{\left\{\Psi_j < \Phi^{-1}(PD)\right\}}$, where (Ψ_j, ξ_j) are standard normal variables, with correlation $K = \eta\sqrt{\rho\beta} + \gamma\sqrt{1-\rho}\sqrt{1-\beta}$.

Thus, the Expected Loss is given by:

$$\begin{aligned} EL &= E\left[\left(1 - e^{\mu+\sigma\,\xi_j}\right)^+ \times 1_{\left\{\Psi_j < \Phi^{-1}(PD)\right\}}\right] \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[1 - e^{\mu+\sigma\,x}\right]^+ \times 1_{\left\{y < \Phi^{-1}(PD)\right\}} \times \varphi_2(x\,;y\,;K)\,dx\,dy \\ &= \int_{-\infty}^{\Phi^{-1}[PD]}\int_{-\infty}^{-\mu/\sigma} \left[1 - e^{\mu+\sigma\,y}\right] \times \varphi_2(x\,;y\,;K)\,dx\,dy \\ &= \Phi_2\left[\Phi^{-1}[PD]\,;-\mu/\sigma\,;K\right] - \int_{-\infty}^{\Phi^{-1}[PD]}\int_{-\infty}^{-\mu/\sigma} e^{\mu+\sigma\,y} \times \varphi_2(x\,;y\,;K)\,dx\,dy \end{aligned}$$

This last integral is computed by the following linear change of variables:

$$\begin{cases} x_1 = x - \sigma K \\ y_1 = y - \sigma \end{cases}$$

Finally, the expected loss is given by:

$$\boxed{EL_{correlated} = \Phi_2\left[\Phi^{-1}(PD)\,;-\frac{\mu}{\sigma}\,;K\right] - e^{\mu+\sigma^2/2} \times \Phi_2\left[\Phi^{-1}(PD) - \sigma K\,;-\frac{\mu}{\sigma} - \sigma\,;K\right]} \qquad (A10)$$

<u>**Aggregated loss**</u>

The aggregated loss $L_{\Psi,\xi}$ can be written as: $L = E\left[LGD_j \times D_j \mid \Psi, \xi\right] = E^Q\left[\left(1 - e^{\tilde{\mu}+\tilde{\sigma}\,\bar{\xi}_j}\right)^+ \times 1_{\left\{\bar{\Psi}_j < \tilde{z}\right\}}\right]$

with: $\tilde{\mu} = \mu + \sigma\sqrt{\beta}\,\xi$, $\tilde{\sigma} = \sigma\sqrt{1-\beta}$ and $\tilde{z} = \dfrac{z - \sqrt{\rho}\,\Psi}{\sqrt{1-\rho}}$

The specific risk factors $(\bar{\Psi}_j, \bar{\xi}_j)$ are standard Gaussian with correlation γ. This expectation has been computed when evaluating the expected loss:

$$\boxed{L = \Phi_2\left[\Phi^{-1}[F(\Psi)]\,;\Phi^{-1}[G(\xi)]\,;\gamma\right] - e^{\mu+\sigma^2/2} \times e^{\sigma\sqrt{\beta}\,\xi - \beta\sigma^2/2} \times \Phi_2\left[\Phi^{-1}[F(\Psi)] - \sigma\gamma\sqrt{1-\beta}\,;\Phi^{-1}[G(\xi)] - \sigma\sqrt{1-\beta}\,;\gamma\right]}$$

with: $F(\Psi) = \Phi\left[\dfrac{\Phi^{-1}(PD) - \sqrt{\rho}\,\Psi}{\sqrt{1-\rho}}\right]$ and $G(\Psi) = \Phi\left[-\dfrac{\mu/\sigma + \sqrt{\beta}\,\xi}{\sqrt{1-\beta}}\right]$

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