A comparative analysis of CDO pricing models

Credit Workshop
Isaac Newton Institute
26 February 2005

Jean-Paul Laurent
Professor ISFA Actuarial School, University of Lyon
& Scientific Consultant, BNP-Paribas
laurent.jeanpaul@free.fr, http://laurent.jeanpaul.free.fr

Joint work with X. Burtschell and J. Gregory

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Comparative analysis of CDO pricing models

- Model dialectics
  - *Benchmark model: one factor Gaussian model*
    - Used by all major investments banks to communicate quotes
    - On a large and liquid market market of synthetic CDO tranches
  - *The oponents*
    - Within the « copula family »
      - Student *t* (O'Kane & Schloegl, Lindskog & McNeil), Clayton (Schönbucher), double *t* (Hull & White), multifactor Gaussian, Marshall-Olkin (Giesecke, Lindskog & McNeil), random factor loadings (Andersen & Sidenius)
    - Intensity models
      - Affine Jump Diffusion (Hutt), Gamma (Joshi), Stochastic Networks (Davis, Backhaus & Frey, Giesecke), Hawkes (Giesecke)
    - Structural models
  - *United We Stand, Divided We Fall*

Comparative analysis of CDO pricing models

- Model dialectics
  - *Try to look for a lowest common denominator*
  - *« Homogeneity » assumption: default times are exchangeable*
    - Can be expressed through joint survival functions
    - Weaker form: the dependence structure is « exchangeable »
    - Expressed through the copula of default times
    - Weaker form: the dependence structure of default indicators is exchangeable
  - *Leads to a one factor representation thanks to de Finetti theorem*
    - Usual de Finetti involves infinite sequences
    - Finite de Finetti, Jaynes (1986)
    - Factor representation involves signed measures

Comparative analysis of CDO pricing models

- Model dialectics
  - *Non homogeneous models*
  - *Usually involves a factor representation*
    - Loading factors will depend on names
- Factor approach holds for many intensity models
  - *Let us consider Joshi (25 February 2005)*
  - *Compensator follows a Gamma process* $\Gamma(t)$
  - *Conditional survival probabilities* $\exp\left(-c_i\Gamma(t)\right)$
  - $\Gamma(t)$ *holds for the factor for time horizon t*
  - *Conditionally on the factor, default indicators are independent*

Comparative analysis of CDO pricing models

- Purpose of the presentation
  - *Assessment of CDO pricing models in a factor framework*
    - Comparisons of different models
    - Comparisons with market quotes
    - Based on tranche premiums
  - *Study the relevance of standard probabilistic tools such as tail dependence or non parametric measures of dependence (Kendall's tau)*
  - *Study how tranche premiums vary with parameters*
    - Stochastic orders theory
  - *Relate semi-analytical pricing approaches and large portfolio approximations (stochastic orders again)*

Comparative analysis of CDO pricing models

- $i = 1, \ldots, n$ names.
- $\tau_1, \ldots, \tau_n$ default times.
- N_i nominal of credit *i*,
- δ_i recovery rate
- $N_i(t) = 1_{\tau_i \leq t}$ default indicator
- $N_i(1-\delta_i)$ loss given default

Comparative analysis of CDO pricing models

- Factor approaches to joint distributions:
  - *V: low dimensional factor*
  - *Conditionally on V, default times are independent.*
  - *Conditional default and survival probabilities:*

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right), \quad q_t^{i|V} = Q\left(\tau_i > t \mid V\right).$$

- Why factor models ?
  - *Tackle with large dimensions*
- Need tractable dependence between defaults:
  - *Parsimonious modelling*
  - *Semi-explicit computations for CDO tranches*

Comparative analysis of CDO pricing models

- Semi-explicit pricing for CDO tranches
  - *Default payments are based on the* accumulated losses *on the pool of credits:*

$$L(t) = \sum_{1 \leq i \leq n} N_i (1 - \delta_i) N_i(t)$$

  - *Tranche premiums only involves call options on the accumulated losses*

$$E\left[\left(L(t) - K \right)^{+} \right]$$

Comparative analysis of CDO pricing models

- Characteristic function: $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
  - *By conditioning upon V and using conditional independence:*

$$\varphi_{L(t)}(u) = E\left[\prod_{1 \le j \le n} \left(1 - p_t^{j|V} + p_t^{j|V} \varphi_{1-\delta_j}(uN_j)\right)\right]$$

  - *Distribution of L(t) can be obtained by FFT*
    - Or other inversion technique
- Only need of conditional (on factor) probabilities $p_t^{i|V}$

Comparative analysis of CDO pricing models

- One factor Gaussian copula:
  - $V, \bar{V}_i, \ i = 1, \ldots, n$ *independent Gaussian,*

$$V_i = \rho_i V + \sqrt{1 - \rho_i^2}\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
  - F_i *marginal distribution function of default times*
  - *Conditional default probabilities:*

$$p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \rho_i^2}}\right)$$

Comparative analysis of CDO pricing models

- Student *t* copula
  - *Embrechts, Lindskog & McNeil, Greenberg et al, Mashal et al, O'Kane & Schloegl, Gilkes & Jobst*

$$\begin{cases} X_i = \rho V + \sqrt{1-\rho^2}\,\bar{V}_i \\ V_i = \sqrt{W} \times X_i \\ \tau_i = F_i^{-1}\left(t_\nu\left(V_i\right)\right) \end{cases}$$

  - $V, \bar{V}_i$ *independent Gaussian variables*
  - $\frac{\nu}{W}$ *follows a* χ_ν^2 *distribution*
  - *Conditional default probabilities (two factor model)*

$$p_t^{i|V,W} = \Phi\left(\frac{-\rho V + W^{-1/2} t_\nu^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right)$$

Comparative analysis of CDO pricing models

- *Clayton* copula
  - *Schönbucher & Schubert, Rogge & Schönbucher, Friend & Rogge, Madan et al*

$$V_i = \psi\left(-\frac{\ln U_i}{V}\right) \qquad \tau_i = F_i^{-1}\left(V_i\right) \qquad \psi(s) = \left(1+s\right)^{-1/\theta}$$

  - *V: Gamma distribution with parameter* θ
  - $U_1, \ldots, U_n$ *independent uniform variables*
  - *Conditional default probabilities (one factor model)*

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

Comparative analysis of CDO pricing models

- Double *t* model (Hull & White)

$$V_i = \rho_i \left(\frac{\nu - 2}{\nu} \right)^{1/2} V + \sqrt{1-\rho_i^2} \left(\frac{\bar{\nu} - 2}{\bar{\nu}} \right)^{1/2} \bar{V}_i$$

  - $V, \bar{V}_i$ *are independent Student t variables*
    - with ν and $\bar{\nu}$ degrees of freedom

$$\tau_i = F_i^{-1}\left(H_i\left(V_i\right)\right)$$

  - *where* H_i *is the distribution function of* V_i

$$p_t^{i|V} = t_{\bar{\nu}} \left(\left(\frac{\bar{\nu}}{\bar{\nu} - 2} \right)^{1/2} \frac{H_i^{-1}\left(F_i(t)\right) - \rho_i \left(\frac{\nu - 2}{\nu} \right)^{1/2} V}{\sqrt{1-\rho_i^2}} \right)$$

Comparative analysis of CDO pricing models

- Shock models (multivariate exponential copulas)
  - *Duffie & Singleton, Giesecke, Elouerkhaoui, Lindskog & McNeil, Wong*
- Modelling of default dates: $V_i = \min\left(V, \bar{V}_i\right)$
  - $V, \bar{V}_i$ *exponential with parameters* $\alpha, 1-\alpha$
  - *Default dates* $\tau_i = S_i^{-1}\left(\exp - \min\left(V, \bar{V}_i\right)\right)$
    - S_i marginal survival function
  - *Conditionally on* V, τ_i *are independent.*
- Conditional default probabilities

$$q_t^{i|V} = 1_{V > -\ln S_i(t)} S_i(t)^{1-\alpha}$$

Comparative analysis of CDO pricing models

- CDO margins (bps pa)
  - *With respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads 100 bp*
  - *5 years maturity*

	equity	mezzanine	senior
0%	5341	560	0.03
10%	3779	632	4.6
30%	2298	612	20
50%	1491	539	36
70%	937	443	52
100%	167	167	91

Comparative analysis of CDO pricing models

- Equity tranche premiums are decreasing wrt ρ
  - *General result ?*
  - *Supermodular function f is such that:*

$$f:\mathbb{R}^n \to \mathbb{R} \qquad \Delta_i^{\varepsilon} f(x) = f(x+\varepsilon e_i) - f(x)$$

$$\forall x \in \mathbb{R}^n, \forall \varepsilon, \delta > 0 \qquad \Delta_i^{\varepsilon} \Delta_j^{\delta} f(x) \geq 0$$

  - *Supermodular order*

$$X = (X_1, \ldots, X_n) \qquad Y = (Y_1, \ldots, Y_n)$$

$$X \leq_{\text{sm}} Y \iff E\left[f(X)\right] \leq E\left[f(Y)\right], \ \forall f \ \text{ supermodular}$$

Comparative analysis of CDO pricing models

- « Supermodular » order of Gaussian vectors
  - *Let X and Y be Gaussian vectors with zero mean*

$$\Sigma^X = \begin{pmatrix} 1 & \sigma_{12}^X & & & & \\ \sigma_{21}^X & 1 & & & & \\ & & 1 & & \sigma_{ij}^X & \\ & & & 1 & & \\ & & \sigma_{ji}^X & & 1 & \\ & & & & & 1 \end{pmatrix} \qquad \Sigma^Y = \begin{pmatrix} 1 & \sigma_{12}^Y & & & & \\ \sigma_{21}^Y & 1 & & & & \\ & & 1 & & \sigma_{ij}^Y & \\ & & & 1 & & \\ & & \sigma_{ji}^Y & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$$\Sigma^X \le \Sigma^Y \overset{\text{def}}{\Leftrightarrow} \sigma_{ij}^X \le \sigma_{ij}^Y, \quad \forall i, j$$

  - *Müller & Scarsini (2000), Müller (2001)*

$$\Sigma^X \le \Sigma^Y \Leftrightarrow X \le_{\text{sm}} Y$$

Comparative analysis of CDO pricing models

- « Stop-Loss » order
  - *Accumulated losses:* $L(t), L'(t)$

$$L(t) \leq_{\text{sl}} L'(t) \overset{\text{def}}{\Leftrightarrow} E\left[\left(L(t)-K\right)^{+}\right] \leq E\left[\left(L'(t)-K\right)^{+}\right], \forall K \geq 0$$

  - *Supermodular order of latent variables implies stop-loss order of accumulated losses*
  - *Thus,equity tranche premium is <u>always</u> decreasing with correlation*
  - *Guarantees uniqueness of « base correlation »*
  - *Monotonicity properties extend to Student t, Marshall-Olkin copulas*

Comparative analysis of CDO pricing models

- Second issue
  - *Equity tranche premium decrease with correlation*
  - *Does* $\rho = 100\%$ *correspond to some lower bound?*
  - $\rho = 100\%$ *corresponds to « comonotonic » default dates:*

$$\left(\tau_1,\ldots,\tau_n\right) \text{ comonotonic } \Leftrightarrow \left(\tau_1,\ldots,\tau_n\right) \overset{d}{=} \left(F_1^{-1}(U),\ldots,F_n^{-1}(U)\right)$$

    - where U is uniform

$$\left(\tau_1,\ldots,\tau_n\right) \leq_{\text{sm}} \left(F_1^{-1}(U),\ldots,F_n^{-1}(U)\right)$$

    - Tchen (1980)
  - $\rho = 100\%$ *is a <u>model free</u> lower bound for the equity tranche premium*

Comparative analysis of CDO pricing models

- Third issue
  - *Does* $\rho = 0\%$ *corresponds to the higher bound on the equity tranche premium?*
  - $\rho = 0\%$ *corresponds to the independence case between default dates*
  - *The answer is no, negative dependence can occur*
  - *Base correlation does not always exists*
    - Even in Gaussian copula models
  - *Factor models are usually associated with positive dependence*
    - i.e. independent default dates are smaller with respect to supermodular order

Comparative analysis of CDO pricing models

- Calibration issues
  - *One parameter copulas*
  - *Well suited for homogeneous portfolios*
  - *Dependence is « monotonic » in the parameter*
- Calibration procedure
  - *Fit Clayton, Student t, double t, Marshall Olkin parameters onto CDO equity tranches*
    - Computed under one factor Gaussian model
    - Or given market quotes on equity trances
  - *Reprice mezzanine and senior CDO tranches*
    - Given the previous parameters

Comparative analysis of CDO pricing models

- CDO margins (bps pa)
  - *With respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads 100 bp*
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Comparative analysis of CDO pricing models

ρ	0%	10%	30%	50%	70%	100%
θ	0	0.05	0.18	0.36	0.66	∞
ρ_6	0%	0%	14%	39%	63%	100%
ρ_{12}	0%	0%	22%	45%	67%	100%
ρ $t(4)$-$t(4)$	0%	12%	34%	55%	73%	100%
ρ $t(5)$-$t(4)$	0%	13%	36%	56%	74%	100%
ρ $t(4)$-$t(5)$	0%	12%	34%	54%	73%	100%
ρ $t(3)$-$t(4)$	0%	10%	32%	53%	75%	100%
ρ $t(4)$-$t(3)$	0%	11%	33%	54%	73%	100%
α	0	28%	53%	69%	80%	100%

Table 5: correspondence between parameters

Comparative analysis of CDO pricing models

ρ	0%	10%	30%	50%	70%	100%
Gaussian	560	633	612	539	443	167
Clayton	560	637	628	560	464	167
Student (6)	676	676	637	550	447	167
Student (12)	647	647	621	543	445	167
$t(4)$-$t(4)$	560	527	435	369	313	167
$t(5)$-$t(4)$	560	545	454	385	323	167
$t(4)$-$t(5)$	560	538	451	385	326	167
$t(3)$-$t(4)$	560	495	397	339	316	167
$t(4)$-$t(3)$	560	508	406	342	291	167
MO	560	284	144	125	134	167

Table 6: mezzanine tranche (bps pa)

Comparative analysis of CDO pricing models

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0.03	4.6	20	36	52	91
Clayton	0.03	4.0	18	33	50	91
Student (6)	7.7	7.7	17	34	51	91
Student (12)	2.9	2.9	19	35	52	91
$t(4)$-$t(4)$	0.03	11	30	45	60	91
$t(5)$-$t(4)$	0.03	10	29	45	59	91
$t(4)$-$t(5)$	0.03	10	29	44	59	91
$t(3)$-$t(4)$	0.03	12	32	47	71	91
$t(4)$-$t(3)$	0.03	12	32	47	61	91
MO	0.03	25	49	62	73	91

Table 7: senior tranche (bps pa)

Comparative analysis of CDO pricing models

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0%	0%	0%	0%	0%	100%
Clayton	0%	0%	2%	15%	35%	100%
Student (6)	3%	3%	4%	6%	13%	100%
Student (12)	0%	0%	0%	1%	4%	100%
$t(4)$-$t(4)$	0%	0%	1%	10%	48%	100%
$t(5)$-$t(4)$	0%	0%	0%	0%	0%	100%
$t(4)$-$t(5)$	0%	100%	100%	100%	100%	100%
$t(3)$-$t(4)$	0%	100%	100%	100%	100%	100%
$t(4)$-$t(3)$	0%	0%	0%	0%	0%	100%
MO	0%	28%	53%	69%	80%	100%

Table 8: coefficient of lower tail dependence (%)

Comparative analysis of CDO pricing models

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0%	1%	6%	16%	33%	100%
Clayton	0%	3%	8%	15%	25%	100%
Student (6)	0%	0%	1%	10%	26%	100%
Student (12)	0%	0%	3%	13%	30%	100%
MO	0%	16%	36%	53%	67%	100%

Table 9: Kendall's τ (%)

Comparative analysis of CDO pricing models

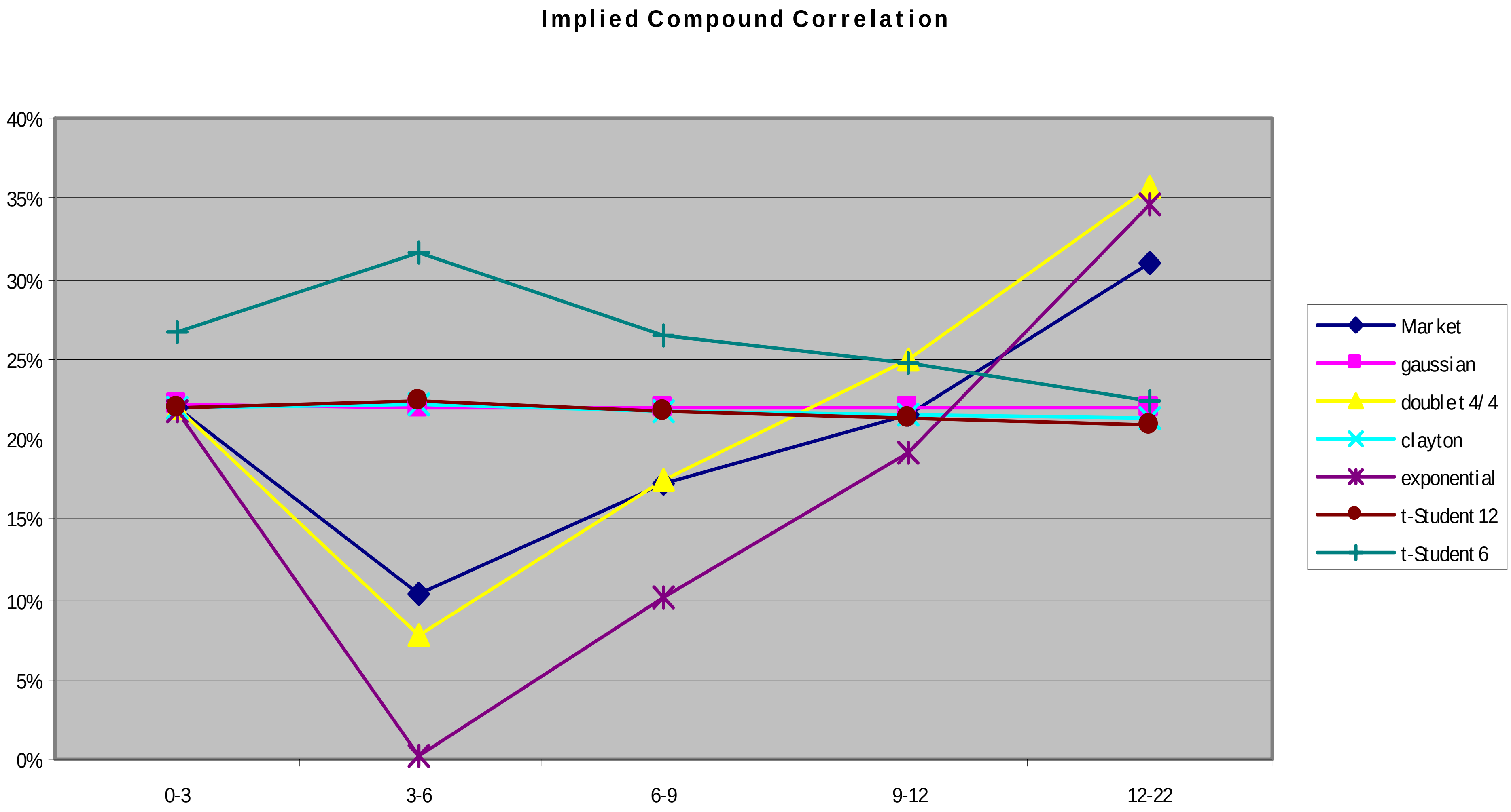
Tranches	Market	Gaussian	*t*(4)-*t*(4)	Clayton	MO	Student (12)	Student (6)
0%-3%	916	916	916	916	916	916	849
3%-6%	101	163	82	163	15	164	186
6%-9%	33	48	34	47	12	47	61
9%-12%	16	17	22	16	12	15	21
12%-22%	9	3	13	2	12	2	3

Table 15: CDO tranche premiums I-TRAXX (bps pa)

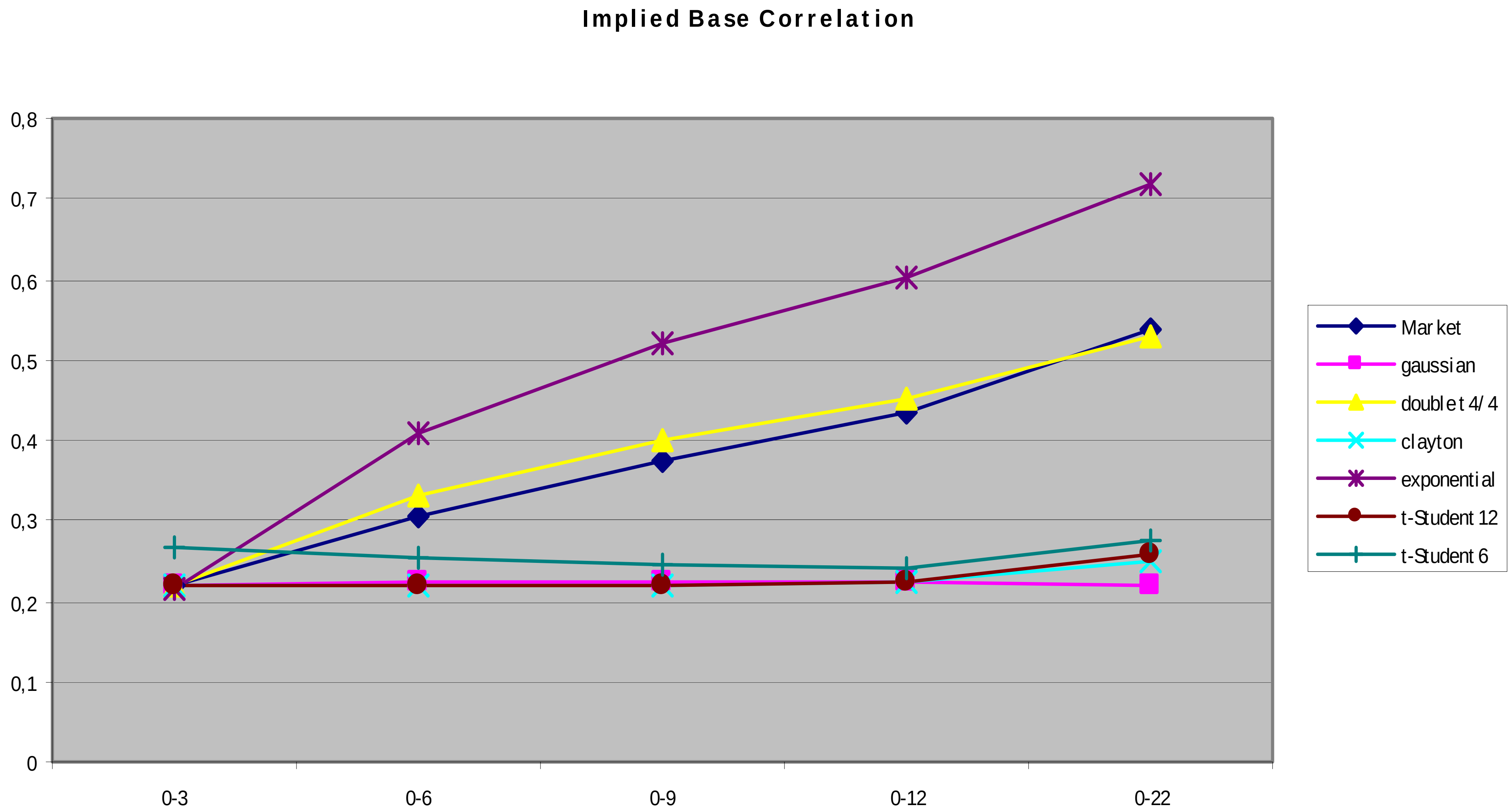
Tranches	Market	Gaussian	*t*(4)-*t*(4)	Clayton	MO	Student (12)	Student (6)
0%-3%	916	916	916	916	916	916	849
0%-6%	466	503	456	504	417	504	490
0%-9%	311	339	305	339	273	340	336
0%-12%	233	253	230	253	203	254	253
0%-22%	128	135	128	135	114	135	135

Table 16: CDO tranche premiums I-TRAXX (bps pa)

Comparative analysis of CDO pricing models



Comparative analysis of CDO pricing models



Model risk: choice of copula

- Related results:
  - *Student vs Gaussian*
    - Frey & McNeil, Mashal et al
    - Calibration on asset correlation
    - Distance between Gaussian and Student is bigger for low correlation levels
    - And extremes of the loss distribution
    - Joint default probabilities increase as number of degrees of freedom decreases
  - *Calibration onto joint default probabilities*
    - or default correlation, or aggregate loss variance
    - O'Kane & Schloegl, Schonbucher
  - *Gaussian, Clayton and Student t are all very similar*

Conclusion

- Factor models of default times:
  - *Simple computation of CDO's*
- Gaussian, Clayton and Student *t* copulas provide very similar patterns
- Shock models (Marshall-Olkin) quite different
- Double *t* provides intermediate results