



Modelling Portfolios of Correlated Credit Sensitive Exposures

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*A comparative analysis of CDO pricing models available on www.defaultrisk.com
Beyond the Gaussian copula: stochastic and local correlation for CDOs coming soon...*



Modelling Portfolios of Correlated Credit Sensitive Exposures

- *One factor Gaussian copula*
 - Factors models, semi-analytical computations
 - Ordering of risks, Base correlation
 - Gaussian extensions, correlation sensitivities
 - Stochastic recovery rates
- *Model dependence/Choice of copula*
 - Student t , double t , Clayton, Marshall-Olkin, Stochastic correlation
 - Calibration methodology, empirical results
 - Distribution of conditional default probabilities
- *Beyond the Gaussian copula*
 - Marginal compound correlation
 - Stochastic correlation and state dependent correlation
 - Local correlation



Semi explicit pricing, conditional default probabilities

- Factor approaches to joint default times distributions:
 - *V: low dimensional factor*
 - *Conditionally on V, default times are independent.*
 - *Conditional default and survival probabilities:*

$$p_t^{i|V} = Q(\tau_i \leq t | V), \quad q_t^{i|V} = Q(\tau_i > t | V).$$

- Why factor models ?
 - *Tackle with large dimensions (i-Traxx, CDX)*
- Need of tractable dependence between defaults:
 - *Parsimonious modelling*
 - *Semi-explicit computations for CDO tranches*
 - *Large portfolio approximations*



Semi explicit pricing, conditional default probabilities

- Semi-explicit pricing for CDO tranches

- Laurent & Gregory [2003]

- *Default payments are based on the accumulated losses on the pool of credits:*

$$L(t) = \sum_{i=1}^n LGD_i 1_{\{\tau_i \leq t\}}, \quad LGD_i = N_i (1 - \delta_i)$$

- *Tranche premiums only involve call options on the accumulated losses*

$$E \left[(L(t) - K)^+ \right]$$

- *This is equivalent to knowing the distribution of $L(t)$*



Semi explicit pricing, conditional default probabilities

- Characteristic function: $\varphi_{L(t)}(u) = E \left[e^{iuL(t)} \right]$
 - *By conditioning upon V and using conditional independence:*

$$\varphi_{L(t)}(u) = E \left[\prod_{1 \leq j \leq n} \left(1 - p_t^{j|V} + p_t^{j|V} \varphi_{1-\delta_j}(uN_j) \right) \right]$$

- *Distribution of $L(t)$ can be obtained by FFT*
 - Or other recursion technique
- Only need of conditional default probabilities $p_t^{i|V}$
- $p_t^{i|V}$ losses on a large homogeneous portfolio
 - *Approximation techniques for pricing CDOs*



Semi explicit pricing, conditional default probabilities

- One factor Gaussian copula:

- $V, \bar{V}_i, i = 1, \dots, n$ independent Gaussian,

$$V_i = \rho_i V + \sqrt{1 - \rho_i^2} \bar{V}_i$$

- *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
- F_i marginal distribution function of default times
- *Conditional default probabilities:*

$$p_t^{i|V} = \Phi \left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \rho_i^2}} \right)$$



One factor Gaussian copula

- CDO margins (bps pa)
 - *With respect to correlation*
 - *Gaussian copula*
 - *Attachment points: 3%, 10%*
 - *100 names*
 - *Unit nominal*
 - *Credit spreads 100 bp*
 - *5 years maturity*

	equity	mezzanine	senior
0%	5341	560	0.03
10%	3779	632	4.6
30%	2298	612	20
50%	1491	539	36
70%	937	443	52
100%	167	167	91



One factor Gaussian copula

- Equity tranche premiums are decreasing wrt ρ
 - *General result*
 - *Equity tranche premium is always decreasing with correlation parameter*
 - See Burtschell et al [2005] for more details about stochastic orders
 - *Guarantees uniqueness of « base correlation »*
 - *Monotonicity properties extend to Student t , Clayton and Marshall-Olkin copulas*



One factor Gaussian copula: extreme cases

- $\rho = 100\%$
 - *Equity tranche premiums decrease with correlation*
 - *Does $\rho = 100\%$ correspond to some lower bound?*
 - *$\rho = 100\%$ corresponds to « comonotonic » default dates:*
 - *$\rho = 100\%$ is a model free lower bound for the equity tranche premium*
- $\rho = 0\%$
 - *Does $\rho = 0\%$ correspond to the higher bound on the equity tranche premium?*
 - *$\rho = 0\%$ corresponds to the independence case between default dates*
 - *The answer is no, negative dependence can occur*
 - *Base correlation does not always exist*



One factor Gaussian copula and extensions

■ Gaussian extensions

- *Intra & intersector correlations*

$$V_i = \rho_{s(i)} W_{k(i)} + \sqrt{1 - \rho_{s(i)}^2} \bar{V}_i$$

- *i, name, s(i) sector*

- ρ systemic correlation

$$W_{s(i)} = \rho W + \sqrt{1 - \rho^2} \bar{W}_{s(i)}$$

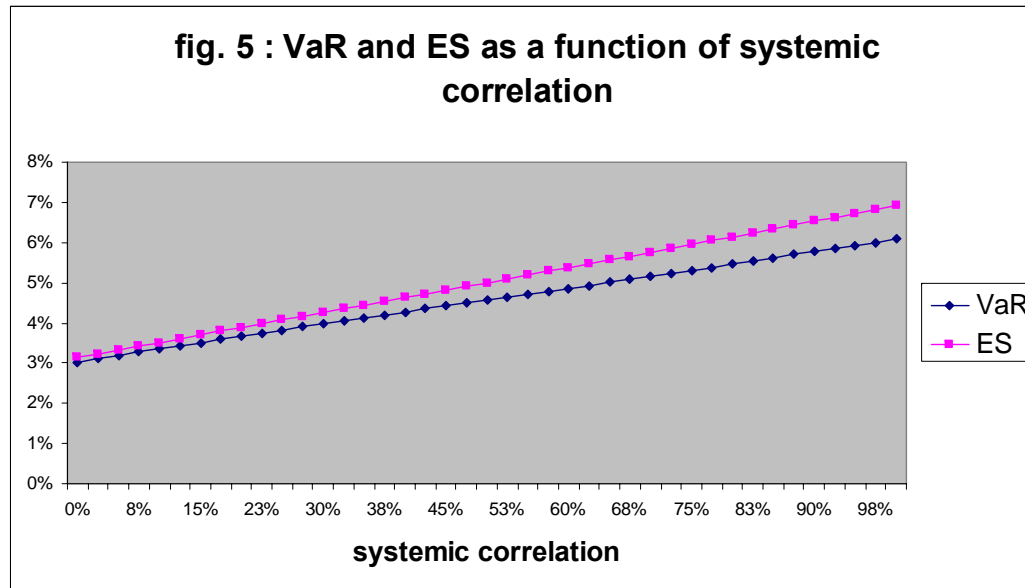
- *Accounting for sector diversification in risk assessment*

- *Risk measures based on unexpected losses, $\alpha = 99.9\%$*

	ζ (VaR)	κ (Expected Shortfall)
$\rho = 100\%$ (Basel II)	6,1%	6,9%
$\rho = 50\%$ (multifactor model)	4,6%	5,0%
Relative variation	-25%	-27%

One factor Gaussian copula and extensions

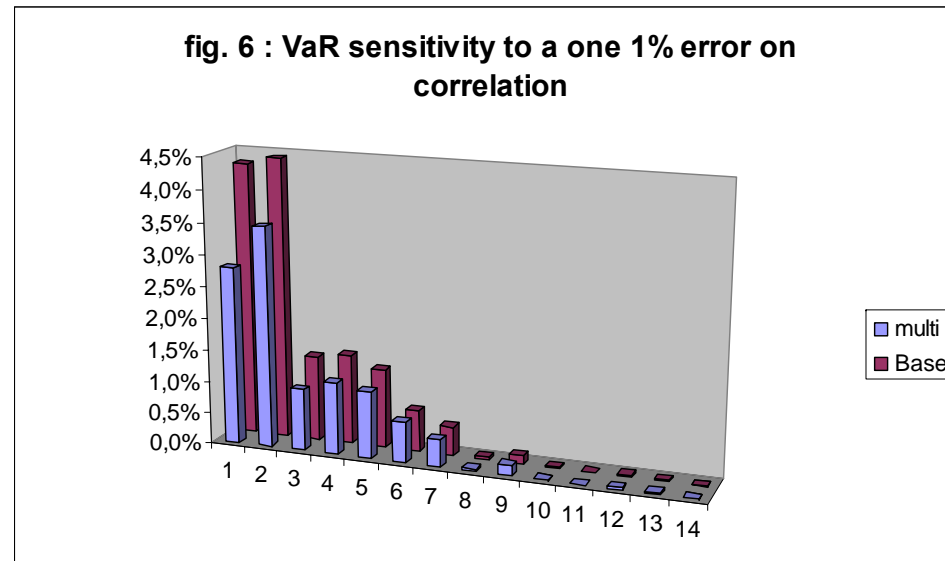
- VaR, Expected Shortfall and systemic correlation



- *Risk measures change almost linearly wrt to systemic correlation*
- *Basel II: no sector diversification*
- *Sector diversification lessens capital requirements*
 - See “Aggregation and credit risk measurement in retail banking”, Chabaane et al [2003]

One factor Gaussian copula and extensions

- VaR and intrasector correlation



- *Elasticity of VaR wrt intrasector correlation parameters*

$$\frac{\rho_J}{\zeta} \times \frac{\partial \zeta}{\partial \rho_J}$$

- *Lines 1 and 2 correspond to subportfolios with highest credit quality*



One factor Gaussian copula and extensions

- Correlation between default dates and recovery rates

- *One factor Gaussian copula for default dates* $\Psi_i = \sqrt{\rho}\Psi + \sqrt{1-\rho}\bar{\Psi}_i$

- *Losses Given Default also have a one factor structure:*

$$\xi_i = \sqrt{\beta}\xi + \sqrt{1-\beta}\bar{\xi}_i$$

- *Merton type LGD:* $\max(0, 1 - e^{\mu + \sigma\xi_i})$

- *A two factor Gaussian model with factors Ψ, ξ*

- *Correlation between defaults & recoveries and amongst recoveries*

- *See Credit Risk Assessment and Stochastic LGD's: an Investigation of Correlation Effects in Recovery Risk: The Next Challenge in Credit Risk Management, Risk Books*

One factor Gaussian copula and extensions

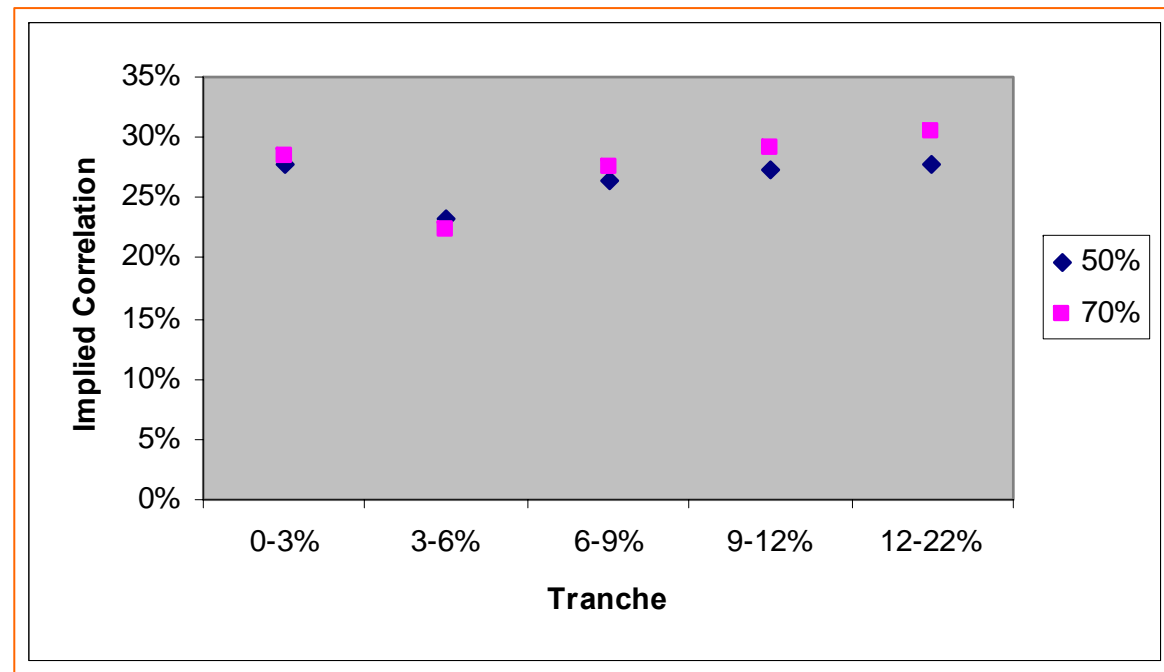
- Correlation between default dates and recovery rates
 - *VaR and ES as a function of correlation parameters*

β η	0%	20%	40%	60%	80%	100%
0%	158,9% 154,8%	161,0% 160,2%	164,2% 165,4%	162,5% 164,7%	159,3% 162,4%	145,9% 152,1%
20%	157,5% 153,9%	175,4% 175,6%	182,6% 183,7%	186,8% 188,6%	186,0% 192,5%	172,8% 179,8%
40%	160,2% 156,0%	194,1% 196,6%	207,9% 211,6%	211,8% 218,7%	212,6% 219,5%	205,7% 217,2%
60%	158,2% 155,2%	207,4% 210,3%	227,0% 231,1%	238,9% 243,0%	240,8% 249,2%	234,1% 243,4%
80%	159,6% 156,0%	223,1% 229,4%	244,1% 249,4%	257,4% 265,1%	264,5% 271,2%	260,5% 273,4%
100%	158,1% 153,9%	238,9% 246,4%	262,7% 268,0%	276,5% 287,3%	283,3% 296,3%	286,8% 296,6%

- *Taking into account correlation between default events and LGD leads to a substantial increase in VaR and Expected Shortfall*

One factor Gaussian copula and extensions

- Correlation between default dates and recovery rates
 - *Correlation smile implied from the correlated recovery rates*
 - *Not as important as what is found in the market*





Model dependence / choice of copula

- Stochastic correlation copula
 - $V, \bar{V}_i, i = 1, \dots, n$ independent Gaussian variables
 - $B_i = 1$ correlation ρ , $B_i = 0$ correlation β

$$V_i = B_i \left(\rho V + \sqrt{1 - \rho^2} \bar{V}_i \right) + (1 - B_i) \left(\beta V + \sqrt{1 - \beta^2} \bar{V}_i \right)$$

$$\tau_i = F_i^{-1}(\Phi(V_i))$$

$$p_t^{i|V} = p \Phi \left(\frac{-\rho V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \rho^2}} \right) + (1 - p) \Phi \left(\frac{-\beta V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \beta^2}} \right)$$



Model dependence / choice of copula

- Student t copula

- *Embrechts, Lindskog & McNeil, Greenberg et al, Mashal et al, O’Kane & Schloegl, Gilkes & Jobst*

$$\begin{cases} X_i = \rho V + \sqrt{1 - \rho^2} \bar{V}_i \\ V_i = \sqrt{W} \times X_i \\ \tau_i = F_i^{-1} (t_v (V_i)) \end{cases}$$

- V, \bar{V}_i independent Gaussian variables
- $\frac{v}{W}$ follows a χ_v^2 distribution
- Conditional default probabilities (two factor model)

$$P_t^{i|V,W} = \Phi \left(\frac{-\rho V + W^{-1/2} t_v^{-1} (F_i(t))}{\sqrt{1 - \rho^2}} \right)$$



Model dependence / choice of copula

- *Clayton copula*

- *Schönbucher & Schubert, Rogge & Schönbucher, Friend & Rogge, Madan et al*

$$V_i = \psi \left(-\frac{\ln U_i}{V} \right) \quad \tau_i = F_i^{-1}(V_i) \quad \psi(s) = (1+s)^{-1/\theta}$$

- Marshall-Olkin construction of archimedean copulas
- *V: Gamma distribution with parameter θ*
- *U_1, \dots, U_n independent uniform variables*
- *Conditional default probabilities (one factor model)*

$$p_t^{i|V} = \exp \left(V \left(1 - F_i(t)^{-\theta} \right) \right)$$



Model dependence / choice of copula

- Double t model (Hull & White)

$$V_i = \rho_i \left(\frac{\nu - 2}{\nu} \right)^{1/2} V + \sqrt{1 - \rho_i^2} \left(\frac{\bar{\nu} - 2}{\bar{\nu}} \right)^{1/2} \bar{V}_i$$

- V, \bar{V}_i are independent Student t variables
 - with ν and $\bar{\nu}$ degrees of freedom

$$\tau_i = F_i^{-1} \left(H_i \left(V_i \right) \right)$$

- where H_i is the distribution function of V_i

$$P_t^{i|V} = t_{\bar{\nu}} \left(\left(\frac{\bar{\nu}}{\bar{\nu} - 2} \right)^{1/2} \frac{H_i^{-1} \left(F_i(t) \right) - \rho_i \left(\frac{\nu - 2}{\nu} \right)^{1/2} V}{\sqrt{1 - \rho_i^2}} \right)$$



Model dependence / choice of copula

- Shock models (multivariate exponential copulas)
 - *Duffie & Singleton, Giesecke, Elouerkhaoui, Lindskog & McNeil, Wong*
- Modelling of default dates: $V_i = \min(V, \bar{V}_i)$
 - V, \bar{V}_i exponential with parameters $\alpha, 1-\alpha$
 - Default dates $\tau_i = S_i^{-1}(\exp(-\min(V, \bar{V}_i)))$
 - S_i marginal survival function
 - *Conditionally on V, τ_i are independent.*
- Conditional default probabilities

$$q_t^{i|V} = 1_{V > -\ln S_i(t)} S_i(t)^{1-\alpha}$$



Model dependence / choice of copula

- Calibration procedure
 - *One parameter copulas*
 - *Fit Clayton, Student t , double t , Marshall Olkin parameters onto CDO equity tranches*
 - Computed under one factor Gaussian model
 - Or given market quotes on equity tranches
 - *Reprice mezzanine and senior CDO tranches*
 - Given the previous parameter



Model dependence / choice of copula

- CDO margins (bps pa)
 - *With respect to correlation*
 - *Gaussian copula*
 - *Attachment points: 3%, 10%*
 - *100 names*
 - *Unit nominal*
 - *Credit spreads 100 bp*
 - *5 years maturity*

	equity	mezzanine	senior
0%	5341	560	0.03
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Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
θ	0	0.05	0.18	0.36	0.66	∞
ρ_6^2			14%	39%	63%	100%
ρ_{12}^2			22%	45%	67%	100%
ρ $t(4)$ - $t(4)$	0%	12%	34%	55%	73%	100%
ρ $t(5)$ - $t(4)$	0%	13%	36%	56%	74%	100%
ρ $t(4)$ - $t(5)$	0%	12%	34%	54%	73%	100%
ρ $t(3)$ - $t(4)$	0%	10%	32%	53%	75%	100%
ρ $t(4)$ - $t(3)$	0%	11%	33%	54%	73%	100%
α	0	28%	53%	69%	80%	100%

Table 5: correspondence between parameters



Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	560	633	612	539	443	167
Clayton	560	637	628	560	464	167
Student (6)			637	550	447	167
Student (12)			621	543	445	167
$t(4)-t(4)$	560	527	435	369	313	167
$t(5)-t(4)$	560	545	454	385	323	167
$t(4)-t(5)$	560	538	451	385	326	167
$t(3)-t(4)$	560	495	397	339	316	167
$t(4)-t(3)$	560	508	406	342	291	167
MO	560	284	144	125	134	167

Table 6: mezzanine tranche (bps pa)



Model dependence / choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0.03	4.6	20	36	52	91
Clayton	0.03	4.0	18	33	50	91
Student (6)			17	34	51	91
Student (12)			19	35	52	91
$t(4)-t(4)$	0.03	11	30	45	60	91
$t(5)-t(4)$	0.03	10	29	45	59	91
$t(4)-t(5)$	0.03	10	29	44	59	91
$t(3)-t(4)$	0.03	12	32	47	71	91
$t(4)-t(3)$	0.03	12	32	47	61	91
MO	0.03	25	49	62	73	91

Table 7: senior tranche (bps pa)

Gaussian, Clayton and Student t CDO premiums are close

Model dependence / choice of copula

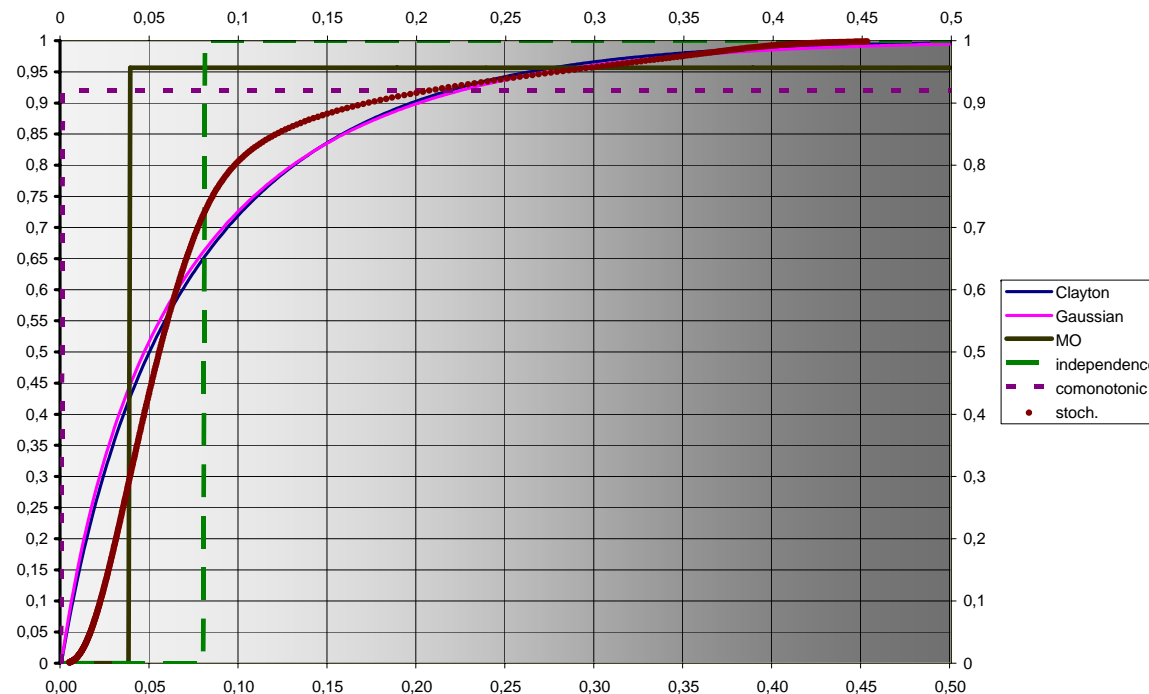
Why Clayton and Gaussian copulas provide same SL premiums?

- Loss distributions depend on the **distribution** of conditional default probabilities

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

$$p_t^{i|V} = \Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1 - \rho^2}}\right)$$

- Distribution of conditional default probabilities are close for Gaussian and Clayton





Matching the correlation skew

Tranches	Market	Gaussian	Clayton	Student (12)	$t(4)-t(4)$	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[3-6%]	101	163	163	164	82	122	14
[6-9%]	33	48	47	47	34	53	11
[9-12%]	16	17	16	15	22	29	11
[12-22%]	9	3	2	2	13	8	11

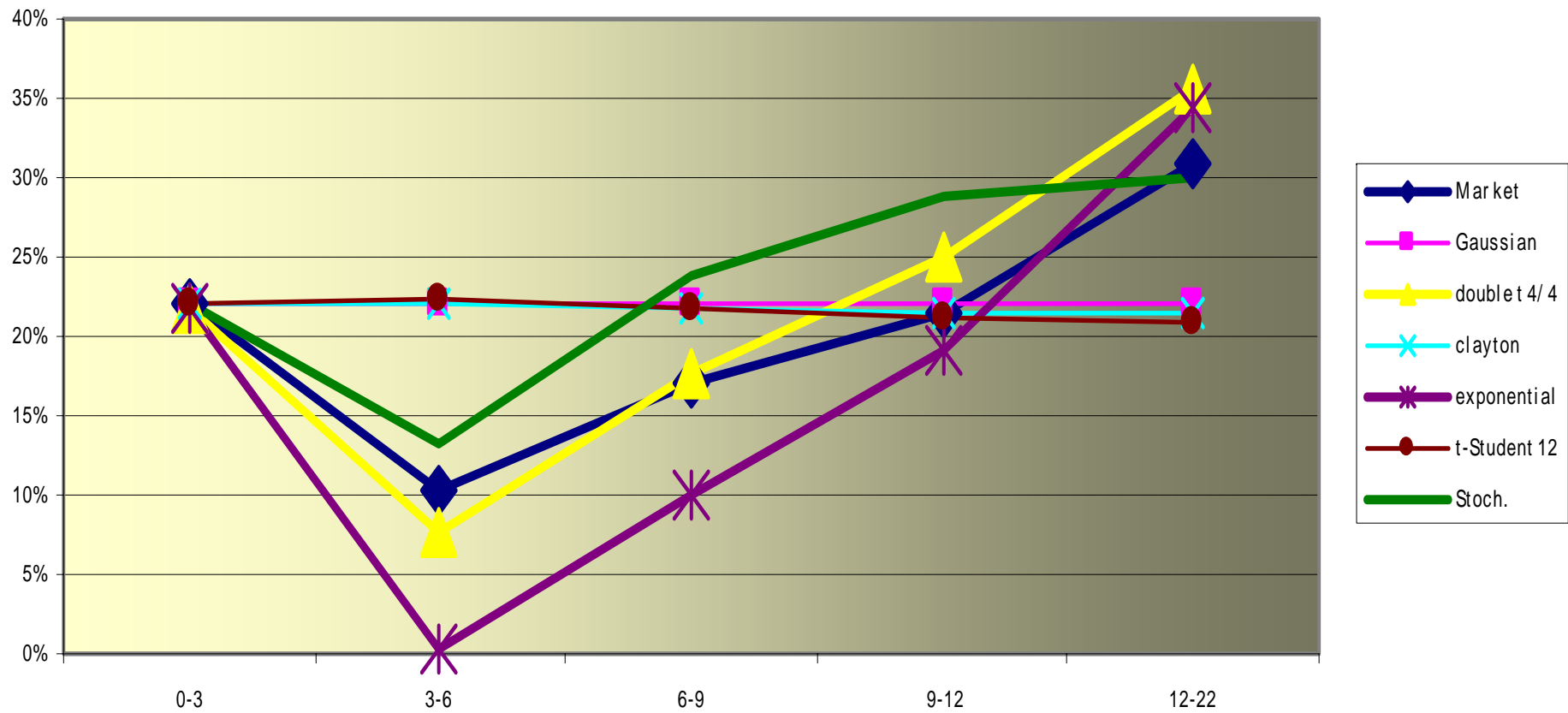
Table 17: CDO tranche premiums iTraxx (bps pa)

Tranches	Market	Gaussian	Clayton	Student (12)	$t(4)-t(4)$	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[0-6%]	466	503	504	504	456	479	418
[0-9%]	311	339	339	340	305	327	272
[0-12%]	233	253	253	254	230	248	203
[0-22%]	128	135	135	135	128	135	113

Table 18: “equity tranche” CDO tranche premiums iTraxx (bps pa)

Matching the correlation skew

implied compound correlation





Beyond the Gaussian copula: stochastic and local correlation

- Stochastic correlation

- Latent variables $V_i = \tilde{\rho}_i V + \sqrt{1 - \tilde{\rho}_i^2} \bar{V}_i, \quad i = 1, \dots, n$

$$\tilde{\rho}_i = (1 - B_s)(1 - B_i)\rho + B_s$$

$\tilde{\rho}_i$, stochastic correlation,

$Q(B_s = 1) = q_s$, systemic state,

$Q(B_i = 1) = q$, idiosyncratic state

- Conditional default probabilities

$$p_t^{V, B_s=0} = (1 - q)\Phi\left(\frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1 - \rho^2}}\right) + qF(t), \quad F(t) \text{ default probability}$$

$$p_t^{V, B_s=1} = \mathbf{1}_{V \leq \Phi^{-1}(F(t))}, \quad \text{comonotonic}$$

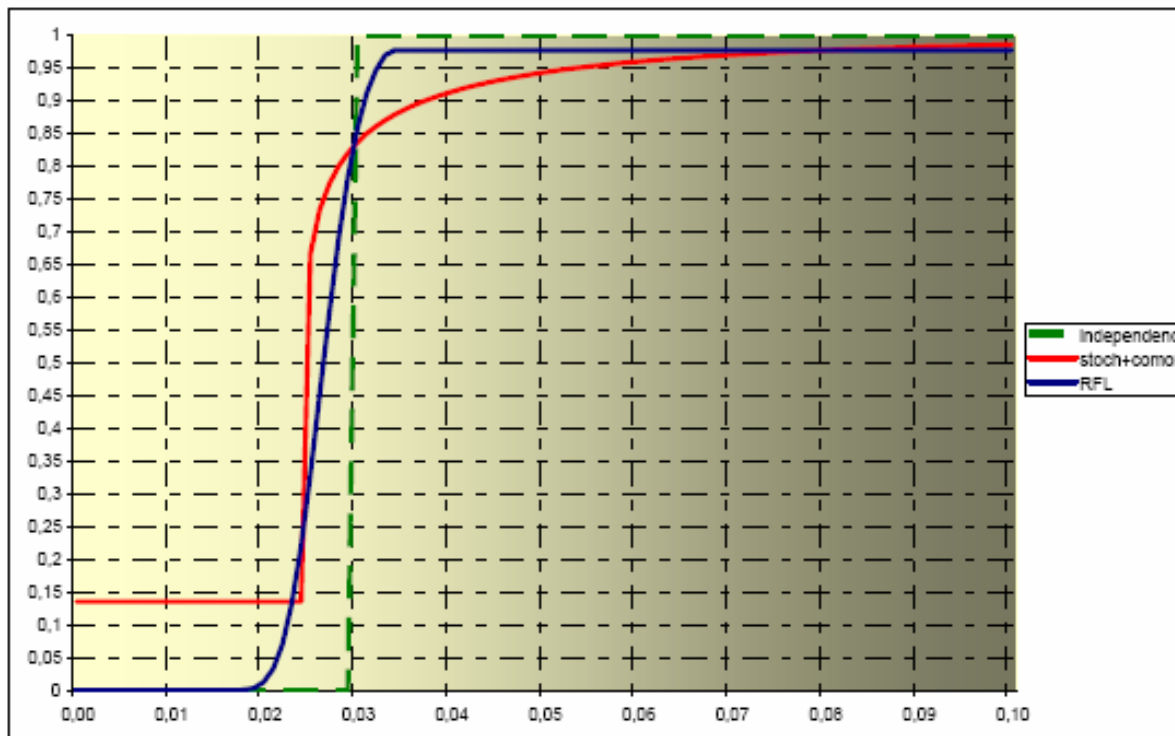


Beyond the Gaussian copula: stochastic and local correlation

- Stochastic correlation $\tilde{\rho}_i = (1 - B_s)(1 - B_i)\rho + B_s$
 - *Semi-analytical techniques for pricing CDOs available*
 - *Large portfolio approximation can be derived*
 - *Allows for Monte Carlo*
 - $\nearrow \rho, \searrow q_s, \searrow q$ *leads to increase senior tranche premiums*
- State dependent correlation $V_i = m_i(V)V + \sigma_i(V)\bar{V}_i, \quad i = 1, \dots, n$
 - *Local correlation* $V_i = -\rho(V)V + \sqrt{1 - \rho^2(V)}\bar{V}_i$
 - *Random factor loadings* $V_i = m + (l1_{V < e} + h1_{V \geq e})V + \nu\bar{V}_i$

Beyond the Gaussian copula: stochastic and local correlation

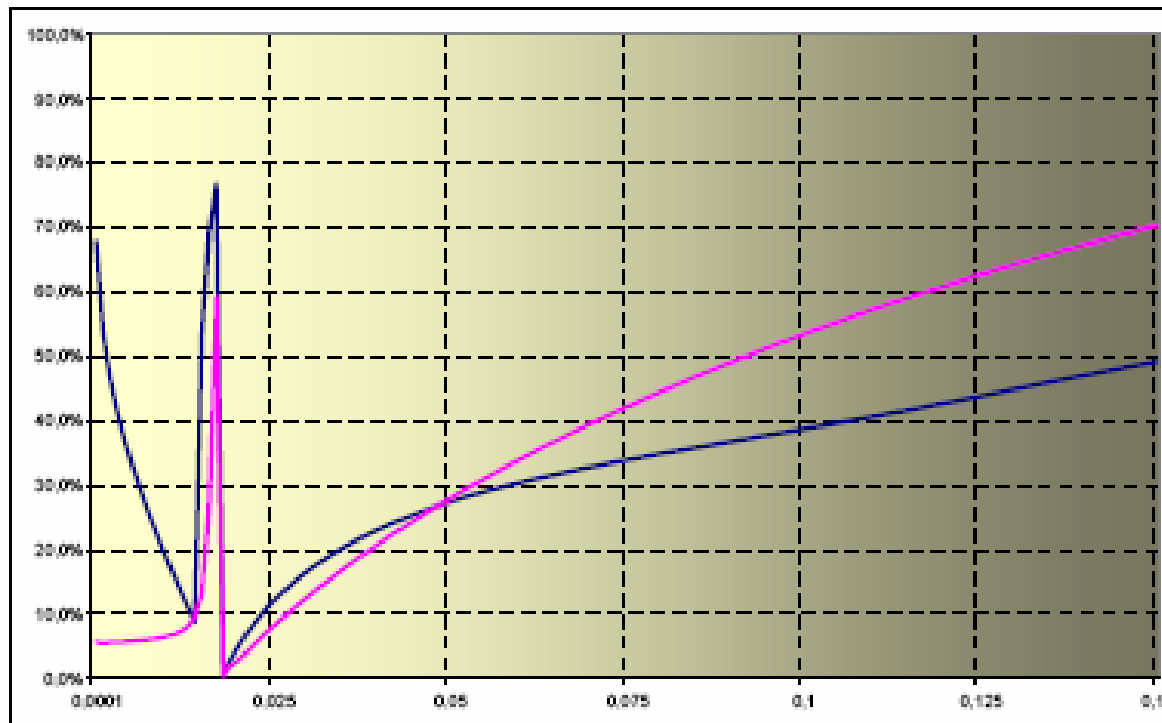
- Distribution functions of conditional default probabilities
 - *stochastic correlation vs RFL*



- *With respect to level of aggregate losses*
- *Also correspond to loss distributions on large portfolios*

Beyond the Gaussian copula: stochastic and local correlation

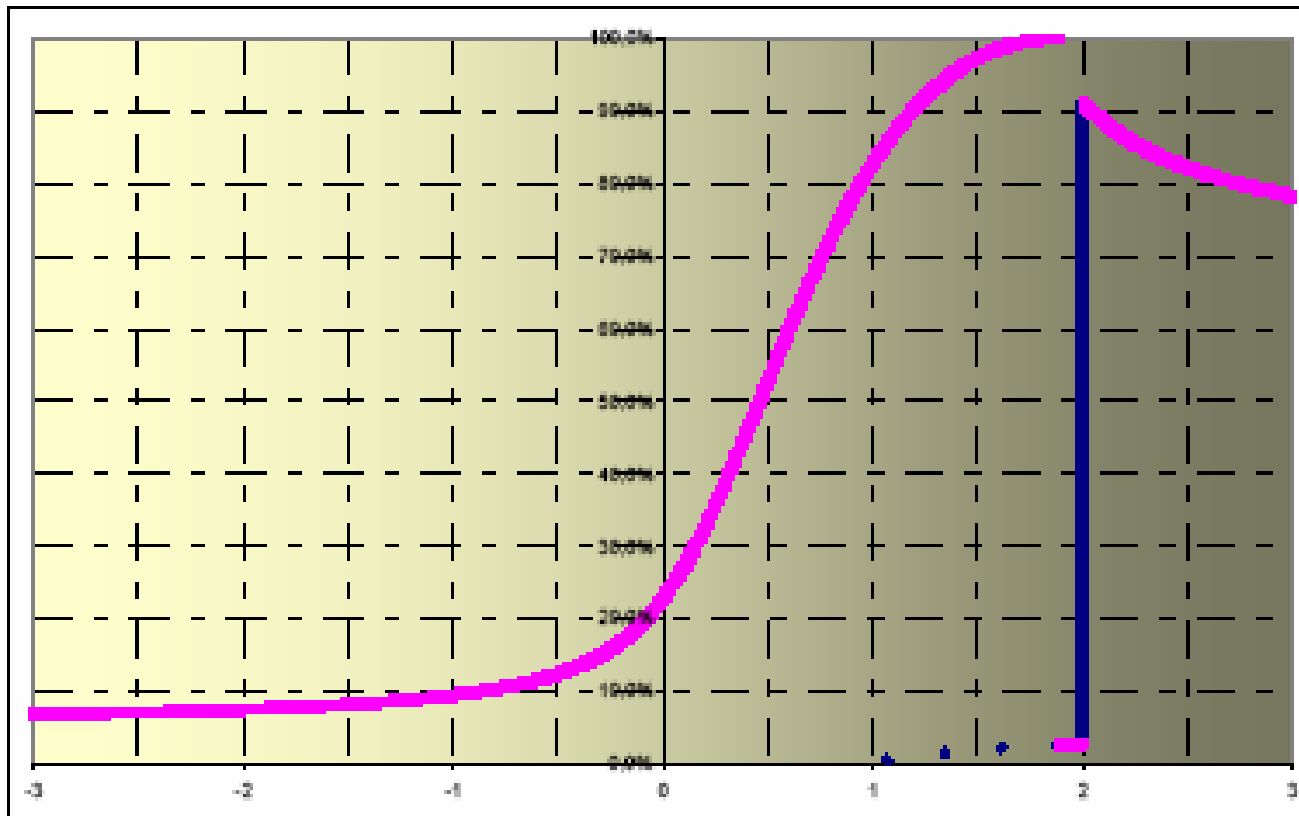
- Marginal compound correlations:
 - *With respect to attachment – detachment point*
 - *Compound correlation of a $[\alpha, \alpha]$ tranche*



- *Stochastic correlation vs RFL*

Beyond the Gaussian copula: stochastic and local correlation

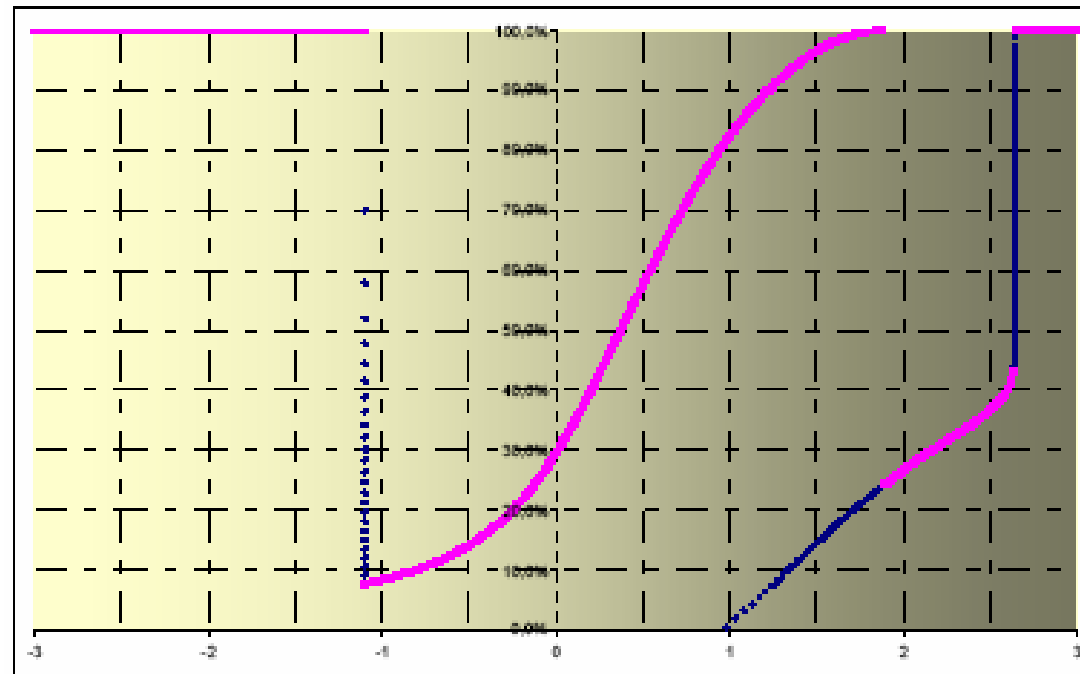
- Local correlation associated with RFL (as a function of the factor)



- Jump at threshold 2, low correlation level 5%, high correlation level 85%
- Possibly two local correlations

Beyond the Gaussian copula: stochastic and local correlation

- Local correlation associated with stochastic correlation model
 - *With respect to factor V*



- *Correlations of 1 for high-low values of V (comonotonic state)*
- *Possibly two local correlations leading to the same prices*
- *As for RFL, rather irregular pattern*



Beyond the Gaussian copula: stochastic and local correlation

- Market fits: stochastic correlation model

Tranche	Market	Model	Market ρ	Model ρ
Index	36			
[0-3%]	24%	25%	16%	14%
[3-6%]	83	84	4%	4%
[6-9%]	27	27	12%	12%
[9-12%]	14	14	17%	17%
[12-22%]	9	9	28%	28%
[22-100%]	4	2	63%	56%

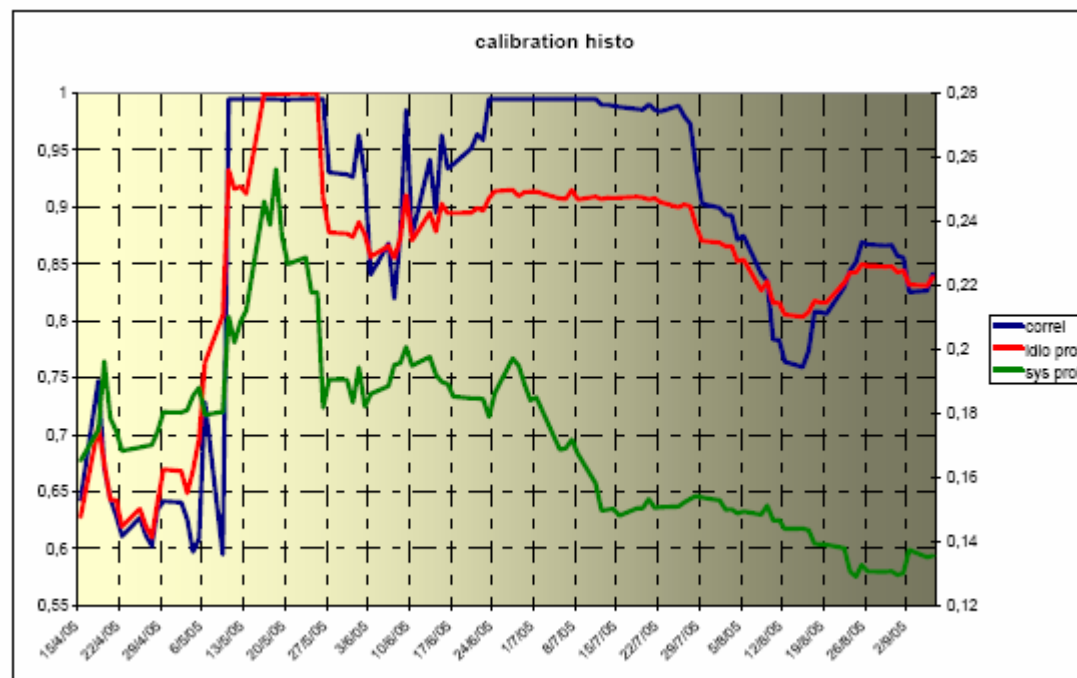
Table 2. Fit of model characterised by equation (2.8) to iTraxx market data on 31-August-2005. All values in bp pa unless otherwise stated. As usual the equity tranche is quoted as an up-front premium, in addition to the contractual 500 bp pa. $q_s = 0.13$, $q = 0.84$, $\rho^2 = 73.5\%$.

Tranche	Market	Model	Market ρ	Model ρ
Index	50			
[0-3%]	40%	38%	10%	13%
[3-7%]	126	139	2%	2%
[7-10%]	36	39	12%	13%
[10-15%]	20	17	20%	19%
[15-30%]	10	10	34%	34%
[30-100%]	2	3	59%	65%

Table 3. Fit of model characterised by equation (2.8) to CDX market data on 31-August-2005. All values in bp pa unless otherwise stated. As usual the equity tranche is quoted as an up-front premium, in addition to the contractual 500 bp pa. $q_s = 0.15$, $q = 0.84$, $\rho^2 = 85.3\%$.

Beyond the Gaussian copula: stochastic and local correlation

- Calibration history (from 15 April 2005)
 - *Implied correlation, implied idiosyncratic and systemic probabilities*



- *Trouble in fitting during the crisis*
- *Since then, decrease in systemic probability*



Conclusion

- Analysis of dependence through Gaussian models
 - *CDO premiums, Risk measures*
 - *Stochastic orders, base correlations*
 - *Analytical techniques, large portfolio approximations*
- Matching the skew with second generation models
 - *RFL, double t*
 - *Conditional default probability distributions are the drivers*
 - *Technique can be extended to structural or intensity models*
- Beyond the Gaussian copula
 - *Stochastic, local & marginal compound correlation*
- Pricing bespoke portfolios, CDO squared with a consistent model