Hedging Issues for CDOs

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Jean-Paul LAURENT
Professor, ISFA Actuarial School, University of Lyon,
Scientific consultant, BNP PARIBAS
http://laurent.jeanpaul.free.fr

Presentation related to papers
A note on the risk management of CDOs (2006)
Hedging default risks of CDOs in Markovian contagion models (2007)
Available on www.defaultrisk.com
Hedging Default and Credit Spread Risks within CDOs

• Purpose of the presentation
  ➢ Not trying to embrace all risk management issues
  ➢ Focus on very specific aspects of default and credit spread risk

• Overlook of the presentation
  ➢ Economic background
  ➢ Tree approach to hedging defaults
  ➢ Hedging credit spread risks for large portfolios
I - Economic Background

- Hedging CDOs context
- About 1 000 papers on defaultrisk.com
- About 10 papers dedicated to hedging issues
  - In interest rate or equity markets, pricing is related to the cost of the hedge
  - In credit markets, pricing is disconnect from hedging
- Need to relate pricing and hedging
- What is the business model for CDOs?
- Risk management paradigms
  - Static hedging, risk-return arbitrage, complete markets
- Static hedging
- Buy a portfolio of credits, split it into tranches and sell the tranches to investors
  - No correlation or model risk for market makers
  - No need to dynamically hedge with CDS
- Only « budget constraint »:
  - Sum of the tranche prices greater than portfolio of credits price
  - Similar to stripping ideas for Treasury bonds
- No clear idea of relative value of tranches
  - Depends of demand from investors
  - Markets for tranches might be segmented
I - Economic Background

• Risk – return arbitrage

• Historical returns are related to ratings, factor exposure
  – CAPM, equilibrium models
  – In search of high alphas
  – Relative value deals, cross-selling along the capital structure

• Depends on the presence of « arbitrageurs »
  – Investors with small risk aversion
    ➢ Trading floors, hedge funds
  – Investors without too much accounting, regulatory, rating constraints
\textbf{I - Economic Background}

- The ultimate step: complete markets
  - As many risks as hedging instruments
  - News products are only designed to save transactions costs and are used for risk management purposes
  - Assumes a high liquidity of the market

- Perfect replication of payoffs by dynamically trading a small number of « underlying assets »
  - Black-Scholes type framework
  - Possibly some model risk

- This is further investigated in the presentation
  - Dynamic trading of CDS to replicate CDO tranche payoffs
• Default risk
  – Default bond price jumps to recovery value at default time.
  – Drives the CDO cash-flows

• Credit spread risk
  – Changes in defaultable bond prices prior to default
    ➢ Due to shifts in credit quality or in risk premiums
  – Changes in the marked to market of tranches

• Interactions between credit spread and default risks
  – Increase of credit spreads increase the probability of future defaults
  – Arrival of defaults may lead to jump in credit spreads
    ➢ Contagion effects (Jarrow & Yu)
Credit deltas in copula models

CDS hedge ratios are computed by bumping the marginal credit curves

- Local sensitivity analysis
- Focus on credit spread risk
- Deltas are copula dependent
- Hedge over short term horizons
  - Poor understanding of gamma, theta, vega effects
  - Does not lead to a replication of CDO tranche payoffs

Last but not least: not a hedge against defaults…
Credit deltas in copula models

- Stochastic correlation model (Burstchell, Gregory & Laurent, 2007)
Main assumptions and results

- Credit spreads are driven by defaults
  - Contagion model
  - Credit spreads are deterministic between two defaults
- Homogeneous portfolio
  - Only need of the CDS index
  - No individual name effect
- Markovian dynamics
  - Pricing and hedging CDOs within a binomial tree
  - Easy computation of dynamic hedging strategies
  - Perfect replication of CDO tranches
We will start with two names only

Firstly in a static framework
  – Look for a First to Default Swap
  – Discuss historical and risk-neutral probabilities

Further extending the model to a dynamic framework
  – Computation of prices and hedging strategies along the tree
  – Pricing and hedging of tranchelets

Multiname case: homogeneous Markovian model
  – Computation of risk-neutral tree for the loss
  – Computation of dynamic deltas

Technical details can be found in the paper:
  – “hedging default risks of CDOs in Markovian contagion models”
II - Tree approach to hedging defaults

• Some notations:
  – $\tau_1, \tau_2$ default times of counterparties 1 and 2,
  – $\mathcal{H}_t$ available information at time $t$,
  – $P$ historical probability,
  – $\alpha_1^P, \alpha_2^P$ : (historical) default intensities:
    \[ P[\tau_i \in [t, t+dt|H_t]] = \alpha_i^P dt, \; i = 1, 2 \]

• Assumption of « local » independence between default events
  – Probability of 1 and 2 defaulting altogether:
    \[ P[\tau_1 \in [t, t+dt], \tau_2 \in [t, t+dt|H_t]] = \alpha_1^P dt \times \alpha_2^P dt \text{ in } (dt)^2 \]
  – Local independence: simultaneous joint defaults can be neglected
Tree approach to hedging defaults

Building up a tree:

- Four possible states: \((D,D), (D,ND), (ND,D), (ND,ND)\)
- Under no simultaneous defaults assumption \(p_{(D,D)} = 0\)
- Only three possible states: \((D,ND), (ND,D), (ND,ND)\)
- Identifying (historical) tree probabilities:

\[
\begin{align*}
\alpha_1^P dt & \quad (D,ND) \\
\alpha_2^P dt & \quad (ND,D) \\
1 - (\alpha_1^P + \alpha_2^P) dt & \quad (ND,ND)
\end{align*}
\]

\[
\begin{align*}
p_{(D,D)} &= 0 \Rightarrow p_{(D,ND)} = p_{(D,D)} + p_{(D,ND)} = p_{(.,.)} = \alpha_1^P dt \\
p_{(D,D)} &= 0 \Rightarrow p_{(ND,D)} = p_{(D,D)} + p_{(ND,D)} = p_{(.,D)} = \alpha_2^P dt \\
p_{(ND,ND)} &= 1 - p_{(.,.)} - p_{(.,D)}
\end{align*}
\]
II - *Tree approach to hedging defaults*

- Stylized cash flows of short term digital CDS on counterparty 1:
  - $\alpha_1^O \, dt$ CDS 1 premium
    
    \[
    \begin{array}{c}
    \alpha_1^P \, dt \\
    \alpha_2^P \, dt \\
    0 \\
    \end{array}
    \quad
    \begin{array}{c}
    1 - \alpha_1^O \, dt & (D, ND) \\
    -\alpha_1^O \, dt & (ND, D) \\
    -\alpha_1^O \, dt & (ND, ND) \\
    \end{array}
    \]

- Stylized cash flows of short term digital CDS on counterparty 2:

  \[
  \begin{array}{c}
  \alpha_1^P \, dt \\
  \alpha_2^P \, dt \\
  0 \\
  \end{array}
  \quad
  \begin{array}{c}
  -\alpha_2^O \, dt & (D, ND) \\
  1 - \alpha_2^O \, dt & (ND, D) \\
  1 - (\alpha_1^P + \alpha_2^P) \, dt & (ND, ND) \\
  \end{array}
  \]
II - Tree approach to hedging defaults

- Cash flows of short term digital first to default swap with premium $\alpha_F^O dt$:

  $\alpha_1^P dt \quad 1 - \alpha_F^O dt \quad (D, ND)$

  $\alpha_2^P dt \quad 1 - \alpha_F^O dt \quad (ND, D)$

  $\quad 1 - (\alpha_1^P + \alpha_2^P) dt \quad -\alpha_F^O dt \quad (ND, ND)$

- Cash flows of holding CDS 1 + CDS 2:

  $\alpha_1^P dt \quad 1 - (\alpha_1^O + \alpha_2^O) dt \quad (D, ND)$

  $\alpha_2^P dt \quad 1 - (\alpha_1^O + \alpha_2^O) dt \quad (ND, D)$

  $\quad 1 - (\alpha_1^P + \alpha_2^P) dt \quad -(\alpha_1^O + \alpha_2^O) dt \quad (ND, ND)$

- Perfect hedge of first to default swap by holding 1 CDS 1 + 1 CDS 2
  - Delta with respect to CDS 1 = 1, delta with respect to CDS 2 = 1
Absence of arbitrage opportunities imply:

- $\alpha^o_F = \alpha^o_1 + \alpha^o_2$

Arbitrage free first to default swap premium

- Does not depend on historical probabilities $\alpha^p_1, \alpha^p_2$

Three possible states: $(D, ND), (ND, D), (ND, ND)$

Three tradable assets: CDS1, CDS2, risk-free asset

For simplicity, let us assume $r = 0$
Three state contingent claims

- Example: claim contingent on state \((D, ND)\)
- Can be replicated by holding
  - \(1 \text{ CDS } 1 + \alpha_1^O dt\) risk-free asset

Replication price = \(\alpha_1^O dt\)

II - Tree approach to hedging defaults
Similarly, the replication prices of the $(ND, D)$ and $(ND, ND)$ claims

\[ \alpha_1^o \ dt \]

\[ \alpha_2^o \ dt \]

\[ 1 - (\alpha_1^o + \alpha_2^o) \ dt \]

\[ 0 \ (D, ND) \]

\[ 1 \ (ND, D) \]

\[ 0 \ (ND, ND) \]

Replication price of:

\[ \alpha_1^o \ dt \times a + \alpha_2^o \ dt \times b + \left(1 - (\alpha_1^o + \alpha_2^o) \ dt\right) c \]

II - Tree approach to hedging defaults
Replication price obtained by computing the expected payoff
- Along a risk-neutral tree

\[ \alpha_1^o dt \times a + \alpha_2^o dt \times b + \left( 1 - (\alpha_1^o + \alpha_2^o) dt \right) c \]

Risk-neutral probabilities
- Used for computing replication prices
- Uniquely determined from short term CDS premiums
- No need of historical default probabilities
II - Tree approach to hedging defaults

- Computation of deltas
  - Delta with respect to CDS 1: $\delta_1$
  - Delta with respect to CDS 2: $\delta_2$
  - Delta with respect to risk-free asset: $p$
    - $p$ also equal to up-front premium

\[
\begin{align*}
  a &= p + \delta_1 \times (1 - \alpha_1^O dt) + \delta_2 \times (-\alpha_2^O dt) \\
  b &= p + \delta_1 \times (-\alpha_1^O dt) + \delta_2 \times (1 - \alpha_2^O dt) \\
  c &= p + \delta_1 \times (-\alpha_1^O dt) + \delta_2 \times (-\alpha_2^O dt)
\end{align*}
\]

- As for the replication price, deltas only depend upon CDS premiums
### II - Tree approach to hedging defaults

- **Dynamic case:**
  - $\lambda_2^o \, dt$ CDS 2 premium after default of name 1
  - $\kappa_1^o \, dt$ CDS 1 premium after default of name 2
  - $\pi_1^o \, dt$ CDS 1 premium if no name defaults at period 1
  - $\pi_2^o \, dt$ CDS 2 premium if no name defaults at period 1

- **Change in CDS premiums due to contagion effects**
  - Usually, $\pi_1^o < \alpha_1^o < \lambda_1^o$ and $\pi_2^o < \alpha_2^o < \lambda_2^o$
II - Tree approach to hedging defaults

- Computation of prices and hedging strategies by backward induction
  - use of the dynamic risk-neutral tree
  - Start from period 2, compute price at period 1 for the three possible nodes
  - + hedge ratios in short term CDS 1,2 at period 1
  - Compute price and hedge ratio in short term CDS 1,2 at time 0
- Example to be detailed:
  - computation of CDS 1 premium, maturity = 2
  - $p_1 dt$ will denote the periodic premium
  - Cash-flow along the nodes of the tree
II - Tree approach to hedging defaults

- Computations CDS on name 1, maturity = 2

\[ 0 = \left( 1 - p_1 \right) \alpha_1^O + \left( -p_1 + \left( 1 - p_1 \right) \kappa_1^O - p_1 \left( 1 - \kappa_1^O \right) \right) \alpha_2^O \]
\[ + \left( -p_1 + \left( 1 - p_1 \right) \pi_1^O - p_1 \pi_2^O - p_1 \left( 1 - \pi_1^O - \pi_2^O \right) \right) \left( 1 - \alpha_1^O - \alpha_2^O \right) \]

- Premium of CDS on name 1, maturity = 2, time = 0, \( p_1 dt \) solves for:
**II - Tree approach to hedging defaults**

- Example: stylized zero coupon CDO tranchelets
  - Zero-recovery, maturity 2
  - Aggregate loss at time 2 can be equal to 0, 1, 2
    - Equity type tranche contingent on no defaults
    - Mezzanine type tranche: one default
    - Senior type tranche: two defaults

\[
\alpha_1^O dt \times \kappa_2^O dt + \alpha_2^O dt \times \kappa_1^O dt
\]

<table>
<thead>
<tr>
<th>State</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D, D)</td>
<td>$1$</td>
</tr>
<tr>
<td>(D, ND)</td>
<td>$\lambda_2^O dt$</td>
</tr>
<tr>
<td>(ND, D)</td>
<td>$\kappa_1^O dt$</td>
</tr>
<tr>
<td>(ND, ND)</td>
<td>$\pi_1^O dt \times \pi_2^O dt$</td>
</tr>
</tbody>
</table>
II - Tree approach to hedging defaults

- mezzanine tranche
  - Time pattern of default payments
    \[
    \alpha_1^O dt + \alpha_2^O dt + (1 - (\alpha_1^O + \alpha_2^O))dt (\pi_1^O + \pi_2^O) dt
    \]
    \[
    \begin{align*}
    \alpha_1^O dt & \quad 1 \quad (D, ND) \\
    \alpha_2^O dt & \quad 1 \quad (ND, D) \\
    1 - (\alpha_1^O + \alpha_2^O) dt & \quad 0 \quad (ND, ND) \\
    \end{align*}
    \]

- Possibility of taking into account discounting effects
- The timing of premium payments
- Computation of dynamic deltas with respect to short or actual CDS on names 1,2
In theory, one could also derive dynamic hedging strategies for index CDO tranches

- Numerical issues: large dimensional, non recombining trees
- Homogeneous Markovian assumption is very convenient

- CDS premiums at a given time $t$ only depend upon the current number of defaults $N(t)$
  - CDS premium at time 0 (no defaults) $\alpha_1^0 dt = \alpha_2^0 dt = \alpha^0 (t = 0, N(0) = 0)$
  - CDS premium at time 1 (one default) $\lambda^0 dt = \kappa_1^0 dt = \alpha^0 (t = 1, N(t) = 1)$
  - CDS premium at time 1 (no defaults) $\pi_1^0 dt = \pi_2^0 dt = \alpha^0 (t = 1, N(t) = 0)$
II - Tree approach to hedging defaults

- Homogeneous Markovian tree

- If we have $N(1) = 1$, one default at $t=1$
- The probability to have $N(2) = 1$, one default at $t=2$…
- Is $1 - \alpha^Q (1,1)$ and does not depend on the defaulted name at $t=1$
- $N(t)$ is a Markov process
- Dynamics of the number of defaults can be expressed through a binomial tree
II - Tree approach to hedging defaults

- From name per name to number of defaults tree

\[
\begin{align*}
\alpha^o(0,0) & \quad (D,ND) \\
\alpha^o(0,0) & \quad (ND,D) \\
1 - 2\alpha^o_i(0,0) & \quad (ND,ND)
\end{align*}
\]

\[
\begin{align*}
\alpha^o(1,1) & \quad (D,D) \\
1 - \alpha^o(1,1) & \quad (D,ND) \\
\alpha^o(1,0) & \quad (D,ND) \\
1 - 2\alpha^o(1,0) & \quad (ND,D)
\end{align*}
\]

Number of defaults tree

\[
\begin{align*}
\alpha^o(1,1) & \quad (D,D) \\
1 - \alpha^o(1,1) & \quad (D,ND) \\
\alpha^o(1,0) & \quad (D,ND) \\
1 - 2\alpha^o(1,0) & \quad (ND,D)
\end{align*}
\]
Easy extension to $n$ names

- Predefault name intensity at time $t$ for $N(t)$ defaults: $\alpha^O( t, N(t) )$
- Number of defaults intensity: sum of surviving name intensities:

$$\lambda(t, N(t)) = (n - N(t)) \alpha^O(t, N(t))$$

- $\alpha^O(0,0)$, $\alpha^O(1,0)$, $\alpha^O(1,1)$, $\alpha^O(2,0)$, $\alpha^O(2,1)$, … can be easily calibrated
- on marginal distributions of $N(t)$ by forward induction.
Previous recombining binomial risk-neutral tree provides a framework for the valuation of payoffs depending upon the number of defaults
  
  - CDO tranches
  - Credit default swap index

What about the credit deltas?
  
  - In a homogeneous framework, deltas with respect to CDS are all the same
  - Perfect dynamic replication of a CDO tranche with a credit default swap index and the default-free asset
  - Credit delta with respect to the credit default swap index
  - \[ \text{Credit delta} = \frac{\text{change in PV of the tranche}}{\text{change in PV of the CDS index}} \]
II - Tree approach to hedging defaults

- Example: number of defaults distribution at 5Y generated from a Gaussian copula
  - Correlation parameter: 30%
  - Number of names: 125
  - Default-free rate: 3%
  - 5Y credit spreads: 20 bps
  - Recovery rate: 40%

- Figure shows the corresponding expected losses for a 5Y horizon
II - Tree approach to hedging defaults

- Calibration of loss intensities
  - For simplicity, assumption of time homogeneous intensities
  - Figure below represents loss intensities, with respect to the number of defaults
  - Increase in intensities: contagion effects
### Dynamics of the 5Y CDS index spread

- In bp pa

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>Weeks</th>
<th>0</th>
<th>14</th>
<th>28</th>
<th>42</th>
<th>56</th>
<th>70</th>
<th>84</th>
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<td>61</td>
<td>58</td>
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<td>336</td>
<td>323</td>
<td>308</td>
<td>294</td>
<td></td>
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</tbody>
</table>
**II - Tree approach to hedging defaults**

- Dynamics of credit deltas:
  - [0,3%] equity tranche, buy protection
  - With respect to the 5Y CDS index
  - For selected time steps

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>OutStanding Nominal</th>
<th>0</th>
<th>14</th>
<th>28</th>
<th>42</th>
<th>56</th>
<th>70</th>
<th>84</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>3.00%</td>
<td>0.967</td>
<td>0.993</td>
<td>1.016</td>
<td>1.035</td>
<td>1.052</td>
<td>1.065</td>
<td>1.075</td>
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<tr>
<td>1</td>
<td>2.52%</td>
<td>0</td>
<td>0.742</td>
<td>0.786</td>
<td>0.828</td>
<td>0.869</td>
<td>0.908</td>
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<tr>
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<td>2.04%</td>
<td>0</td>
<td>0.439</td>
<td>0.484</td>
<td>0.532</td>
<td>0.583</td>
<td>0.637</td>
<td>0.691</td>
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<tr>
<td>3</td>
<td>1.56%</td>
<td>0</td>
<td>0.206</td>
<td>0.233</td>
<td>0.265</td>
<td>0.301</td>
<td>0.343</td>
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<tr>
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<td>0</td>
<td>0.082</td>
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<td>0.121</td>
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<tr>
<td>6</td>
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<td>0.006</td>
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<tr>
<td>7</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- Hedging strategy leads to a perfect replication of equity tranche payoff
- Prior to first defaults, deltas are above 1!
- When the number of defaults is > 6, the tranche is exhausted
**II - Tree approach to hedging defaults**

- **Credit deltas of the tranche**
  - Sum of credit deltas of premium and default legs

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>OutStanding Nominal</th>
<th>Premium leg</th>
<th>Default leg</th>
<th>Weeks</th>
<th>OutStanding Nominal</th>
<th>Premium leg</th>
<th>Default leg</th>
</tr>
</thead>
<tbody>
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**II - Tree approach to hedging defaults**

- **Credit deltas of the premium leg of the equity tranche**
  - Premiums based on outstanding nominal
  - Arrival of defaults reduces the commitment to pay
    - Smaller outstanding nominal
    - Increase in credit spreads (contagion) involve a decrease in expected outstanding nominal
  - **Negative deltas**
    - This is only significant for the equity tranche
      - Associated with much larger spreads

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<th>Weeks</th>
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**II - Tree approach to hedging defaults**

Credit deltas for the default leg of the equity tranche:

- Are actually between 0 and 1
- Gradually decrease with the number of defaults
  - Concave payoff, negative gammas
- Credit deltas increase with time
  - Consistent with a decrease in time value
  - At maturity date, when number of defaults < 6, delta=1

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<th>28</th>
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Dynamics of credit deltas
- Junior mezzanine tranche [3,6%]
- Deltas lie in between 0 and 1
- When the number of defaults is above 12, the tranche is exhausted

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<th>Weeks</th>
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Dynamics of credit deltas (junior mezzanine tranche)
- Gradually increase and then decrease with the number of defaults
- Call spread payoff (convex, then concave)
- Initial delta = 16% (out of the money option)

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• Comparison analysis
  – After six defaults, the [3,6%] should be like a [0,3%] equity tranche
  – However, credit delta is much lower
    ➢ 12% instead of 84%
  – But credit spreads after six defaults are much larger
    ➢ 127 bps instead of 19 bps
  – Expected loss of the tranche is much larger
  – Which is associated with smaller deltas
**Dynamics of credit deltas ([6,9%] tranche)**

- Initial credit deltas are smaller (deeper out of the money call spread)

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II - Tree approach to hedging defaults

- Small dependence of credit deltas with respect to recovery rate
  - Equity tranche, $R=30\%$

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- Equity tranche, $R=40\%$

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</tbody>
</table>
II - Tree approach to hedging defaults

- Small dependence of credit deltas with respect to recovery rate
  - Initial delta with respect to the credit default swap index
  - Only a small dependence of credit deltas with respect to recovery rates

> Which is rather fortunate
### II - Tree approach to hedging defaults

- Dependence of credit deltas with respect to correlation
  - Default leg, equity tranche

#### ρ=10%

<table>
<thead>
<tr>
<th>Nb Defaults</th>
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<th>0</th>
<th>14</th>
<th>28</th>
<th>42</th>
<th>56</th>
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<th>42</th>
<th>56</th>
<th>70</th>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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</table>
II - Tree approach to hedging defaults

\[ \begin{align*}
\rho = 10\%, \quad N(14) = 0, \quad \delta = 97\% & \quad \rho = 10\%, \quad N(14) = 1, \quad \delta = 93\% \\
\rho = 30\%, \quad N(14) = 0, \quad \delta = 84\% & \quad \rho = 30\%, \quad N(14) = 1, \quad \delta = 61\%
\end{align*} \]

- Smaller correlation
  - Prior to first default, higher expected losses on the tranche
    - Should lead to smaller deltas
  - But smaller contagion effects
    - When shifting from zero to one default
    - The expected loss on the index jumps due to…
      - Default arrival and jumps in credit spreads
      - Smaller jumps in credit spreads for smaller correlation
    - Smaller correlation is associated with smaller jumps in the expected loss of the index
    - Leads to higher deltas
      - Since we have negative gamma
What do we learn from this hedging approach?

- Thanks to stringent assumptions:
  - credit spreads driven by defaults
  - homogeneity
  - Markov property
- It is possible to compute a dynamic hedging strategy
  - Based on the CDS index
- That fully replicates the CDO tranche payoffs
  - Model matches market quotes of liquid tranches
  - Very simple implementation
  - Credit deltas are easy to understand
- Improve the computation of default hedges
  - Since it takes into account credit contagion
- Credit spread dynamics needs to be improved
When dealing with the risk management of CDOs, traders concentrate upon credit spread and correlation risk. They neglect default risk.

What about default risk? For large indices, default of one name has only a small direct effect on the aggregate loss.

Is it possible to build a framework where hedging default risk can be neglected?

And where one could only consider the hedging of credit spread risk? See paper “A Note on the risk management of CDOs.”
Main and critical assumption
- Default times follow a multivariate Cox process
  - For instance, affine intensities
  - Duffie & Garleanu, Mortensen, Feldhütter, Merrill Lynch

\[ \tau_i = \inf \left\{ t \in \mathbb{R}^+, U_i \geq \exp \left( - \int_0^t \lambda_{i,u} \, du \right) \right\}, \quad i = 1, \ldots, n \] (2.2)

where \( \lambda_1, \ldots, \lambda_n \) are strictly positive, \( F \) - progressively measurable processes, \( U_1, \ldots, U_n \) are independent random variables uniformly distributed on \([0,1]\) under \( Q \) and \( F \) and \( \sigma(U_1, \ldots, U_n) \) are independent under \( Q \).

No contagion effects
III - Hedging credit spread risks for large portfolios

- No contagion effects
  - credit spreads drive defaults but defaults do not drive credit spreads
  - For a large portfolio, default risk is perfectly diversified
  - Only remains credit spread risks: parallel & idiosyncratic

- Main result
  - With respect to dynamic hedging, default risk can be neglected
  - Only need to focus on dynamic hedging of credit spread risks
    - With CDS
  - Similar to interest rate derivatives markets
Formal setup

- $\tau_1, \ldots, \tau_n$ default times
- $N_i(t) = 1_{\{\tau_i \leq t\}}, i = 1, \ldots, n$ default indicators
- $H_t = \bigvee_{i=1,\ldots,n} \sigma(N_i(s), s \leq t)$ natural filtration of default times
- $F_t$ background (credit spread filtration)
- $G_t = H_t \vee F_t$ enlarged filtration, $P$ historical measure
- $l_i(t,T), i = 1, \ldots, n$ time $t$ price of an asset paying $N_i(T)$ at time $T$
III - Hedging credit spread risks for large portfolios

- Sketch of the proof
- Step 1: consider some smooth shadow risky bonds
  - Only subject to credit spread risk
  - Do not jump at default times
- Projection of the risky bond prices on the credit spread filtration

**Definition 3.2** The default free $T$ forward loss process associated with name $i \in \{0, \ldots, n\}$, denoted by $p^i(\cdot, T)$ is such that for $0 \leq t \leq T$:

$$p^i(t, T) \overset{A}{=} \mathbb{E}^Q \left[ p^i(T) \mid \mathcal{F}_t \right] = \mathbb{E}^Q \left[ N_i(T) \mid \mathcal{F}_t \right] = \mathcal{Q}(\tau_i \leq T \mid \mathcal{F}_t). \quad (3.2)$$

**Lemma 3.1** $p^i(t, T)$, $i = 1, \ldots, n$ are projections of the forward price processes $l^i(t, T)$ on $\mathcal{F}_t$:

$$p^i(t, T) = \mathbb{E}^Q \left[ l^i(t, T) \mid \mathcal{F}_t \right], \quad (3.3)$$

for $i = 1, \ldots, n$ and $0 \leq t \leq T$. 
III - Hedging credit spread risks for large portfolios

- Step 2: Smooth the aggregate loss process
- ... and thus the tranche payoffs
  - Remove default risk and only consider credit spread risk
  - Projection of aggregate loss on credit spread filtration

**Definition 3.1** We denote by $p^i(.)$, the default-free running loss process associated with name $i \in \{0, \ldots, n\}$, which is such that for $0 \leq t \leq T$:

$$p^i(t) \triangleq E^Q[N_i(t) \mid \mathcal{F}_t] = Q(\tau_i \leq t \mid \mathcal{F}_t) = 1 - \exp(-\Lambda_{i,t}).$$ (3.1)

**Definition 3.5** Default-free aggregate running loss process The default free aggregate running loss at time $t$ is such that for $0 \leq t \leq T$:

$$p_n(t) \triangleq \frac{1}{n} \sum_{i=1}^{n} p^i(t).$$ (3.7)
III - Hedging credit spread risks for large portfolios

- Step 3: compute perfect hedge ratios of the smoothed payoff
  - With respect to the smoothed risky bonds
    - Smoothed payoff and risky bonds only depend upon credit spread dynamics
    - Both idiosyncratic and parallel credit spread risks
    - Similar to a multivariate interest rate framework
    - Perfect hedging in the smooth market

Assumption 2 There exists some bounded \( \mathcal{F} \)-predictable processes \( \theta_1(.), \ldots, \theta_n(.) \) such that:

\[
(p_n(T) - K)^+ = E^Q [(p_n(T) - K)^+] + \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{T} \theta_i(t)dp^i(t, T) + z_n, \tag{4.2}
\]

where \( z_n \) is \( \mathcal{F}_T \)-measurable, of \( Q \)-mean zero and \( Q \)-strongly orthogonal to \( p^1(., T), \ldots, p^n(., T) \).
III - Hedging credit spread risks for large portfolios

- Step 4: apply the hedging strategy to the true defaultable bonds
- **Main result**
  - Bound on the hedging error following the previous hedging strategy
  - When hedging an actual CDO tranche with actual defaultable bonds
  - Hedging error decreases with the number of names
  
  ➢ Default risk diversification

**Proposition 1** Under Assumptions (1) and (2), the hedging error $\varepsilon_n$ defined as:

$$\varepsilon_n = (l_n(T) - K)^+ - E^Q [(l_n(T) - K)^+] - \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{T} \theta_i(t) dl^i(t, T),$$

is such that $E^P[|\varepsilon_n|]$ is bounded by:

$$\frac{1}{\sqrt{2n}} \left(1 + \left(E^Q \left[ \frac{dP}{dQ} \right]^2 \right)^{1/2} \right) + \frac{1}{n} \left(E^Q \left[ \left( \frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \left(\sum_{i=1}^{n} (Q(\tau_i \leq T) + E^Q [B_i | T]) \right)^{1/2} + E^P[|\varepsilon_n|].$$

(4.5)
III - Hedging credit spread risks for large portfolios

• Provides a hedging technique for CDO tranches
  – Known theoretical properties
  – Takes into account idiosyncratic and parallel gamma risks
  – Good theoretical properties rely on no simultaneous defaults, no contagion effects assumptions
  – Empirical work remains to be done

• Thought provocative
  – To construct a practical hedging strategy, do not forget default risk
  – Equity tranche [0,3%]
  – iTraxx or CDX first losses cannot be considered as smooth
Hedging credit spread risk for large portfolios

• Linking pricing and hedging?

• The black hole in CDO modeling?

• Standard valuation approach in derivatives markets
  ➢ Complete markets
  ➢ Price = cost of the hedging/replicating portfolio

• Mixing of dynamic hedging strategies
  – for credit spread risk

• And diversification/insurance techniques
  – For default risk
Two different models have been investigated:

- Contagion homogeneous Markovian models:
  - Perfect hedge of default risks
  - Easy implementation
  - Poor dynamics of credit spreads
  - No individual name effects

- Multivariate Cox processes:
  - Rich dynamics of credit spreads
  - But no contagion effects
  - Thus, default risk can be diversified at the index level
  - Replication of CDO tranches is feasible by hedging only credit spread risks.