Fast Analytic Techniques for Pricing Synthetic CDOs

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Fast Analytic Techniques for Pricing Synthetic CDOs

- Pricing of CDO tranches
  - *Premiums involves loss distributions*
  - *Computation of loss distributions in factor models*
- Model risk: choice of copula
  - *Default probabilities in Gaussian, Student, Clayton and Shock models*
  - *Empirical comparisons*
- Risk analysis
  - *Sensitivity with respect to credit curves*
  - *Correlation parameters*

Pricing of CDO tranches

- $i = 1, \ldots, n$ names.
- $\tau_1, \cdots, \tau_n$ default times.
- N_i nominal of credit *i*,
- δ_i recovery rate
- *Default indicator* $N_i(t) = 1_{\tau_i \le t}$, $N_i(1 - \delta_i)$ *loss given default*
- Default payments are based on the *accumulated losses* on the pool of credits

$$L(t) = \sum_{1 \le i \le n} N_i(1 - \delta_i) N_i(t)$$

Pricing of CDO tranches

- Tranches with thresholds $0 \leq A \leq B \leq \sum N_j$
  - *Mezzanine: losses are between A and B*
- Cumulated payments at time *t on mezzanine tranche*

$$M(t) = (L(t) - A)) 1_{[A,B]}(L(t)) + (B - A) 1_{]B,\infty[}(L(t))$$

- *Payments on default leg:*

$\Delta M(t) = M(t) - M(t^-)$ at time $t \leq T$

- *Payments on premium leg:*
  - periodic premium,
  - proportional to outstanding nominal: $B - A - M(t)$

Pricing of CDO tranches

- Upfront premium: $E\left[\int_0^T B(t)dM(t)\right]$
    - *B(t) discount factor, T maturity of CDO*
- Integration by parts $B(T)E[M(T)] + \int_0^T E[M(t)]dB(t)$
    - *Where* $E[M(t)] = (B-A)Q(L(t) > B) + \int_A^B (x-A)dF_{L(t)}(x)$
- Premium only involves loss distributions
- Contribution of names to the PV of the default leg
    - *See « Basket defaults swaps, CDO's and Factor Copulas » available on www.defaultrisk.com*

Pricing of CDO tranches

- Factor approaches to joint distributions:
  - V*: low dimensional factor*
  - *Conditionally on V, default times are independent.*
  - *Conditional default and survival probabilities:*

$$p_t^{i|V} = Q(\tau_i \leq t \mid V), \quad q_t^{i|V} = Q(\tau_i > t \mid V).$$

- Why factor models ?
  - *Tackle with large dimensions*
- Need tractable dependence between defaults:
  - *Parsimonious modeling*
  - *Semi-explicit computations for CDO tranches*

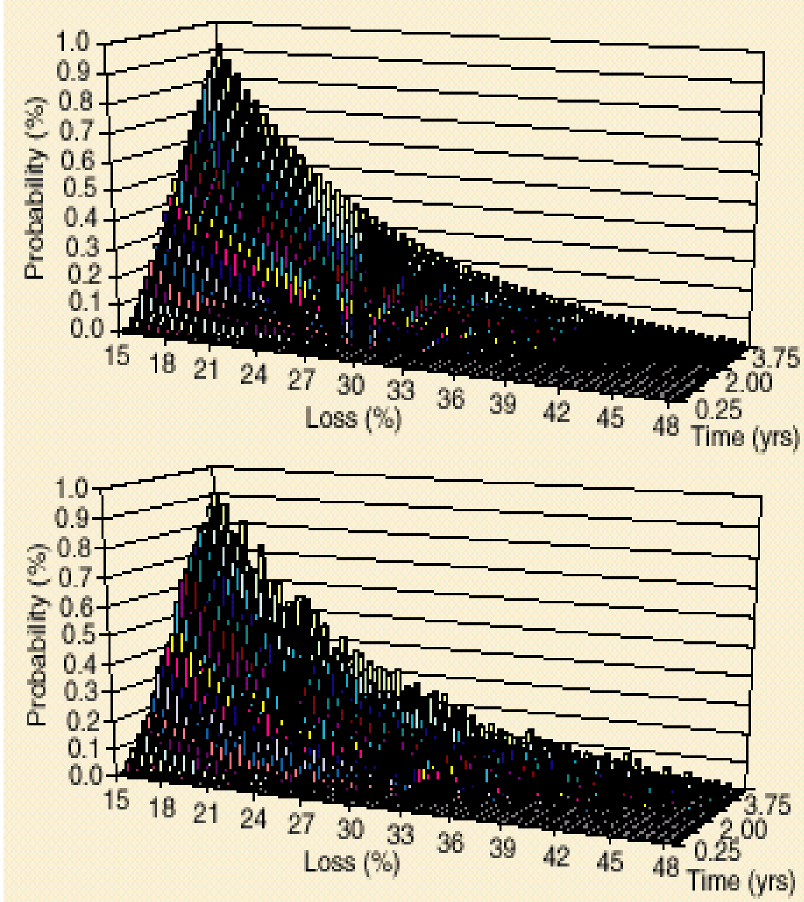
Pricing of CDO tranches

- Accumulated loss at *t*: $L(t) = \sum_{1 \leq i \leq n} N_i(1-\delta_i) N_i(t)$
  - *Where* $N_i(t) = 1_{\tau_i \leq t}$, $N_i(1-\delta_i)$ *loss given default.*
- Characteristic function: $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
  - *By conditioning:* $\varphi_{L(t)}(u) = E\left[\prod_{1 \leq j \leq n} \left(1 - p_t^{j|V} + p_t^{j|V} \varphi_{1-\delta_j}(uN_j)\right)\right]$
  - *Distribution of L(t) can be obtained by FFT*
    - Or other inversion technique
  - *Only need of conditional probabilities*

Pricing of CDO tranches

- CDO premiums only involve loss distributions
- One hundred names, same nominal.
- Recovery rates: 40%
- Credit spreads uniformly distributed between 60 and 250 bp.
- Gaussian copula, correlation: 50%
- 10^5 Monte Carlo simulations

3. Loss distribution



Loss distribution over time for the table B example with 50% correlation for the semi-explicit approach (top) and Monte Carlo simulation (bottom)

Pricing of CDO tranches

B. Pricing of five-year maturity CDO tranches

	Equity (0–3%)		Mezzanine (3–14%)		Senior (14–100%)	
	SE	MC	SE	MC	SE	MC
0%	8,219.4	8,228.5	816.2	814.3	0.0	0.0
20%	4,321.1	4,325.3	809.4	806.9	13.7	13.7
40%	2,698.8	2,696.7	734.3	731.4	33.4	33.2
60%	1,750.6	1,738.5	641.0	637.8	54.1	53.7
80%	1,077.5	1,067.9	529.5	526.9	77.0	76.6
100%	410.3	406.6	371.2	367.0	110.4	109.6

Premiums in basis points per annum as a function of correlation for 5-year maturity CDO tranches on a portfolio with credit spreads uniformly distributed between 60 and 250bp. The recovery rates are 40%

- *Semi-explicit vs MonteCarlo*
- *One factor Gaussian copula*
- *CDO tranches margins with respect to correlation parameter*

Model risk: choice of copula

- One factor Gaussian copula:
  - $V, \bar{V}_i,\ i = 1, \ldots, n$ *independent Gaussian,*

$$V_i = \rho_i V + \sqrt{1 - \rho_i^2}\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
  - F_i *marginal distribution function of default times*
  - *Conditional default probabilities:* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \rho_i^2}}\right)$

Model risk: choice of copula

- Student *t* copula
  - *Embrechts, Lindskog & McNeil, Greenberg et al, Mashal et al, O'Kane & Schloegl, Gilkes & Jobst*

$$\begin{cases} X_i = \rho V + \sqrt{1-\rho^2}\,\bar{V}_i \\ V_i = \sqrt{W} \times X_i \\ \tau_i = F_i^{-1}\left(t_\nu\left(V_i\right)\right) \end{cases}$$

  - $V, \bar{V}_i$ *independent Gaussian variables*
  - $\frac{\nu}{W}$ *follows a* χ_ν^2 *distribution*
  - *Conditional default probabilities (two factor model)*

$$p_t^{i|V,W} = \Phi\left(\frac{-\rho V + W^{-1/2} t_\nu^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right)$$

Model risk: choice of copula

- *Clayton* copula
  - *Schönbucher & Schubert, Rogge & Schönbucher, Friend & Rogge, Madan et al*

$$V_i = \psi\left(-\frac{\ln U_i}{V}\right) \quad \tau_i = F_i^{-1}(V_i) \quad \psi(s) = (1+s)^{-1/\theta}$$

  - *V: Gamma distribution with parameter* θ
  - $U_1, \ldots, U_n$ *independent uniform variables*
  - *Conditional default probabilities (one factor model)*

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

  - *Frailty model: multiplicative effect on default intensity*
  - *Copula:* $C(u_1, \ldots, u_n) = \left(u_1^{-\theta} + \ldots + u_n^{-\theta} - n + 1\right)^{-1/\theta}$

Model risk: choice of copula

- Shock models for previous models
  - *Duffie & Singleton, Giesecke, Elouerkhaoui, Lindskog & McNeil, Wong*
- Modeling of default dates: $\tau_i = \min(\bar{\tau}_i, \tau)$
  - $Q(\tau_i = \tau_j) \geq Q\left(\tau \leq \min(\bar{\tau}_i, \bar{\tau}_j)\right) > 0$ *simultaneous defaults.*
  - *Conditionally on* τ, τ_i *are independent.*

$$Q(\tau_1 \leq t_1, \dots, \tau_n \leq t_n \mid \tau) = \prod_{1 \leq i \leq n} Q(\tau_i \leq t_i \mid \tau)$$

- Conditional default probabilities (one factor model)

$$p_t^{i|\tau} = 1_{\tau > t} Q(\bar{\tau}_i \leq t) + 1_{\tau \leq t}$$

Model risk: choice of copula

- Calibration issues
  - *One parameter copulas*
  - *Well suited for homogeneous portfolios*
    - See later on for sector effects
  - *Dependence is « monotonic » in the parameter*
- Calibration procedure
  - *Fit Clayton, Student, Marshall Olkin parameters onto first to default or CDO equity tranches*
    - Computed under one factor Gaussian model
  - *Reprice n^{th} to default, mezzanine and senior CDO tranches*
    - Given the previous parameters

Model risk: choice of copula

- First to default swap premium vs number of names
  - *From n=1 to n=50 names*
  - *Unit nominal*
  - *Credit spreads = 80 bp*
  - *Recovery rates = 40 %*
  - *Maturity = 5 years*
  - *Basket premiums in bppa*
  - *Gaussian correlation parameter= 30%*
- MO is different
- Kendall's tau ?

Names	Gaussian	Student (6)	Student (12)	Clayton	MO
1	80	80	80	80	80
5	332	339	335	336	244
10	567	578	572	574	448
15	756	766	760	762	652
20	917	924	920	921	856
25	1060	1060	1060	1060	1060
30	1189	1179	1185	1183	1264
35	1307	1287	1298	1294	1468
40	1417	1385	1403	1397	1672
45	1521	1475	1500	1492	1875
50	1618	1559	1591	1580	2079
Kendall	19%			8%	33%

Model risk: choice of copula

- From first to last to default swap premiums
  - *10 names, unit nominal*
  - *Spreads of names uniformly distributed between 60 and 150 bp*
  - *Recovery rate = 40%*
  - *Maturity = 5 years*
  - *Gaussian correlation: 30%*
- Same FTD premiums imply consistent prices for protection at all ranks
- Model with simultaneous defaults provides very different results

Rank	Gaussian	Student (6)	Student (12)	Clayton	MO
1	723	723	723	723	723
2	277	278	276	274	160
3	122	122	122	123	53
4	55	55	55	56	37
5	24	24	25	25	36
6	11	10	10	11	36
7	3.6	3.5	4.0	4.3	36
8	1.2	1.1	1.3	1.5	36
9	0.28	0.25	0.35	0.39	36
10	0.04	0.04	0.06	0.06	36

Model risk: choice of copula

- CDO margins (bp)
  - *With respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads 100 bp*
  - *5 years maturity*

	equity	mezzanine	senior
0 %	5341	560	0.03
10 %	3779	632	4.6
30 %	2298	612	20
50 %	1491	539	36
70 %	937	443	52
100%	167	167	91

Model risk: choice of copula

ρ	0%	10%	30%	50%	70%	100%
Gaussian	560	633	612	539	443	167
Clayton	560	637	628	560	464	167
Student (6)	676	676	637	550	447	167
Student (12)	647	647	621	543	445	167
MO	560	284	144	125	134	167

Table 8: mezzanine tranche (bp pa)

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0.03	4.6	20	36	52	91
Clayton	0.03	4.0	18	33	50	91
Student (6)	7.7	7.7	17	34	51	91
Student (12)	2.9	2.9	19	35	52	91
MO	0.03	25	49	62	73	91

Table 9: senior tranche (bp pa)

Model risk: choice of copula

- Related results:
  - *Student vs Gaussian*
    - Frey & McNeil, Mashal et al
    - Calibration on asset correlation
    - Distance between Gaussian and Student is bigger for low correlation levels
    - And extremes of the loss distribution
    - Joint default probabilities increase as number of degrees of freedom decreases
  - *Calibration onto joint default probabilities*
    - or default correlation, or aggregate loss variance
    - O'Kane & Schloegl, Schonbucher
  - *Gaussian, Clayton and Student t are all very similar*

Model risk: choice of copula

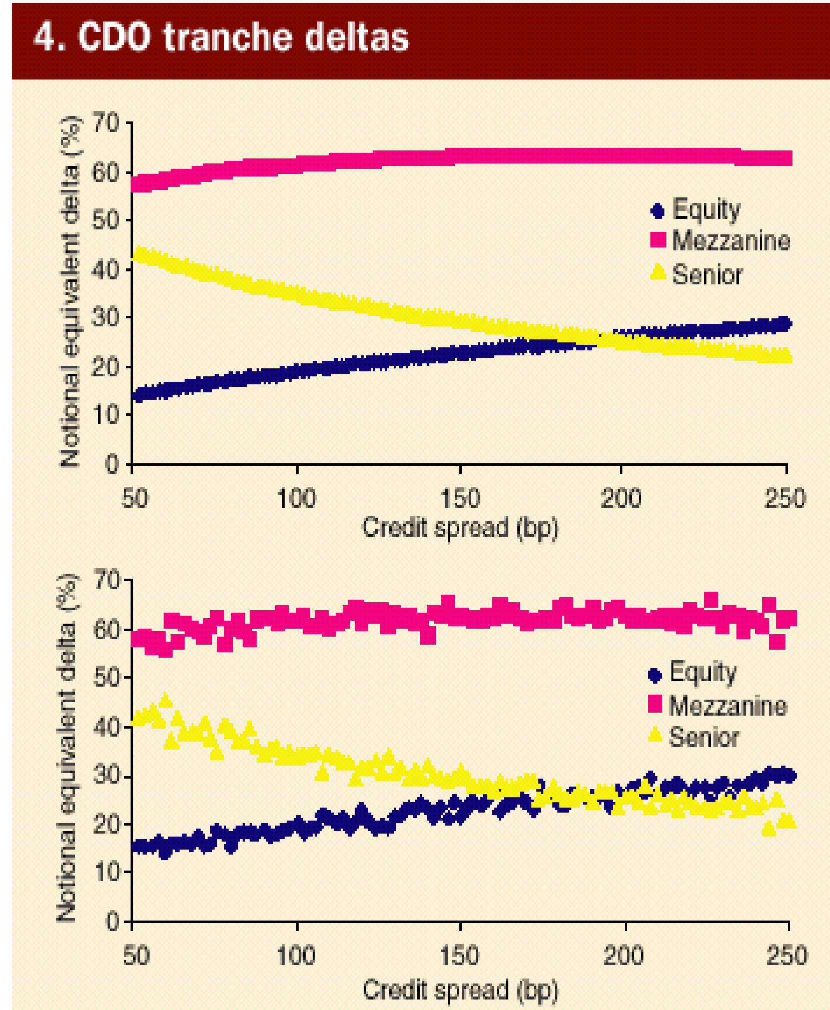
- Related results:
  - *Calibration to the correlation smile*
    - Gilkes & Jobst, Greenberg et al : Student and Gaussian very similar
  - *Clayton vs Gaussian*
    - Madan et al
    - For well chosen parameters, Clayton and Gaussian are close
    - Calibration on Kendall's tau ?
- Conclusion:
  - *Mapping of parameters for Gaussian, Clayton, Student*
    - Such that CDO tranches, joint default probabilities, default correlation, loss variance, spread sensitivities are well matched
    - Even though dynamic properties are different

Risk analysis: sensitivity with respect to credit curves

- Computation of Greeks
  - *Changes in credit curves of individual names*
  - *Changes in correlation parameters*
- Greeks can be computed up to an integration over factor distribution
  - *Lengthy but easy to compute formulas*
  - *The technique is applicable to Gaussian and non Gaussian copulas*
  - *See « I will survive », RISK magazine, June 2003, for more about the derivation.*

Risk analysis: sensitivity with respect to credit curves

- Hedging of CDO tranches with respect to credit curves of individual names
- Amount of individual CDS to hedge the CDO tranche
- Semi-analytic : some seconds
- Monte Carlo more than one hour and still shaky



CDO tranche deltas using the analytical method (top) and Monte Carlo (bottom) for a correlation of 50%

Risk analysis: correlation parameters

- CDO premiums (bp pa)
  - *with respect to correlation*
  - *Gaussian copula*
  - *Attachment points: 3%, 10%*
  - *100 names, unit nominal*
  - *5 years maturity, recovery rate 40%*
  - *Credit spreads uniformly distributed between 60 and 150 bp*
- Equity tranche premiums decrease with correlation
- Senior tranche premiums increase with correlation
- Small correlation sensitivity of mezzanine tranche

ρ	equity	mezzanine	senior
0 %	6176	694	0.05
10 %	4046	758	5.8
30 %	2303	698	23
50 %	1489	583	40
70 %	933	470	56

Risk analysis: correlation parameters

- Gaussian copula with sector correlations

$$\begin{pmatrix} 1 & \beta_1 & \beta_1 & & & & & & \\ \beta_1 & 1 & \beta_1 & & & & & \gamma & \\ \beta_1 & \beta_1 & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & \beta_m & \beta_m \\ & \gamma & & & & & \beta_m & 1 & \beta_m \\ & & & & & & \beta_m & \beta_m & 1 \end{pmatrix}$$

  - *Analytical approach still applicable*
  - *"In the Core of Correlation", Risk Magazine, October*

Risk analysis: correlation parameters

- TRAC-X Europe
  - *Names grouped in 5 sectors*
  - *Intersector correlation: 20%*
  - *Intrasector correlation varying from 20% to 80%*
  - *Tranche premiums (bp pa)*
- Increase in intrasector correlation
  - *Less diversification*
  - *Increase in senior tranche premiums*
  - *Decrease in equity tranche premiums*

$$
\begin{pmatrix}
1 & 60\% & 60\% & & & & & & & \\
60\% & 1 & 60\% & & & & & & 20\% & \\
60\% & 60\% & 1 & & & & & & & \\
 & & & 1 & & & & & & \\
 & & & & \cdot & & & & & \\
 & & & & & \cdot & & & & \\
 & & & & & & 1 & & & \\
 & & & & & & & 1 & 60\% & 60\% \\
 & 20\% & & & & & & 60\% & 1 & 60\% \\
 & & & & & & & 60\% & 60\% & 1
\end{pmatrix}
$$

	0-3%	3-6%	6-9%	9-12%	12-22%
20%	1273.9	287.5	93.4	33.3	6.0
30%	1226.6	294.4	102.7	39.9	7.9
40%	1168.9	303.5	114.0	47.3	10.3
50%	1100.5	314.2	127.6	56.3	13.3
60%	**1020.9**	**325.8**	**143.8**	**67.2**	**17.0**
70%	929.1	337.5	163.6	80.8	21.6
80%	821.9	349.3	188.0	98.8	27.2

Risk analysis: correlation parameters

- Implied flat correlation
  - *With respect to intrasector correlation*
- * premium cannot be matched with flat correlation
  - *Due to small correlation sensitivities of mezzanine tranches*
- Negative correlation smile

$$\begin{pmatrix} 1 & 60\% & 60\% & & & & & & & \\ 60\% & 1 & 60\% & & & & & & 20\% & \\ 60\% & 60\% & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & \cdot & & & & & \\ & & & & & \cdot & & & & \\ & & & & & & 1 & & & \\ & & & & & & & 1 & 60\% & 60\% \\ & 20\% & & & & & & 60\% & 1 & 60\% \\ & & & & & & & 60\% & 60\% & 1 \end{pmatrix}$$

	0-3%	3-6%	6-9%	9-12%	12-22%
20%	20.0%	20.0%	20.0%	20.0%	20.0%
30%	22.2%	22.6%	22.1%	22.2%	22.0%
40%	25.0%	27.6%	25.2%	24.6%	24.2%
50%	28.5%	*	29.7%	27.3%	26.8%
60%	32.8%	*	40.5%	30.6%	29.8%
70%	44.9%	*	*	34.8%	33.1%
80%	44.8%	*	*	41.3%	37.1%

Risk analysis: correlation parameters

- Pairwise correlation sensitivities
  - *not to be confused with sensitivities to factor loadings*

$$V_i = \rho_i V + \sqrt{1 - \rho_i^2}\bar{V}_i$$

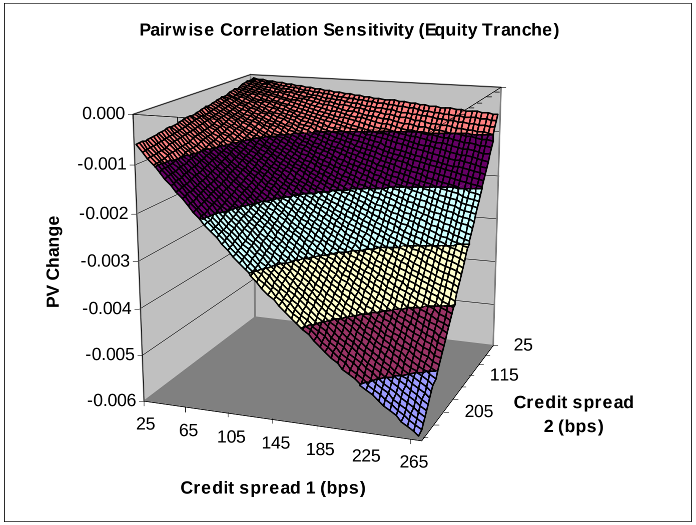
  - *Correlation between names i and j:* $\rho_i \rho_j$
  - *Sensitivity wrt factor loading: shift in* ρ_i
  - *All correlations involving name i are shifted*

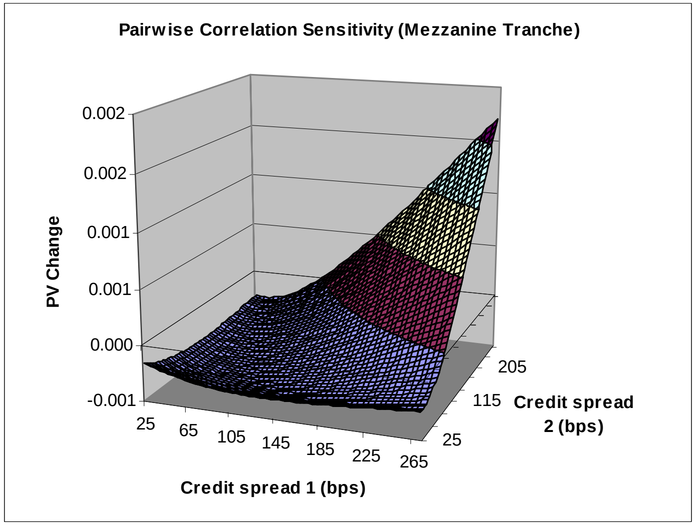
- Pairwise correlation sensitivities
  - *Local effect*

$$\begin{pmatrix} 1 & \rho_{12} & & & & & & \\ \rho_{21} & 1 & & & & & & \\ & & 1 & & \rho_{ij}+\delta & & & \\ & & & . & & & & \\ & & \rho_{ij}+\delta & & . & & & \\ & & & & & 1 & & \\ & & & & & & 1 & . \\ & & & & & & . & 1 \end{pmatrix}$$

Risk analysis: correlation parameters

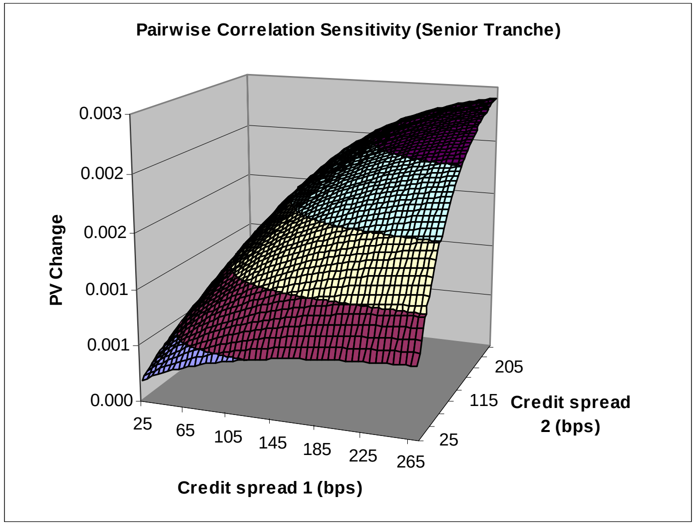
- Pairwise Correlation sensitivities
  - *Protection buyer*
- 50 names
  - *spreads 25, 30,…, 270 bp*
- Three tranches:
  - *attachment points: 4%, 15%*
- Base correlation: 25%
- Shift of pair-wise correlation to 35%
- Correlation sensitivities wrt the names being perturbed
- equity (top), mezzanine (bottom)
  - *Negative equity tranche correlation sensitivities*
  - *Bigger effect for names with high spreads*





Risk analysis: correlation parameters

- Senior tranche correlation sensitivities
  - *Positive sensitivities*
  - *Protection buyer is long a call on the aggregated loss*
    - Positive vega
  - *Increasing correlation*
    - Implies less diversification
    - Higher volatility of the losses



- Names with high spreads have bigger correlation sensitivities

Conclusion

- Factor models of default times:
  - *Simple computation of CDO's*
    - Tranche premiums and risk parameters
- Gaussian, Clayton and Student *t* copulas provide very similar patterns
- Shock models (Marshall-Olkin) quite different
- Possibility of extending the 1F Gaussian copula model
  - *To deal with intra and inter-sector correlation*
  - *Compute correlation sensitivities*