A comparative analysis of CDO pricing models

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Abstract

We compare some popular CDO pricing models, related to the bottom-up approach. Dependence between default times is modelled through Gaussian, stochastic correlation, Student t, double t, Clayton and Marshall-Olkin copulas. We detail the model properties and compare the semi-analytic pricing approach with large portfolio approximation techniques. We study the independence and perfect dependence cases and the uniqueness of base correlation. The ability of the models to fit the correlation skew observed in CDO market quotes is also assessed. Eventually, we relate CDO premiums and the distribution of conditional default probabilities which appears as a key input in the copula specification.

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This is the updated version of a paper that first circulated in 2005. While the main results remain unchanged, we tried to make the importance of the theory of stochastic orders for the pricing of CDO tranches and model comparison issues more understandable. We also updated various results based on stochastic orders, references and corrected miscellaneous typos. To ease the reading, most technical parts are postponed in the Appendix. This working paper version is closely related to chapter 15, pages 389-427 in « *The Definitive Guide to CDOs* » (Risk Books) edited by G. Meissner in 2008. The two versions, among other things, share the same title. We also refer to Cousin and Laurent [2008a,b] which can be seen as follow-ups of this paper. The authors thank F. Benatig, L. Carlier, L. Cousot, A. Cousin, S. Figlewski, S. Hutt, P. Laurence, M. Leeming, Y. Malevergne, G. Meissner, C. Miglietti, P. Miralles, M. Musiela, T. Rehmann, O. Scaillet and participants at ICBI Risk Management Conference in Geneva, RISK Europe in London, at Séminaire Bachelier in Paris, at the Isaac Newton Institute Credit Workshop in Cambridge and at the Lyon-Lausanne actuarial seminar for useful feedback. The usual disclaimer applies. This paper is for academic purpose only and is not intended to reflect the way BNP Paribas prices CDO tranches.
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Introduction

This paper provides a comparison of some popular CDO pricing models, within the "bottom-up" approach and more precisely based on copulas in order to define the dependence structure between default times[4]. We use a factor approach leading to semi-analytic pricing expressions that ease model risk assessment. We focus on "copula models" since there are predominantly used in the credit derivatives markets, though the factor approach also applies to various intensity models (see Mortensen [2006] or Eckner [2007] for examples). Let us stress that we rely on the standard pricing methodology for credit derivatives, i.e. a risk neutral measure is considered as given and may not be explicitly connected to self financed replicating strategies.

In the "bottom-up" approach, CDO tranche premiums depend upon the individual credit risk of names in the underlying portfolio and the dependence structure between default times. There are currently several "bottom-up" approaches to CDO pricing. One may start from a specification of dependent default intensities. A typical example is provided by Duffie and Gârleanu [2001]. An alternative route is the structural approach, corresponding to a multivariate hitting time model, as illustrated by Hull *et al.* [2005]. The previous approaches involve a calibration to marginal default distributions. On the other hand, the copula approach directly specifies the dependence structure, though in a somehow ad-hoc way. While the Gaussian copula model, introduced to the credit field by Li [2000] has become an industry standard, its theoretical foundations, for instance credit spread dynamics, and more practical issues such as fitting CDO tranche quotes, have been questioned. For this purpose, other dependence structure such as Clayton, Student *t*, double *t*, or Marshall-Olkin copulas have been proposed. The scope of copula modelling recently expanded; this is reported and analyzed in Cousin and Laurent [2008b]. The paper neither accounts for new research aiming at closing the gap between "top-down" and bottom-up approaches. In the top down approach, the starting point is the modelling of the aggregate loss process, from which one tries to derive in a consistent way individual name dynamics. We refer here among other papers to Bielecki, Crepey and Jeanblanc [2008] or Halperin and Tomecek [2008] following earlier research by Gieseke and Goldberg [2008].

The factor approach is quite standard in credit risk modelling (see for instance Crouhy *et al*. [2000], Merino and Nyfeler [2002], Pykhtin and Dev [2002], Gordy [2003] and Frey and McNeil [2003]). In the case of homogeneous portfolios, it is often coupled with large portfolio approximation techniques. In such a framework, Gordy and Jones [2003] analyse the risks within CDO tranches. In order to deal with numerical issues, Gregory and Laurent [2003] and Laurent and Gregory [2005] have described a semi-analytical approach, based on factor models, for the pricing of basket credit derivatives and CDOs. This topic is also discussed by, among others, Andersen *et al*. [2003] and Hull and White [2004]. We will further rely on this factor approach, which also provides an easy to deal framework for model comparisons. Other contributions dedicated to comparing various copulas in the credit field are Das and Deng [2004], Burstchell *et al.* [2007], the aforementioned papers of Cousin and Laurent [2008a,b] or the book by Cherubini *et al*. [2004]. The models studied here are the following:

- The Gaussian copula, more precisely, its one factor sub-case. This model is widely used by the financial industry. Despite numerous critics and the credit market crisis, it remains the benchmark tool for pricing and risk managing CDO tranches.
- A stochastic correlation extension of the Gaussian copula.
- The Student *t* extension of the Gaussian copula with six and twelve degrees of freedom.
- A double *t* one factor model as introduced by Hull and White [2004].
- The Clayton copula model that can also be seen as a frailty model with a Gamma distribution.
- A multivariate exponential model associated with multiple defaults. The associated copula is the Marshall-Olkin copula.

We refer to Andersen [2007] or Cousin and Laurent [2008a,b] for a discussion of other recent extensions of the factor copula approach. The critics may find unpleasant to deal with copula models, which are

[4] Some pros and cons of the "bottom-up" and "top-down" approaches can be found in the book edited by Lipton and Rennie [2007]. As it appears, the game is not over.

usually associated with poor dynamics of the aggregate loss process. Since our focus here is CDO pricing, we are only concerned with marginal distributions of the aggregate loss at different time horizons and not with the law of the aggregate loss process, which is involved in more sophisticated products, such as CPDOs, forward starting CDOs, Leverage Super Senior tranches. Of course, due the credit crisis and market environment, these more complex products are scarcely traded nowadays.

The paper aims at providing a comparison methodology for pricing models which are not embedded. We found that the theory of stochastic orders, familiar to actuaries or people involved in reliability theory, to be quite helpful in the credit field, especially in order to compare dependence structures between default times. Some of the tools used here may look a bit abstract, such as the supermodular order, for the reader unfamiliar with these topics. For the paper to be self-contained, we recall the mathematical definitions as the paper goes around. The reader may also refer to books by Denuit *et al*. [2005] or Müller and Stoyan [2002], among others, for further details and comments.

As for the dependence structure, the paper focuses on parametric models, i.e. the copulas considered belong to a common family, which usually involves a small number of parameters, unlike the non parametric implied copula approach introduced by Hull and White [2006].

For simplicity and to ease model comparisons, we will thereafter restrict to cases where the copula of default times is a symmetric function with respect to its coordinates. For instance, in the Gaussian copula case, this means that the correlation parameter is constant, whatever the couples of names[5]. Comparing CDO pricing models is easier due to the small number of parameters involved. We study the dependence of CDO tranche premiums with respect to the choice of dependence or "correlation" parameter. This involves some results in the theory of stochastic orders. For example, we can show that first to default swaps or base correlation CDO tranche premiums are monotonic with respect to the relevant dependence parameter (see McGinty and Ahluwalia [2004] for a discussion about base correlations). We also discuss some extreme cases such as independence and comonotonicity (or "perfect positive dependence") between default times. The theory of stochastic orders also provides some comparison results between CDO tranche premiums depending on the granularity of the reference credit portfolio. As an example, we can easily compute exact prices of CDO base or senior tranches and those computed under a large portfolio approximation.

We then compare CDO pricing models based under different copula assumptions. We show that well-appraised dependence indicators such as Kendall's τ or the tail dependence parameter fail to explain the differences between CDO tranche premiums. Therefore, these indicators should be used with great caution when considering the dependence structure within credit portfolios. On the other hand, the distribution of the conditional default probabilities appears as a key input. This explains for instance that, for a given time horizon, the Clayton copula and the one factor Gaussian copula almost lead to the same CDO tranche premiums. The conditional default probabilities are also of first importance in large portfolio approximations that dramatically simplify the computation of CDO tranche premiums. These findings appear to be the main contribution of this paper and might help designing more suitable models in the future.

Eventually, we study the ability of the studied models to fit market quotes: Double *t* and stochastic correlation models appear to provide the better fits, while for instance the Clayton and the Gaussian copula provide some strikingly similar CDO tranche premiums. However, due to space and time constraints, we only provide some examples based on specific trading dates and do not conduct a systematic time series analysis of the fitting properties of the models at hand.

The paper is organized as follows: we firstly briefly recall the semi-analytical pricing approach of basket credit derivatives or CDO tranches in a factor framework. The second section reviews the models under study. The third section is devoted to applications of the theory of stochastic orders to the pricing of CDO tranches. Though the third section is more theoretical in nature, it has quite important practical

[5] Practitioners then talk of "flat correlation". The symmetry assumption does not preclude the case of heterogeneous credit spreads for different names.

implications: we are able to show the existence of a unique implied dependence correlation parameter in most cases. For instance, we give a formal proof of the uniqueness of implied base correlations, a result of importance for practitioners. Some comparison results between large portfolio approximations and semi-analytic approaches are provided and granularity issues are discussed. The fourth section contains empirical investigations. Our comparison methodology relies on the uniqueness of implied dependence parameters for base correlation tranches. Firstly, we study how the different models at hand differ as far as the pricing of basket default swaps and CDO tranches is concerned. We then discuss the ability of the different models to reproduce market quotes on standardized CDO tranches based on the iTraxx index. Eventually, we provide an analysis of the differences between the studied models based on the distribution of conditional default probabilities.

I) Semi-analytical pricing of basket default swaps and CDOs

In this section, we recall how the factor or conditional independence approach can be associated with tractable computations for basket default swaps and CDO tranches. These are detailed in Laurent and Gregory [2005].

Throughout the paper, we will consider *n* obligors and denote the random vector of default times as $\left(\tau_1,\ldots,\tau_n\right)$. We will denote by *F* and *S* respectively the joint distribution and survival functions such that for all $\left(t_1,\ldots,t_n\right)\in\mathbb{R}^n$, $F\left(t_1,\ldots,t_n\right)=Q\left(\tau_1\le t_1,\ldots,\tau_n\le t_n\right)$ and $S\left(t_1,\ldots,t_n\right)=Q\left(\tau_1> t_1,\ldots,\tau_n> t_n\right)$ where *Q* represents some pricing probability measure. $F_1,\ldots,F_n$ represent the marginal distribution functions and $S_1,\ldots,S_n$ the corresponding survival functions. *C* denotes the copula of default times which is such that $F(t_1,\ldots,t_n)=C\left(F_1(t_1),\ldots,F_n(t_n)\right)$[6]. We denote by E_i, $i=1,\ldots,n$ the nominal exposures[7] associated with *n* credits, with δ_i $i=1,\ldots,n$ being the corresponding recovery rates and by $M_i=E_i\left(1-\delta_i\right)$ the loss given default for name *i*. We will thereafter assume that recovery rates are deterministic[8] and concentrate upon the dependence of default times.

We will consider a latent factor *V* such that conditionally on *V*, the default times are independent. The factor approach makes it simple to deal with a large number of names and leads to very tractable pricing results. We will denote by $p_t^{i|V}=Q\left(\tau_i\le t\middle|V\right)$ and $q_t^{i|V}=Q\left(\tau_i> t\middle|V\right)$ the conditional default and survival probabilities. Conditionally on V, the joint survival function is:

$$S(t_1,\cdots,t_n|V)=\prod_{1\le i\le n} q_{t_i}^{i|V} .$$

[6] Let *F* be a joint distribution function defined on $\mathbb{R}^n$ and $F_1,\ldots,F_n$ be the corresponding marginal distribution functions. Then, there exists a distribution function *C* over $\left[0,1\right]^n$ such that for all $x=\left(x_1,\ldots,x_n\right)\in\mathbb{R}^n$, $F(x)=C\left(F_1(x_1),\ldots,F_n(x_n)\right)$. If $F_1,\ldots,F_n$ are all continuous, then *C* is uniquely defined. Conversely, if *C* is an *n*-copula and $F_1,\ldots,F_n$ are univariate distribution functions, $x\to C\left(F_1(x_1),\ldots,F_n(x_n)\right)$ defines a joint distribution function.

[7] We further assume that the nominals are positive, thus leaving aside the case of long-short CDOs. Let us however emphasize that important results such as Monotonicity of base tranches with respect to some dependence parameter may not be fulfilled for a long-short CDO. For instance, uniqueness of base correlation is not guaranteed. In the case of index tranches, the nominal exposures are usually equal i.e. $E_i=\frac{1}{n}$.

[8] Up to the 2008 credit crisis, participants to the CDO market mostly relied on this assumption, with a commonly assumed value of $\delta_i=40\%$ for CDX.NA.IG and iTraxx indices.

Basket Default Swaps and especially CDO tranches are now standardized products. As for the pricing of the CDO tranche, we need to consider the aggregate loss process defined as $L(t)=\sum_{i=1}^{n} M_i N_i(t)$, where $N_i(t)$ are the default indicators processes associated with the different names and M_i the corresponding losses given default. It can be shown that we only need the marginal distributions of $L(t)$ up to maturity in order to price the default and the premium leg of a CDO tranche. The computation of the default payment leg involves $E\left[\left(L(t)-K\right)^{+}\right]$ where *K* are the attachment points of the tranches. Semi-analytical techniques allow for quick computation of the aggregate loss distribution. This can be done by considering its characteristic function. Thanks to the conditional independence assumption, and since recovery rates are deterministic, the characteristic function of the aggregate loss can be written as: $\varphi_{L(t)}(u)=E\left[e^{iuL(t)}\right]=E\left[\prod_{1\le j\le n}\left(q_t^{j|V}+p_t^{j|V}e^{iuM_j}\right)\right]$. The computation of the expectation involves a numerical integration over the distribution of the factor *V*, which can be easily achieved numerically provided that the dimension of V is small[9]. Eventually, the distribution of the aggregate loss can be computed from the characteristic function or by recursion techniques. For more details about these approaches, we refer to Laurent and Gregory [2005], Andersen *et al*. [2003], Hull and White [2004]. Jackson *et al*. [2007] discuss the efficiency of different methods for the computation of loss distributions. Lately, El Karoui, Jiao and Kurtz [2008] and Bastide, Benhamou and Ciuca [2007] suggested and studied quick to compute and accurate approximations of CDO tranche prices in a factor framework based on Stein's method and zero bias transformation. Whenever, one cannot use these pricing techniques, Monte Carlo simulation is required. For this purpose, we also provide ways to draw randomly default times under the considered models. Chen and Glasserman [2008] consider efficient importance sampling schemes applicable when the factor assumption does not hold.

For modelling purpose, we emphasize that the only inputs to a factor copula model are the conditional (distribution of) default probabilities $p_t^{i|V}$, which include all pricing requirements. Since the premises of the choice of copula are unclear, some contemptors talk about ad-hoc models without sound economic foundation. Nevertheless, the distribution of conditional default probability is closely related to the one of large and well diversified credit portfolios. Therefore, rather than looking for a suitable choice of copula, in most cases, the relevant quantity to be considered is the distribution of conditional default probability. Dependence between default events, marginal distributions of aggregate losses and eventually CDO tranche quotes stems for a proper selection of the previous constituent.

II) The models under study

There are now a number of books dedicated to copulas such as Joe [1997], Nelsen [1999] or Cherubini *et al*. [2004]. As for the insurance case, we can also refer to the paper by Frees and Valdez [1998]. We detail below some "factor copulas" that are useful in the pricing of basket credit derivatives and CDOs. We will thereafter restrict ourselves to one parameter copulas to ease comparisons. The symmetry assumption is made about the copula of default times and not about the joint distribution of default times. This assumption can be related but is weaker than the exchangeability assumption. For instance, we may have constant correlations in a Gaussian copula but different marginal default probabilities and recovery rates. An analysis of heterogeneity effects within the Gaussian copula can be found in Gregory and Laurent [2004]. Regarding stochastic orders and related probability concepts such as comonotonicity, exchangeability, tail dependence, Kendall's τ, we refer to the papers of Müller [1997], Denuit *et al*. [2001], Dhaene and Goovaerts [1997], Hu and Wu [1999] and the above-mentioned books. To ease the reading of this paper, mathematical definitions and properties are briefly recalled in the core text, footnotes and in the Appendix.

[9] In the examples below, the dimension of V will be equal to one or two. Gössl [2007] considers some factor reduction techniques in a Gaussian copula framework.

II.1 One factor Gaussian copula

The default times are modelled from a Gaussian vector $(V_1,\ldots,V_n)$. As in Li [2000], the default times are given by: $\tau_i = F_i^{-1}(\Phi(V_i))$ for $i=1,\ldots,n$ where F_i^{-1} denotes the generalized inverse of F_i and Φ is the Gaussian cdf. In the one factor case, $V_i = \rho V + \sqrt{1-\rho^2}\overline{V}_i$ where $V, \overline{V}_i$ are independent Gaussian random variables and $0 \le \rho \le 1$ [10]. Then:

$$p_t^{i|V} = \Phi\left(\frac{-\rho V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho^2}}\right).$$

$\rho = 0$ corresponds to independent default times while $\rho = 1$ is associated with the comonotonic case[11]. When $\rho = 1$, we simply have $p_t^{i|V} = 1_{\{V \le \Phi^{-1}(F_i(t))\}}$.

There is no upper or lower tail dependence when $\rho < 1$ while the coefficient of tail dependence is equal to 1 when $\rho = 1$ [12]. The relation between Kendall's τ [13] and linear correlation parameter ρ^2 is given by: $\rho_K = \frac{2}{\pi}\arcsin \rho^2$. An important result is that the one factor Gaussian copula is increasing in the supermodular order with respect to the correlation parameter ρ (definition and additional comments regarding the supermodular order are postponed in the Appendix). This result was proved by Bäuerle and Müller [1998] and further generalized by Müller and Scarsini [2000], Müller [2001]. Since default times are increasing functions of the V_i's, the default times do also increase, with respect to the supermodular order, as the correlation parameter increases. Loosely speaking, default times are more dependent when the correlation parameter increases, which is rather intuitive, though the formal proofs are quite involved. The notion of dependence with respect to the supermodular order makes sense especially for non Gaussian vectors, such as default times, as will be detailed below.

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[10] As a consequence, the correlation between V_i and V_j is equal to ρ^2. Let us remark that some papers rather write the latent variables as $V_i = \sqrt{\rho}V + \sqrt{1-\rho}\overline{V}_i$.

[11] Comonotonicity can be considered as "perfect dependence" between the components of a random vector. More formally, let $X = (X_1,\ldots,X_n)$ be a random vector with marginal distribution functions $F_1,\ldots,F_n$. *X* is said to be comonotonic if it has the same distribution as $(F_1^{-1}(U),\ldots,F_n^{-1}(U))$ where *U* is a $[0,1]$ uniform random variable and F_i^{-1} is the generalized inverse of F_i. Moreover, a random vector is comonotonic if and only if the associated copula is the upper Fréchet copula, such that for all $u = (u_1,\ldots,u_n) \in [0,1]^n$, $C^+(u_1,\ldots,u_n) = \min(u_1,\ldots,u_n)$. The Fréchet copula acts as an upper bound, since for any copula *C*, we have $C(u) \le C^+(u)$ for all $u \in [0,1]^n$.

[12] Let *X* and *Y* be two random variables, with distribution functions F_X, F_Y, and let *C* denote the copula associated with (X,Y). The coefficient of upper tail dependence is such that:

$$\lim_{u\to 1} Q\left(X > F_X^{-1}(u) \middle| Y > F_Y^{-1}(u)\right) = \lim_{u\to 1}\frac{C(u,u)+1-2u}{1-u},$$

whenever the limit exists. We say that there is upper tail dependence if the coefficient is positive. From the definition, it can be seen that the coefficient of upper tail dependence is always less or equal to 1. It is equal to 1 for the upper Fréchet copula C^+. We can also consider the coefficient of lower tail dependence defined as:

$$\lim_{u\to 0} Q\left(X \le F_X^{-1}(u) \middle| Y \le F_Y^{-1}(u)\right) = \lim_{u\to 0}\frac{C(u,u)}{u}.$$

This coefficient is also less or equal to 1 and is equal to one for the upper Fréchet copula C^+.

[13] Given a bivariate copula C, Kendall's τ is given by $\rho_K = 4\iint_{[0,1]^2} C(u,v)dC(u,v) - 1$.

II.2 Stochastic Correlation

There has been much interest in simple extensions of the Gaussian copula model (see Andersen and Sidenius [2005], Schloegl [2005]) in order to match "correlation smiles" in the CDO market. Let us present the simplest version of such a model. The latent variables are given by:

$$V_i = B_i\left(\rho V + \sqrt{1-\rho^2}\overline{V}_i\right) + \left(1-B_i\right)\left(\beta V + \sqrt{1-\beta^2}\overline{V}_i\right),$$

for $i = 1,\dots,n$, where B_i are Bernoulli random variables, $V, \overline{V}_i$ are standard Gaussian random variables, all these being jointly independent and ρ, β are some correlation parameters, $0 \le \beta \le \rho \le 1$. We denote by $p = Q\left(B_i = 1\right)$. The above model is a convex sum of one factor Gaussian copulas, involving a mixing distribution over factor exposure. In our examples, there are here two states for each name, one corresponding to a high correlation and the other to a low correlation. We could equivalently write the latent variables as:

$$V_i = \left(B_i\rho + \left(1-B_i\right)\beta\right)V + \sqrt{1-\left(B_i\rho + \left(1-B_i\right)\beta\right)^2}\,\overline{V}_i,$$

This makes clear that we deal with a stochastic correlation Gaussian model. We have a factor exposure ρ with probability p and β with probability $1-p$. It can be easily checked that the marginal distributions of the V_i's are Gaussian. As above, we define the default dates as $\tau_i = F_i^{-1}\left(\Phi\left(V_i\right)\right)$ for $i = 1,\dots,n$.

The default times are independent conditionally on *V* and we can write the conditional default probabilities:

$$p_t^{i|V} = p\Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right) + (1-p)\Phi\left(\frac{-\beta V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\beta^2}}\right).$$

We denote by C_γ^G the bivariate Gaussian copula with covariance term γ. We can check that the bivariate copula of default times can be written as:

$$p^2 C_{\rho^2}^G\left(u,v\right) + 2p(1-p)C_{\rho\beta}^G\left(u,v\right) + (1-p)^2 C_{\beta^2}^G\left(u,v\right),$$

for u,v in $\left[0,1\right]$. As a consequence, the previous model might be seen as a mixture of Gaussian copulas, involving all combinations of correlations. The tail dependence coefficient is equal to zero if $\beta \le \rho < 1$, to p^2 if $\beta < \rho = 1$ and to 1 if $\beta = \rho = 1$. It is also possible to provide an analytical though lengthy expression for Kendall's τ as:

$$\frac{2}{\pi}\times\begin{pmatrix} p^4\arcsin\left(\rho^2\right) + 2p^2(1-p)^2\arcsin\left(\rho\beta\right) + (1-p)^4\arcsin\left(\beta^2\right) \\ +4p^3(1-p)\arcsin\left(\dfrac{\rho^2+\rho\beta}{2}\right) + 2p^2(1-p)^2\arcsin\left(\dfrac{\rho^2+\beta^2}{2}\right) + 4p(1-p)^3\arcsin\left(\dfrac{\beta^2+\rho\beta}{2}\right)\end{pmatrix}$$

Since the supermodular order is closed under mixtures, it can be proved that increasing $\left(\rho,\beta,p\right)$ leads to an increase in dependence in the supermodular sense. The proof is postponed in the Appendix. The reader can find further examples of the stochastic correlation approach in Burtschell *et al*. [2007].

II.3 Student *t* copula

The Student *t* copula is a simple extension of the Gaussian copula. It has been considered for credit and risk issues by a number of authors, including Andersen *et al*. [2003], Demarta and McNeil [2005], Embrechts *et al*. [2003], Frey and McNeil [2003], Greenberg *et al*. [2004], Mashal and Zeevi [2003], Mashal *et al*. [2003], Schloegl and O'Kane [2005].

In the Student *t* approach, the underlying vector $(V_1,\ldots,V_n)$ follows a Student *t* distribution with ν degrees of freedom. In the symmetric case which we are going to consider, we have $V_i=\sqrt{Z}X_i$ where $X_i=\rho V+\sqrt{1-\rho^2}\bar{V}_i$, $V,\bar{V}_i$ are independent Gaussian random variables, Z is independent from $(X_1,\ldots,X_n)$ and follows an inverse Gamma distribution with parameters equal to $\frac{\nu}{2}$ (or equivalently $\frac{\nu}{Z}$ follows a χ^2_ν distribution). Let us remark that the covariance between V_i and V_j, $i\neq j$ is equal to $\frac{\nu}{\nu-2}\rho^2$ for $\nu>2$. We further denote by t_ν the distribution function of the standard univariate Student *t*, that is the univariate cdf of the V_i's. We then have $\tau_i=F_i^{-1}(t_\nu(V_i))$. It can be seen that conditionally on (V,Z) default times are independent and:

$$p_t^{i|V,W}=\Phi\left(\frac{-\rho V+Z^{-1/2}t_\nu^{-1}(F_i(t))}{\sqrt{1-\rho^2}}\right).$$

Thus we deal with a two factor model. As for the Gaussian copula, we have Kendall's τ expressed as: $\rho_K=\frac{2}{\pi}\arcsin\rho^2$. The Student *t* copula has upper and lower tail dependence with equal coefficients, being equal to $2t_{\nu+1}\left(-\sqrt{\nu+1}\times\sqrt{\frac{1-\rho^2}{1+\rho^2}}\right)$. Let us remark that even for $\rho=0$, we still have tail dependence. Thus, $\rho=0$ does not correspond to the independence case. In fact, there is always tail dependence whatever the parameters ρ and ν. Thus, we cannot match the product copula[14] by using the Student *t* copula. However, when $\rho=1$, all the V_i's are equal and this corresponds to the comonotonic case. Since the supermodular order is closed under mixtures and using the supermodular order of Gaussian copulas, we readily obtain that the Student *t* copula is positively ordered with respect to the parameter ρ in the supermodular sense.

II.4 Double *t* copula

This model is also a simple extension of the one factor Gaussian copula. It has been considered for the pricing of CDOs by Hull and White [2004]. As for the Gaussian copula, it belongs to the class of additive factor copulas. We refer to Cousin and Laurent [2008b] and the references therein for further examples and discussion.

The default times are modelled from a latent random vector $(V_1,\ldots,V_n)$. The latent variables are such that $V_i=\rho\left(\frac{\nu-2}{\nu}\right)^{1/2}V+\sqrt{1-\rho^2}\left(\frac{\bar{\nu}-2}{\bar{\nu}}\right)^{1/2}\bar{V}_i$ where $V,\bar{V}_i$ are independent random variables following Student *t* distributions with ν and $\bar{\nu}$ degrees of freedom and $\rho\geq 0$. Since the Student distribution is not stable under convolution, the V_i's do not follow Student distributions; the copula associated with $(V_1,\ldots,V_n)$ is not a Student copula. Thus, this model differs from the previous one. As for the one factor Gaussian copula model, $\rho=0$ is associated with independent default times and $\rho=1$ with comonotonic default times.

The default times are then given by: $\tau_i=F_i^{-1}(H_i(V_i))$ for $i=1,\ldots,n$ where H_i is the distribution function of V_i [15]. Then:

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[14] Random variables $X_1,\ldots,X_n$ are independent if and only if the associated copula is the product copula $C^\perp$ such that: $\forall(u_1,\ldots,u_n)\in[0,1]^n$, $C^\perp(u_1,\ldots,u_n)=u_1\times\ldots\times u_n$.

[15] H_i must be computed numerically and depends upon ρ.

$$p_t^{i|V} = t_{\bar{\nu}}\left(\left(\frac{\bar{\nu}}{\bar{\nu}-2}\right)^{1/2}\frac{H_i^{-1}\left(F_i(t)\right)-\rho\left(\frac{\nu-2}{\nu}\right)^{1/2}V}{\sqrt{1-\rho^2}}\right).$$

It is possible to derive some tail dependence parameters in the double *t* model. Using Malevergne and Sornette [2004], we can express the coefficient of tail dependence (the coefficients of upper and lower tail dependence are equal) as:

$$\lambda = \frac{1}{1+\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)^{\nu}},$$

when $\nu = \bar{\nu}$. If $\nu < \bar{\nu}$, then the tail of the factor *V* is bigger than the tail of the idiosyncratic risk $\bar{V}_i$. As a consequence, the coefficient of tail dependence is equal to one. In the tails, the idiosyncratic risk can be neglected, and extreme movements are driven solely by the factor. On the other hand, if $\nu > \bar{\nu}$, then the tail of the factor is smaller than the tail of the idiosyncratic risk and there is no tail dependence between the default times[16]. As the dependence parameter increases, it can be proved that the double *t* copula increases with respect to the supermodular order, in the homogeneous case, i.e. when default probabilities are name independent (see Cousin and Laurent [2008a]).

II.5 Clayton copula

Schönbucher and Schubert [2001], Schönbucher [2002], Gregory and Laurent [2003], Rogge and Schönbucher [2003], Madan *et al*. [2004], Laurent and Gregory [2005], Schloegl and O'Kane [2005], Friend and Rogge [2005] have been considering this model in a credit risk context.

Let us proceed to a formal description of the model. We consider a positive random variable V, which is called a frailty, following a standard Gamma distribution with shape parameter $1/\theta$ where $\theta > 0$. Its probability density is given by $f(x) = \frac{1}{\Gamma(1/\theta)}e^{-x}x^{(1-\theta)/\theta}$ for $x > 0$. We denote by ψ the Laplace transform of f. We get $\psi(s) = \int_0^\infty f(x)e^{-sx}dx = (1+s)^{-1/\theta}$. We then define some latent variables V_i's as:

$$V_i = \psi\left(-\frac{\ln U_i}{V}\right),$$

where $U_1, \ldots U_n$ are independent uniform random variables also independent from V. Eventually, the default times are such that:

$$\tau_i = F_i^{-1}(V_i),\ i = 1,\ldots,n$$

The previous equations imply a one factor representation where V is the factor. The conditional default probabilities can be expressed as:

$$p_t^{i|V} = \exp\left(V\left(1-F_i(t)^{-\theta}\right)\right)$$

Low levels of the latent variable are associated with shorter survival default times. For this reason, *V* is called a "frailty".

Let us remark that the V_i's have uniform marginal distributions. Since the default times are increasing functions of these V_i's, the copula of default times is the joint distribution of the V_i's. We readily check that $Q(V_1 < u_1,\ldots,V_n < u_n) = \psi\left(\psi^{-1}(u_1)+\cdots\psi^{-1}(u_n)\right) = \left(u_1^{-\theta}+\cdots+u_n^{-\theta}-n+1\right)^{-1/\theta}$, for any

[16] Let us remark that, unlike what is sometimes stated, we can have no tail dependence between default times even if the factor has fat tails. Moreover, since the aggregate loss is bounded by the portfolio nominal, it always has thin tails.

$\left(u_1,\ldots,u_n\right)\in\left[0,1\right]^n$. The distribution function of the V_i's is known as the Clayton copula. The Clayton copula is Archimedean and the generator of the copula is $\varphi(t)=t^{-\theta}-1$, i.e.

$$C_\theta(u_1,\ldots,u_n)=\varphi^{-1}\left(\varphi(u_1)+\cdots+\varphi(u_n)\right).$$

From Embrechts *et al*. [2003], we obtain Kendall's τ for a Clayton copula as: $\rho_K=\dfrac{\theta}{\theta+2}$ where $\theta\in\left[-1,\infty\right)\setminus\left\{0\right\}$. The Clayton copula exhibits lower tail dependence for $\theta>0$, $\lambda_L=2^{-1/\theta}$ and no upper tail dependence i.e. $\lambda_U=0$ [17]. When $\theta=0$, we obtain the product copula, i.e. default times are independent. When $\theta=+\infty$, the Clayton copula turns out to be the upper Fréchet bound corresponding to the case where default times are comonotonic.

As the parameter θ increases, the Clayton copula increases with respect to the supermodular order (Wei and Hu [2002]).

II.6 Multivariate exponential models and the Marshall-Olkin copula

The reliability theory denotes these as "shock models". There are also known as multivariate exponential models in Marshall and Olkin [1967]. They were introduced to the credit domain by Duffie and Singleton [1998] and also discussed by Li [2000], Wong [2000]. More recently, Elouerkhaoui [2003a,b], Giesecke [2003], Lindskog and McNeil [2003] considered the use of such models.

We present here the simplest form of the model corresponding to a single fatal shock[18]. We consider some latent variables $V_i=\min\left(V,\bar{V}_i\right)$, $i=1,\ldots,n$ where $V,\bar{V}_i$, $i=1,\ldots,n$ are independent exponentially distributed random variables with parameters $\alpha,1-\alpha$, $\alpha\in\left]0,1\right[$. The corresponding survival copula[19] belongs to the Marshall-Olkin family (see Nelsen [1999], pages 46-49) and can be expressed as:

$$\hat{C}\left(u_1,\ldots,u_n\right)=\min\left(u_1^\alpha,\ldots,u_n^\alpha\right)\prod_{i=1}^n u_i^{1-\alpha}$$

The default times are then defined as:

$$\tau_i=S_i^{-1}\left(\exp\left(-\min\left(V,\bar{V}_i\right)\right)\right)$$

Since $t\to S_i^{-1}\left(\exp\left(-t\right)\right)$ are increasing functions, the copula of default times is the same as the copula of $\min\left(V,\bar{V}_i\right)$. We can also check that the survival function of τ_i is indeed S_i. From the definition of default times, we readily see that default times are conditionally independent upon *V* and the conditional survival probabilities are given by:

$$q_t^{i|V}=1_{V>-\ln S_i(t)}S_i(t)^{1-\alpha}.$$

There is upper and lower tail dependence with the same coefficient equal to α. It can be shown (see Embrechts *et al*. [2003]) that Kendall's τ is given by: $\rho_K=\dfrac{\alpha}{2-\alpha}$. $\alpha=0$ corresponds to the independence and $\alpha=1$, implies that $\tau_i=S_i^{-1}\left(V\right)$ i.e. default dates are comonotonic.

Let us consider the case of equal marginal distributions of default times. Then, $Q\left(\tau_i=\tau_j\right)\geq Q\left(V<\min\left(\bar{V}_i,\bar{V}_j\right)\right)>0$. Thus the model allows for simultaneous defaults with positive probability.

[17] General results about the tail dependence within Archimedean copulas can be found in Charpentier and Segers [2007].

[18] The reader can find some extensions to the case of non fatal shocks in Cousin and Laurent [2008a].

[19] The survival copula of default times, $\hat{C}$ is such that $S(t_1,...,t_n)=\hat{C}\left(S_1(t_1),\ldots,S(t_n)\right)$.

It can be proved that increasing α leads to an increase in the dependence between default dates with respect to the supermodular order. The proof is postponed in the Appendix.

III) Ordering of CDO tranche premiums

III.1 Monotonic CDO premiums with respect to dependence parameters

Increasing the correlation parameter ρ within Gaussian double t and Student t copulas, increasing the parameters ρ, β or p in the stochastic correlation model, increasing the parameter θ in the Clayton copula or the parameter α (that represents the relative magnitude of the common shock) in the exponential model leads to an increase in dependence between default times. As a consequence, it can be proven that CDO tranche premiums of equity or senior type, i.e. either with an attachment point equal to zero or a detachment point equal to 100% are monotonic with respect to the dependence parameter. We will thereafter concentrate on equity tranches (i.e. first loss tranches) that are usually associated with the base correlation approach.

For simplicity, let us consider the Gaussian copula case, though the results extend readily to the other models studied. We can formally prove that equity tranche premiums are decreasing with respect to the correlation parameter. This has an obvious practical importance, since it guarantees the uniqueness of base correlations whatever the maturity of the CDO or the marginal distributions of default times.

To emphasize the dependence of the aggregate loss distributions upon the correlation parameter, let us denote by $L_\rho(t)$ the aggregate loss for time *t*, associated with some correlation parameter ρ . Then, for all time horizons *t*, and attachment points *K*, we can prove that:

$$0 \le \rho \le \rho' \Rightarrow E\left[\left(L_\rho(t)-K\right)^+\right] \le E\left[\left(L_{\rho'}(t)-K\right)^+\right]$$ [20].

Let us also remark that in all the studied models, $E\left[L(t)\right] = \sum_{i=1}^{n} M_i F_i(t)$. Thus, the expected loss on the reference portfolio is the sum of the expected losses on the names and is invariant with respect to the correlation structure. From call-put parity, we have:

$$\rho \le \rho' \Rightarrow E\left[\min\left(K, L_\rho(t)\right)\right] \ge E\left[\min\left(K, L_{\rho'}(t)\right)\right].$$

$E\left[\min\left(K, L(t)\right)\right]$ is known as the expected loss for a base tranche with detachment point K or as the "base expected loss". The mapping $(t, K) \rightarrow E\left[\min\left(K, L(t)\right)\right]$ is usually known as the "loss surface" (see Krekel and Partenheimer [2006]). Thus, we can state that the base expected loss decreases with the correlation parameter ρ [21]. Since the present value of the default leg of an equity tranche involves a discounted average of such expectations (see Laurent and Gregory [2005]), we conclude that the value of the default leg of an equity tranche decreases when the correlation parameter increases.

To complete the analysis, we also need to consider the behaviour of the premium leg of a CDO tranche with respect to the dependence parameter. We recall that the premium paid is proportional to the

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[20] Let X and Y be two random variables with finite mean. We say that X precedes Y in *stop-loss order* if $E\left[\left(X-K\right)^+\right] \le E\left[\left(Y-K\right)^+\right]$ for all K . It can be shown that if two random vectors $X = \left(X_1, \ldots, X_n\right)$ and $Y = \left(Y_1, \ldots, Y_n\right)$ are ordered for the supermodular order, then $\sum_{i=1}^{n} M_i X_i$ is smaller than $\sum_{i=1}^{n} M_i Y_i$ for the stop-loss order. Since the one factor Gaussian copula is increasing in the supermodular order with respect to the correlation parameter, we obtain the stated inequalities.

[21] Let us remark that though for simplicity, we have assumed deterministic recovery rates, the previous result also holds for a random vector of non negative losses given default $\left(M_1, \ldots, M_n\right)$, independent of the latent variables driving the default times.

outstanding nominal of the tranche, that is $\left(K-L_\rho(t)\right)^+$ in the case of an equity tranche with detachment point K. Using the same line of reasoning as above, we have:

$$\rho \le \rho' \Rightarrow E\left[\left(K-L_\rho(t)\right)^+\right] \le E\left[\left(K-L_{\rho'}(t)\right)^+\right].$$

We conclude that the value of the premium leg increases with the correlation parameter. Since meanwhile, the value of the default leg decreases, the value of a buy protection on an equity tranche decreases when the correlation parameter increases. Using the same lines of reasoning, the previous result also holds for the stochastic correlation[22], Student *t*, Clayton and Marshall-Olkin copulas with respect to the corresponding dependence parameters. Cousin and Laurent [2008a] show that this also applies to the double *t* model.

Note that since $0 \le \rho \le \rho' \Rightarrow E\left[\left(L_\rho(t)-K\right)^+\right] \le E\left[\left(L_{\rho'}(t)-K\right)^+\right]$, the value of the default leg of a senior tranche with attachment point K always increases with the correlation parameter. The premium paid on a senior tranche is proportional to the outstanding nominal of the tranche, which is trickier than for an equity tranche. To illustrate this, let us consider a [30%-100%] tranche on the portfolio of 100 names. At inception, the protection buyer can expect a maximum payout of 70%. Thus, the corresponding premium is based upon that outstanding nominal. Now, let us assume that the first default is painless. For instance, the Fannie Mae subordinated debt auction was associated with a recovery rate of 99.9%. For simplicity, we consider a recovery rate of 100%. After that first default, the maximum loss is equal to 99% and the maximum payout of the super-senior tranche is now equal to 69%. A reasonable contract feature is thus that the premium payment to be proportional to that maximum payout. The previous effect corresponds to a moving detachment point, so-called "amortization from the top". Instead of being $\sum_{i=1}^{n} E_i$, the detachment point is equal to $\sum_{i=1}^{n} E_i\left(1-\delta_i N_i(t)\right)$[23]. The computation of the premium leg of a super senior tranche is detailed in the Appendix.

As a consequence, the value of the premium leg of a senior tranche decreases with the correlation parameter. Therefore, the value of a buy protection senior tranche always increases when the correlation parameter increases.

The usefulness of the supermodular order is made clear from the above discussion: it provides some monotonicity results on CDO tranche premiums with respect to the copula dependence parameter.

III.2 Comonotonic case

We study possible bounds on CDO tranche premiums. Tchen [1980] proved that the random vector of default times $\left(\tau_1,\ldots,\tau_n\right)$ is always smaller, with respect to the supermodular order, that the comonotonic vector of default times $\left(F_1^{-1}(U),\ldots,F_n^{-1}(U)\right)$, where U follows a (standard) uniform distribution. As a consequence, the case of perfect dependence or "comonotonicity" actually provides a model free lower

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[22] Thanks to the previous ordering results on the stochastic correlation model, it can be checked that default times within that framework are more dependent, with respect to the supermodular order, than default times computed under a one factor Gaussian copula with a correlation parameter β and less dependent that default times computed with a correlation parameter ρ. As a consequence, base correlations associated with the stochastic correlation model are uniquely defined and lie between β and ρ. Another extension of the one factor Gaussian copula model is the inter-intra sector model of Gregory and Laurent [2004]. This involves a Gaussian copula with different correlations between names. As for the stochastic correlation case, the square of base correlation is uniquely defined and lies between the smallest and the largest pairwise correlations.

[23] That moving detachment point is a non negative, piecewise constant and non increasing process.

bound on equity tranche premiums. We also recall that this model free bound is reached within the Gaussian copula and a correlation parameter equal to 100%.

Though the comonotonic case is well defined on mathematical grounds, it is rather counterintuitive, especially when the default probabilities differ. For instance, one can perfectly predict all default dates after the first default and the individual credit names deltas are rather hectic (see Morgan and Mortensen [2007]).

The CDO tranche premiums computations are fairly easy in this comonotonic case. For notational simplicity, we assume that the names are ordered, with name 1 associated with the highest default probability and name n associated with the lowest default probability, i.e. $F_1(t) \geq \ldots \geq F_n(t)$. The loss at time t can only take $n+1$ values: $0, M_1, M_1 + M_2, \ldots, M_1 + \cdots + M_n$. The probability of no losses occurring is $1 - F_1(t)$, while the probability of a loss equal to $M_1 + \cdots + M_{i-1}$ is $F_{i-1}(t) - F_i(t)$ and eventually the probability of a loss equal to $M_1 + \cdots + M_n$ is $F_n(t)$. Let us remark that the base expected loss $E\left[\min\left(K, L(t)\right)\right]$ is piecewise linear in K.

In the comonotonic case, the expected loss for a base tranche is actually increasing and concave in the detachment point, as one could expect from an arbitrage-free model. The corresponding lower bound on the base expected loss is not similar to the usual one as computed in Walker [2007]. These no arbitrage bounds are related to the prices of traded tranches which is not the case here. They do not depend upon the recovery rate assumptions, while "model-free" has to be understood here with respect to the dependence structure between default dates. Let us also notice that our bound depends upon the single name default probabilities, which is not the case when dealing directly with base expected losses related to standard detachment points.

The previous results shed some light on the computational issues with the senior tranches on the CDX and iTraxx indices during March 2008[24]. Trading firms experienced difficulties in deriving base correlations assuming a Gaussian copula and a fixed recovery rate of 40%. Traded premiums of some base tranches in the CDO market were below those computed with a correlation parameter equal to 100%. Such an issue cannot be solved by a choice of a more suitable dependence structure between default dates but only by changing the recovery rate assumptions. To further illustrate this issue, let us consider a double *t* copula with a correlation parameter equal to 100%. It is also associated with comonotonic default dates and thus with the same prices as in the one factor Gaussian copula and a correlation parameter of 100%. This reasoning also applies to recent parametric models such as Albrecher *et al.* [2007] where comonotonic default dates are obtained for a correlation parameter of 100%. Another example is provided by the implied copula as discussed in Hull and White [2006]. Any implied copula is associated with smaller dependence than in the comonotonic case. As a consequence, senior tranche premium are always smaller than in the one factor Gaussian copula with a correlation parameter of 100%. Clearly, if in the market, [60-100%] tranches are traded at a positive premium, one needs to leave away the assumption of a fixed recovery rate of 40%.

III.3 Independence case

The dual case of independence case leads to upper bounds on equity tranche premiums in the studied models. For all models at hand, except for Student *t* (see below), this is a consequence of corollary (3.5) in

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[24] Another example to illustrate the practical relevance of comonotonicity is the pricing of senior tranches of CDOs of subprimes. Ashcraft and Schuermann [2008], Crouhy and Turnbull [2008] argue that since the assets were already well-diversified, idiosyncratic risks were wiped-off, leaving only exposure to factor risk, meaning that a correlation of 100% should have been taken into account. Though the economic ideas are quite valid, one should notice that since CDOs of subprimes are CDO squared that involve mezzanine tranches, the premiums are not monotonic with respect to correlation. Note also that according to Basel II, capital requirements are additive, which is also associated with comonotonicity between credit portfolio losses.

Bäuerle and Müller [1998]. The Student *t* copula must be treated slightly differently (see Appendix). Moreover, the independence bound is reached respectively for $\rho = 0$ (Gaussian and double *t*), $\theta = 0$ (Clayton) and for $\alpha = 0$ (Marshall-Olkin). As discussed below, that has several consequences regarding the existence of a base correlation or more generally of implied parameters given some market quotes of base tranches.

One issue is whether the independence case is associated with a *model free* upper bound on equity tranche premiums. The answer is negative. For instance, let us consider a Gaussian copula with constant negative correlation equal to $-\frac{1}{n-1}$. This leads to admissible correlation matrix; as a consequence of Müller and Scarsini [2000], the corresponding copula is smaller than the product copula with respect to the supermodular order. Thus the equity tranche premium will be greater than when computed under the independence assumption.

One may have noticed that the previous counterexample, though built over a Gaussian copula is not associated with a factor model, due to the negative pairwise correlation parameters. On the other hand, all previous factor models allow some comparison with the independence case using the supermodular order. It is well known that the factor structure and the exchangeability property lead to "positive dependence". For instance, one can easily state that the covariances between default indicators $\operatorname{cov}\left(1_{\{\tau_i \le t\}}, 1_{\{\tau_j \le t\}}\right) = \operatorname{var}\left(p_t^{|V}\right) \ge 0$. We can provide a rather general result which states that equity tranche premiums computed in factor models are lower than those computed under the assumption of independent default times. The precise results and proofs are postponed in the Appendix.

From the previous remarks, we can state some important properties of base correlations. Whenever it exists, the base correlation is unique. This results from the monotonicity of equity tranche premiums with respect to the Gaussian correlation parameter stated in subsection III.1. However, given improper recovery rate assumptions or in the case of negative association between default times, it may be that no base correlation can be found. A detailed counterexample is provided in the Appendix. Since base correlation may not exist, even for arbitrage-free CDO tranche premiums, it differs from the implied volatility in the Black-Scholes model.

III.4 Large portfolio approximations

Large portfolio approximations are well known in the credit portfolio field (see Vasicek [2002], Schönbucher [2002] or Schloegl and O'Kane [2005]). The Basel II agreement talks about "infinitely granular" portfolios. In this subsection, we show that true equity tranche premiums are smaller that those computed under a large portfolio approximation.

We now recall a useful result from Dhaene *et al*. [2002]. Let $Z = (Z_1, \ldots, Z_n)$ be a random vector and V a random variable. Then:

$$E\left[Z_1 | V\right] + \cdots E\left[Z_n | V\right] \le_{cx} Z_1 + \cdots + Z_n ,$$

where $\le_{cx}$ is the convex order[25]. Using conditional Jensen inequality, this readily extends to the case where V is a random vector (see Appendix), which is useful to deal with the Student *t* case, associated with a two factor model.

Let us apply this result to the credit case. Here, $Z_i = M_i 1_{\{\tau_i \le t\}}$ and $L(t) = Z_1 + \ldots + Z_n$ are respectively the individual loss on name i and the aggregate loss at time t. As above M_i denotes the deterministic loss

[25] Let X and Y be two random variables. We say that X is smaller than Y with respect to the convex order and we denote $X \le_{cx} Y$ if $E\left[f(X)\right] \le E\left[f(Y)\right]$, for all convex functions such that the expectations are well defined.

given default on name i. We have: $E\left[Z_i \middle| V\right] = M_i p_t^{i|V}$ where $p_t^{i|V} = Q\left(\tau_i \le t \middle| V\right)$ denotes the conditional default probability of name i. Then, the approximation of the loss is provided by $E\left[L(t)\middle|V\right] = \sum_{i=1}^{n} M_i p_t^{i|V}$ which is a deterministic function of the factor V[26]. Let us remark that we do not assume that the marginal default probabilities $F_i(t), i = 1,\ldots,n$ nor the losses given default $M_i, i = 1,\ldots,n$ are equal, thus this differs from the large homogeneous portfolio approximation (see Appendix for further discussion). For instance in the one factor Gaussian copula with "flat correlation" and deterministic recovery rates, the large portfolio approximation is given by $E\left[L(t)\middle|V\right] = \sum_{i=1}^{n} M_i \Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right)$[27]. As a consequence the computation of the expected loss on an equity tranche $E\left[\min\left(K, \sum_{i=1}^{n} M_i p_t^{i|V}\right)\right]$ can be done by a simple quadrature without any inversion of the characteristic function or recursion techniques. Moreover, since $\sum_{i=1}^{n} M_i p_t^{i|V} \le_{cx} L(t)$, we have $E\left[\min\left(K, L(t)\right)\right] \le E\left[\min\left(K, \sum_{i=1}^{n} M_i p_t^{i|V}\right)\right]$. Thus, the true value of the default leg of an equity tranche is smaller than the one computed under the large portfolio approximation. Using the same reasoning, we also have $E\left[\left(K - \sum_{i=1}^{n} M_i p_t^{i|V}\right)^+\right] \le E\left[\left(K - L(t)\right)^+\right]$. Therefore, the true value of the premium leg on an equity tranche is larger than the one computed under the large portfolio approximation. We conclude that true equity tranche premiums are smaller that those computed under a large portfolio approximation. Clearly, this is a model dependent upper bound.

One can further look at the dependence of the CDO premiums computed under the large portfolio approximation with respect to the dependence parameter. For simplicity, let us focus on the one factor Gaussian copula. We recall that the large portfolio approximation is provided by: $LP_\rho(t) = \sum_{i=1}^{n} M_i \Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right)$. It can be shown that:

$$0 \le \rho \le \rho^* \Rightarrow E\left[\left(LH_\rho(t) - K\right)^+\right] \le E\left[\left(LH_{\rho^*}(t) - K\right)^+\right].$$

In other words, the monotonicity results stated for finitely granular portfolios also hold for the infinitely granular limit. The proof is postponed in the Appendix.

III.5 The case of basket default swaps

Let us consider the case of a homogeneous first to default swap, i.e. all names have the same nominal and recovery rate[28]. It can be treated as a homogeneous CDO equity tranche with detachment point

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[26] After the March 2008 crisis, some emphasis was put on using stochastic recovery rates. In the simplest case where M_i are deterministic functions of the factor V, denoted by $M_i(V)$, which corresponds to neglecting idiosyncratic risk in recovery rates, the large portfolio approximation is given by $\sum_{i=1}^{n} M_i(V) p_t^{i|V}$ (see Amraoui and Hitier [2008]).

[27] Since here V is an arbitrary random vector, one will rather consider the factor in the one dimensional case (stochastic correlation, double *t*, Clayton and Marshall-Olkin copulas). The Student *t* copula is associated with a two factor model and the large portfolio approximation can then be written as: $\sum_{i=1}^{n} M_i \Phi\left(\frac{-\rho V + Z^{-1/2} t_\nu^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right)$.

[28] We refer to Laurent and Gregory [2005] for an analysis of basket default swaps in the non homogeneous case.

equal to the common loss given default. Thus, the previous results stated for CDO tranches apply. For instance, increasing the correlation parameter in the one factor Gaussian copula model always leads to a decrease in the first to default swap premium.

IV) Comparing Basket Default Swaps and CDO premiums

In order to conduct model comparisons, we proceeded the following way. Since the studied copulas depend upon a one dimensional parameter, we have chosen that parameter so that either first to default (for basket default swaps) or equity tranches premiums (for CDO tranches) are equal. Such a correspondence between parameters is meaningful since equity tranche premiums are monotonic with respect to the relevant dependence parameter (see previous section). We then compute the premiums of basket default swaps and various CDO tranches and study the departures between the different models and also between model and market quotes.

IV.1 First to default swaps with respect to the number of names

We firstly computed first to default swap premiums under different models as a function of the number of names in the basket, from 1 to 50. We assumed flat and equal CDS premiums of 80 bps, recovery rates of 40% and 5 year maturity. The default free rates are provided in the Appendix. The dependence parameters were set to get equal premiums for 25 names. They are reported in Table 1.

	Gaussian	Student (6)	Student (12)	Clayton	MO
dependence	$\rho^2 = 30\%$	$\rho^2 = 11.9\%$	$\rho^2 = 21.6\%$	$\theta = 0.173$	$\alpha = 49\%$

Table 1: dependence parameters for the pricing of first to default swaps.

Table 2 reports the first to default premiums. Let us remark that Gaussian, Student *t* and Clayton copulas lead to quite similar premiums, while the Marshall-Olkin deviate quite significantly. The second line in the table corresponds to a plain CDS on a single name and thus all models provide the same input premium of 80 bps. We can also notice that the premiums always increase with the number of names[29].

Names	Gaussian	Student (6)	Student (12)	Clayton	MO
1	80	80	80	80	80
5	332	339	335	336	244
10	567	578	572	574	448
15	756	766	760	762	652
20	917	924	920	921	856
25	1060	1060	1060	1060	1060
30	1189	1179	1185	1183	1264
35	1307	1287	1298	1294	1468
40	1417	1385	1403	1397	1672
45	1521	1475	1500	1492	1875
50	1618	1559	1591	1580	2079

Table 2: First to default premiums with respect to the number of names (bps pa).

Table 3 provides Kendall's τ for the different models. As can be seen, even once the models have been calibrated on a first to default swap premium with 25 names, the non linear correlations are quite different.

	Gaussian	Student (6)	Student (12)	Clayton	MO
ρ_K	19%	8%	14%	8%	32%

[29] This feature is model independent: the survival function of first to default time in a homogeneous basket is given by: $S_n^1(t) = Q\left(\tau_1 > t, \ldots, \tau_n > t\right) \geq Q\left(\tau_1 > t, \ldots, \tau_n > t, \tau_{n+1} \geq t\right) = S_{n+1}^1(t)$.

Table 3: Kendall's τ for the studied models.

IV.2 *k*-th to default swaps

We then considered 10 names with credit spreads evenly distributed between 60 bps and 150 bps, a constant recovery rate of 40% and maturity still equal to 5 years. Table 4 reports the dependence parameters. They are set so that the first to default premiums are equal for all models.

Gaussian	Clayton	Student (6)	Student (12)	MO
$\rho^2 = 30\%$	$\theta = 0.1938$	$\rho^2 = 16.5\%$	$\rho^2 = 23.6\%$	$\alpha = 36\%$

Table 4: dependence parameters for the pricing of *k*-th to default swaps.

The columns of Table 5 provides first, second… until last to default premiums. As in the previous example, the differences between Gaussian, Student *t* and Clayton copulas are minor while the Marshall-Olkin copula leads to strikingly different results for higher order basket default swaps.

Rank	Gaussian	Clayton	Student (6)	Student (12)	MO
1	723	723	723	723	723
2	275	274	278	276	173
3	122	123	122	122	71
4	55	56	55	55	56
5	24	25	24	25	55
6	11	11	10	10	55
7	4.7	4.3	3.5	4.0	55
8	1.5	1.5	1.1	1.3	55
9	0.39	0.39	0.25	0.35	55
10	0.06	0.06	0.04	0.06	55

Table 5: First to last to default swap premiums (bps pa) for different models.

Once again, Kendall's τ is poorly related to the premium structure (see Table 6).

	Gaussian	Clayton	Student (6)	Student (12)	MO
ρ_K	19%	9%	11%	15%	22%

Table 6: Kendall's τ for the studied models.

IV.3 CDO tranche premiums under different models

As a practical example, we considered 100 names, all with a recovery rate of $\delta = 40\%$ and equal unit nominal. The credit spreads are all equal to 100 bps. They are assumed to be constant until the maturity of the CDO. The attachment points of the tranches are $A = 3\%$ and $B = 10\%$. The CDO maturity is equal to five years. The default free rates are provided in the Appendix.

We considered CDO margins for equity, mezzanine and senior tranches[30] for the different models. We firstly considered the Gaussian model and computed the margins with respect to the correlation parameter ρ^2. These results show a strong negative dependence of the equity tranche with respect to the correlation parameter, a positive dependence of the senior tranche and a bumped curve for the mezzanine, which is not as sensitive to the correlation parameter.

ρ^2	equity	mezzanine	Senior
0 %	5341	560	0.03

[30] Corresponding to $[0-3\%]$, $[3-10\%]$ and $[10-100\%]$ tranches.

10 %	3779	632	4.6
30 %	2298	612	20
50 %	1491	539	36
70 %	937	443	52
100%	167	167	91

Table 7: CDO margins (bp pa) Gaussian copula with respect to the correlation parameter.

In order to compare the different pricing models, we set the dependence parameters to get the same equity tranche premiums. This gives the following correspondence table:

ρ^2	0%	10%	30%	50%	70%	100%
θ	0	0.05	0.18	0.36	0.66	∞
ρ_6^2			14%	39%	63%	100%
ρ_{12}^2			22%	45%	67%	100%
ρ^2 *t*(4)-*t*(4)	0%	12%	34%	55%	73%	100%
ρ^2 *t*(5)-*t*(4)	0%	13%	36%	56%	74%	100%
ρ^2 *t*(4)-*t*(5)	0%	12%	34%	54%	73%	100%
ρ^2 *t*(3)-*t*(4)	0%	10%	32%	53%	75%	100%
ρ^2 *t*(4)-*t*(3)	0%	11%	33%	54%	73%	100%
α	0	27%	53%	68%	80%	100%

Table 8: correspondence between parameters for the pricing of CDO tranches.

For instance, when the Gaussian copula parameter is equal to 30%, we must set the Clayton copula parameter to 0.18 in order to get the same equity tranche premium[31].

Once the equity tranches were matched, we computed the premiums of the mezzanine and senior tranche with the different models. It can be seen that Clayton and Student *t* provide results that are close to the Gaussian case. For instance, for a Gaussian correlation of 30%, the senior tranche premium computed under the Gaussian assumption is equal to 20bps, while we obtained 18 bps under the Clayton assumption and 19 bps with a Student *t* with 12 degrees of freedom.

ρ	0%	10%	30%	50%	70%	100%
Gaussian	560	633	612	539	443	167
Clayton	560	637	628	560	464	167
Student (6)			637	550	447	167
Student (12)			621	543	445	167
t(4)-*t*(4)	560	527	435	369	313	167
t(5)-*t*(4)	560	545	454	385	323	167
t(4)-*t*(5)	560	538	451	385	326	167
t(3)-*t*(4)	560	495	397	339	316	167
t(4)-*t*(3)	560	508	406	342	291	167
MO	560	284	144	125	134	167

Table 9: mezzanine tranche premiums (bps pa) computed under the various models for different levels of Gaussian copula correlation.

[31] We could not match the independence case with the Student *t* copula. Even for a zero correlation parameter, there is still tail dependence. As a consequence, no correlation parameter in the Student *t* copula allows a fit to the equity tranche premium computed under Gaussian copula and correlation equal to 0 or 10%.

ρ	0%	10%	30%	50%	70%	100%
Gaussian	0.03	4.6	20	36	52	91
Clayton	0.03	4.0	18	33	50	91
Student (6)			17	34	51	91
Student (12)			19	35	52	91
t(4)-*t*(4)	0.03	11	30	45	60	91
t(5)-*t*(4)	0.03	10	29	45	59	91
t(4)-*t*(5)	0.03	10	29	44	59	91
t(3)-*t*(4)	0.03	12	32	47	71	91
t(4)-*t*(3)	0.03	12	32	47	61	91
MO	0.03	25	49	62	73	91

Table 10: senior tranche premimus (bps pa) computed under the various models for different levels of Gaussian copula correlation.

As for the basket default swap premiums, the premiums computed under the Marshall-Olkin copula are fairly different, except of course for the extreme cases of independence and comonotonicity. The double *t* model lies between these two extremes, i.e. Gaussian and Marshall-Olkin copulas.

Let us now consider a non-parametric measure of dependence such as Kendall's τ. We used the analytical formulas for the Gaussian, Clayton, Student and Marshall-Olkin copulas. Table 11 shows that the level of dependence associated with the Marshall-Olkin copula is bigger than in the Gaussian, Clayton or Student *t* copulas. Though Gaussian and Clayton copulas lead to similar CDO premiums, Kendall's τ are quite different.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	0%	6%	19%	33%	49%	100%
Clayton	0%	3%	8%	15%	25%	100%
Student (6)			9%	25%	44%	100%
Student (12)			14%	30%	47%	100%
MO	0%	16%	36%	52%	67%	100%

Table 11: Kendall's τ (%) for the studied models and for different levels of Gaussian copula correlation.

Let us remark that Kendall's τ increases with the correlation parameter. Since the copulas are positively ordered with respect to the dependence parameter, $\theta_1 \le \theta_2$ implies that $\rho_{K,C_{\theta_1}} \le \rho_{K,C_{\theta_2}}$ where $\rho_{K,C}$ denotes Kendall's τ associated with copula C. Moreover, $\rho_{K,C^+} = 1$.

Table 12 provides the tail dependence coefficients associated with the different models. The different columns in the table correspond to the different Gaussian correlation coefficients involved in the previous tables, i.e. 0%, 10%, 30%, 50%, 70% and 100%. Since the copulas are positively ordered with respect to the dependence parameter (as a consequence of the supermodular order), $\theta_1 \le \theta_2$ implies $C_{\theta_1} \prec C_{\theta_2}$ which in turn implies that the tail dependence coefficients are positively ordered with respect to the relevant dependence parameter. We can check the increase of the tail dependence coefficients from 0 to 100% on each row.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	0%	0%	0%	0%	0%	100%
Clayton	0%	0%	2%	15%	35%	100%
Student (6)			5%	12%	25%	100%
Student (12)			1%	4%	13%	100%
t(4)-*t*(4)	0%	0%	1%	10%	48%	100%
t(5)-*t*(4)	0%	0%	0%	0%	0%	100%
t(4)-*t*(5)	0%	100%	100%	100%	100%	100%
t(3)-*t*(4)	0%	100%	100%	100%	100%	100%

t(4)-*t*(3)	0%	0%	0%	0%	0%	100%
MO	0%	27%	53%	68%	80%	100%

Table 12: coefficient of lower tail dependence (%) for the studied models and for different levels of Gaussian copula correlation.

It can be noticed that for a 30% Gaussian correlation, the level of tail dependence is rather small for Gaussian, Clayton and Student *t* copulas. This is also the case for the *t*(4)-*t*(4) model which however leads to quite different senior tranche premiums. The tail dependence is much bigger for the Marshall-Olkin copula. The previous table shows no obvious link between tail dependence and the price of the senior tranche. The reason for this is rather simple. It can be seen that the probability of a default payment occurring on the senior tranche over a 5 year time horizon, $Q\left(L(5) \geq 10\%\right) \simeq 30\%$. Thus, we are still far way from the tail of the loss distribution.

We also considered the bivariate default probabilities corresponding to the CDO maturity, $Q\left(\tau_i \leq 5, \tau_j \leq 5\right)$ for $i \neq j$. From the symmetry of the distributions, these do not depend of the chosen couple of names. The univariate default probability for a five years horizon is $Q\left(\tau_i \leq 5\right) = 8.1\%$. In the independence case, the bivariate default probability is $\left(8.1\%\right)^2 = 0.66\%$. The bivariate default probabilities are very close for the Gaussian, Clayton and Student *t* copulas. We have stronger bivariate default probabilities for the double *t* models and even larger for the Marshall-Olkin copula. Let us remark that since the marginal default probabilities are given, the variance of the loss distribution and the linear correlation between default indicators only involve the bivariate default probability. The larger the bivariate default probabilities, the larger will be the variance of the loss distribution and the linear correlation between default indicators.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	0.66%	0.91%	1.54%	2.41%	3.59%	8.1%
Clayton	0.66%	0.88%	1.45%	2.24%	3.31%	8.1%
Student (6)			1.41%	2.31%	3.52%	8.1%
Student (12)			1.49%	2.36%	3.56%	8.1%
t(4)-*t*(4)	0.66%	1.22%	2.38%	3.49%	4.67%	8.1%
t(5)-*t*(4)	0.66%	1.16%	2.27%	3.38%	4.57%	8.1%
t(4)-*t*(5)	0.66%	1.18%	2.28%	3.37%	4.54%	8.1%
t(3)-*t*(4)	0.66%	1.34%	2.57%	3.69%	5.02%	8.1%
t(4)-*t*(3)	0.66%	1.31%	2.55%	3.70%	4.87%	8.1%
MO	0.66%	2.63%	4.53%	5.65%	6.53%	8.1%

Table 13: bivariate default probabilities (5 year time horizon) for the studied models and for different levels of Gaussian copula correlation.

To further study some possible discrepancies between Gaussian, Clayton and Student *t* copulas, we kept the previous correspondence table between parameters and recomputed the tranche premiums for different input credit spreads. We want here to check whether the Gaussian copula can provide a good fit to Clayton and Student *t* copula premiums uniformly over credit spread curves. In tables 14, 15, 16 below, credit spreads have been shifted from 100 bps to 120 bps.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	6476	4530	2695	1731	1085	200
Clayton	6476	4565	2748	1781	1132	200
Student (6)			2765	1765	1104	200
Student (12)			2730	1748	1093	200

Table 14: equity tranche premiums (bps pa) after a shift of credit spreads.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	853	857	765	652	527	200
Clayton	853	867	794	687	564	200
Student (6)			807	672	537	200
Student (12)			782	661	531	200

Table 15: mezzanine tranche premiums (bps pa) after a shift of credit spreads.

ρ^2	0%	10%	30%	50%	70%	100%
Gaussian	0.2	8	28	46	64	109
Clayton	0.2	7	25	42	60	109
Student (6)			23	44	63	109
Student (12)			26	45	64	109

Table 16: senior tranche premiums (bps pa) after a shift of credit spreads.

We can see that the same set of parameters still enables to provide quite similar premiums for the different models, especially for the senior tranche. These overall results are not surprising keeping in mind the results in Greenberg *et al*. [2004]. Demarta and McNeil [2005] also use some proximity between the *t*-EV copula and Gumbel or Galambos copulas for suitable choices of parameters. Breymann *et al*. [2003] show some similarity between Student *t* and Clayton copulas as far as extreme returns are concerned[32].

IV.4 Market and model CDO tranche premiums

While the previous results relied on constant credit spreads, we now consider another example related to the Dow Jones iTraxx Europe index. The CDO maturity is equal to five years. The attachment detachment points correspond to the standard iTraxx CDO tranches, i.e. 3%, 6%, 9%, 12% and 22%. The index is based on 125 names. The 5 year credit spreads of the names lie in between 9 bps and 120 bps with an average of 29 bps and a median of 26 bps. The credit spreads and the default free rates are detailed in the Appendix. To ease comparisons, we assumed constant credit spreads with respect to maturity.

We discuss the ability of each copula to produce a smile on pricing tranches on this index as is observed in the market. We calibrated the different models on the market quote for the [0-3%] equity tranche. The parameters used for a three state stochastic correlation model were $\gamma^2 = 6.6\%$ with probability 0.66, $\beta^2 = 20\%$ with probability 0.1 and $\rho^2 = 80\%$ with probability 0.24. Better fits are presumably possible as we did not perform an optimization to match the market prices. Let us remark that we could not fit a Student *t* model with 6 degrees of freedom on the equity tranche market quote. We provide results both for tranches as quoted in the market and for "equity type" tranches.

Tranches	Market	Gaussian	Clayton	Student (12)	*t*(4)-*t*(4)	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[3-6%]	101	163	163	164	82	122	14
[6-9%]	33	48	47	47	34	53	11
[9-12%]	16	17	16	15	22	29	11
[12-22%]	9	3	2	2	13	8	11

Table 17: iTraxx CDO tranche premiums (bps pa) using market and model quotes.

Tranches	Market	Gaussian	Clayton	Student (12)	*t*(4)-*t*(4)	Stoch.	MO
[0-3%]	916	916	916	916	916	916	916
[0-6%]	466	503	504	504	456	479	418

[32] This also shows that the dynamics of the credit spreads implied by the copula is not relevant for the pricing of CDOs. From Schönbucher and Schubert [2001], we know that Gaussian and Clayton copulas differ quite significantly from that point of view.

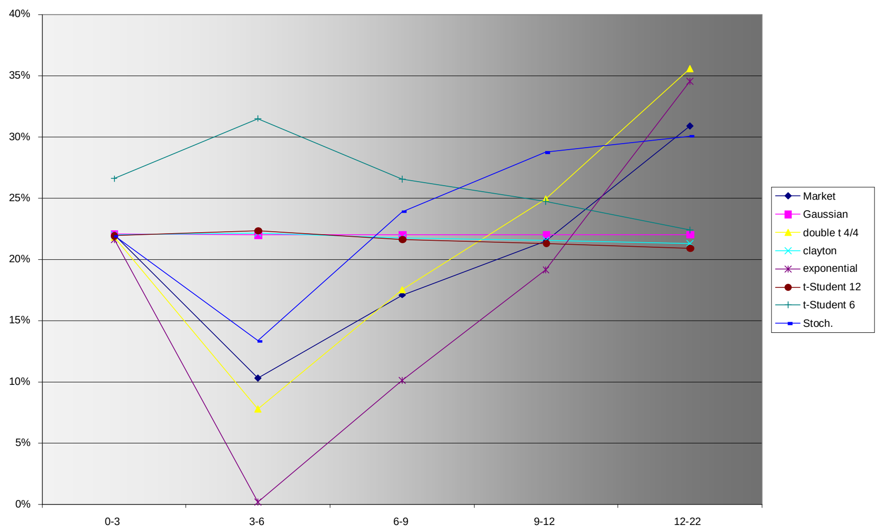
[0-9%]	311	339	339	340	305	327	272
[0-12%]	233	253	253	254	230	248	203
[0-22%]	128	135	135	135	128	135	113

Table 18: iTraxx “equity tranche” CDO premiums (bps pa) using market and model quotes.

Most practitioners deal with implied Gaussian correlation, that is the flat correlation in the one factor Gaussian copula model associated with a given premium. Table 19 and Graph 1 show that correlation parameters are smaller for mezzanine tranches leading to a so called “correlation smile”. Friend and Rogge [2005], Greenberg *et al*. [2004], Finger [2005] also report such an effect meaning that the Gaussian copula fails to price exactly the observed prices of iTraxx tranches. It can be seen that Clayton or Student t copulas are still close to Gaussian and thus do not create any correlation smile. This is consistent with previous empirical studies (see also Schönbucher [2002], Schloegl and O’Kane [2005]). The Marshall-Olkin model underestimates the prices of the mezzanine tranches and overestimates the super senior. The double t model provides a better overall fit but overestimates the senior tranches. The stochastic correlation model fits reasonably to the market prices, in particular the equity and junior super senior. It overestimated the mezzanine tranche premiums and would therefore underestimate the super senior [22-100%] region. This could be associated to the lack of extreme or a fat tail risk on the loss distribution.

Tranches	Market	Gaussian	Clayton	Student (12)	$t(4)$-$t(4)$	Stoch.	MO
[0-3%]	22%	22%	22%	22%	22%	22%	22%
[3-6%]	10%	22%	22%	22%	8%	13%	0%
[6-9%]	17%	22%	22%	22%	18%	24%	10%
[9-12%]	22%	22%	23%	21%	25%	29%	19%
[12-22%]	31%	22%	21%	21%	36%	30%	35%

Table 19: implied compound correlation for iTraxx tranches.

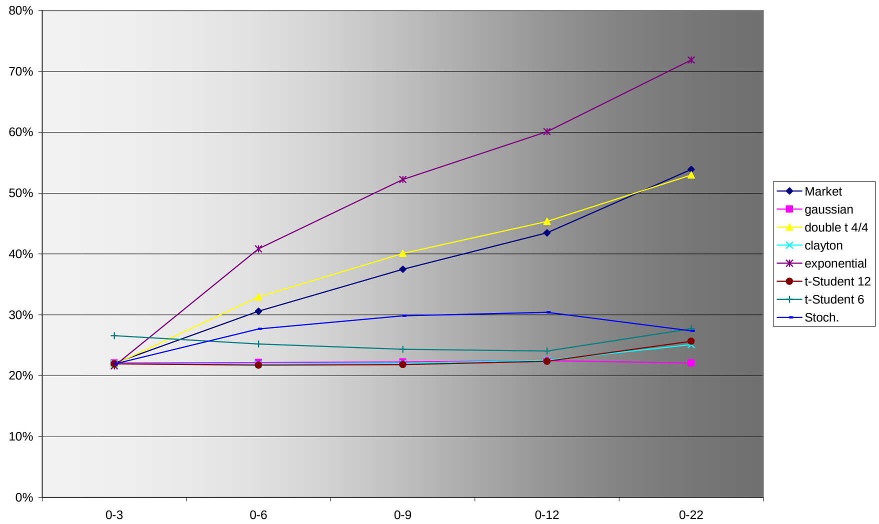


Graph 1: implied compound correlation for iTraxx CDO tranches based on market and model quotes. Tranches are on the x - axis, compound correlations on the y - axis.

Table 20 and Graph 2 show the “equity-type” implied correlations or “base correlations”. We believe the best criteria to assess the ability of a model to fit the market is the difference in compound correlation. The relative pricing error on each tranche should be reasonably close to this although there can be problems for tranches that are rather insensitive to correlation. Base correlation may not be appropriate because small mispricings lower on the capital structure cause dramatic deviations on high base correlation tranches. This can be seen in Graph 2 where reasonable fits to compound can be seen to look extremely poor in terms of their implied base correlations. For example in the stochastic correlation model, the [0-22%] mispricing on base correlation is 27% whereas the [12-22%] tranche is priced within 1bp.

Tranches	Market	Gaussian	Clayton	Student (12)	*t*(4)-*t*(4)	Stoch.	MO
[0-3%]	22%	22%	22%	22%	22%	22%	22%
[0-6%]	31%	22%	22%	22%	33%	28%	41%
[0-9%]	37%	22%	22%	22%	40%	30%	52%
[0-12%]	43%	22%	23%	23%	45%	30%	60%
[0-22%]	54%	22%	25%	26%	53%	27%	72%

Table 20: implied base correlation for iTraxx tranches.



Graph 2: implied base correlation for iTraxx CDO tranches computed from market and model quotes. Tranches are on the x - axis, base correlations on the y - axis.

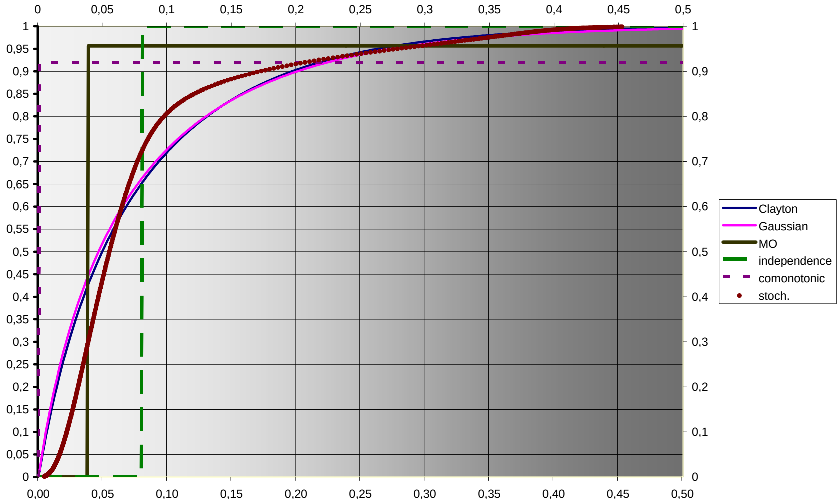
IV.5 Conditional default probability distributions drive CDO tranche premiums

The pricing of basket default swaps or CDOs only involve loss distributions over different time horizons. The characteristic function of the aggregate loss only involves the conditional default probabilities $p_t^{i|V}$. When these are identically distributed, the characteristic function can be written as:

$$\varphi_{L(t)}(u) = \int \prod_{1 \le j \le n} \left(1 - p + pe^{iuM_j}\right) G(dp),$$

where *G* is the distribution function of the conditional default probabilities[33]. In other words, two models associated with the same distributions of conditional default probabilities will lead to the same joint distribution of default indicators and eventually to the same CDO premiums. As an example, let us consider Gaussian, stochastic correlation, Clayton and Marshall-Olkin copulas. We have $p_t^{i|V} = \Phi\left(\dfrac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right)$, $p_t^{i|V} = p\Phi\left(\dfrac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}}\right) + (1-p)\Phi\left(\dfrac{-\beta V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\beta^2}}\right)$, *V* Gaussian for the Gaussian and stochastic correlation copulas, $p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$, *V* standard Gamma for the Clayton copula and $p_t^{i|V} = 1 - 1_{V > -\ln S_i(t)} S_i(t)^{1-\alpha}$, *V* exponential for the Marshall-Olkin copula.

[33] $G(p) = Q\left(p_t^{i|V} \le p\right)$ for $0 \le p \le 1$.



Graph 3: distribution functions of conditional default probabilities for different models.

Let us go back to the previous CDO example with flat credit curves of 100bps. For a Gaussian correlation of 30%, the correspondence table gives $\theta = 0.18$ and $\alpha = 53\%$. Graph 3 provides the distribution functions of the 5 year conditional default probabilities. It can be seen that the distribution functions are almost identical in the Gaussian and Clayton copula cases. For the Marshall-Olkin copula, the conditional default probability only takes values 1 and $1 - S_i(t)^{1-\alpha}$ which leads to a step distribution function. The independence case is associated with a Dirac mass at the marginal default probability while the conditional default probability is a Bernoulli variable in the comonotonic case. It is quite clear that the differences between Marshall-Olkin copula on one hand, Gaussian and Clayton copulas on the other hand are quite substantial. We also provide the distribution of conditional default probabilities for a stochastic correlation model. Here, $\beta^2 = 10\%$ with probability 0.8 and $\rho^2 = 90\%$ with probability 0.2. We can see that the stochastic correlation model lies in between Marshall-Olkin and Gaussian. An interesting area of research consists in building the distribution of $p_t^{i|V}$ from the market prices which could give some insight on choice of model. Such construction can be found in Hull and White [2006]. The practical relevance of conditional default probabilities is also emphasized in Burtschell *et al.* [2007] or Cousin and Laurent [2008b]. A general investigation of the use of conditional default probabilities in the pricing of CDOs and connexions with the theory of stochastic orders is done in Cousin and Laurent [2008a].

Conclusion

We discussed the choice of dependence structure in basket default swap and CDO modelling. We compared some popular copula models against the one factor Gaussian copula that is currently the industry standard. We considered an assessment methodology based on the matching of basket default swap premiums and CDO tranches. The main results are the following:

- For pricing purposes, and once correctly calibrated, Student *t* and Clayton copula models provide rather similar results, close to the Gaussian copula.
- The Marshall-Olkin copula associated with large probabilities of simultaneous defaults leads to strikingly different results and a dramatic fattening of the tail of the loss distributions.
- The double *t* model lies in between and provides a better fit to market quotes. We found that related models such as the random factor loadings model of Andersen and Sidenius [2005] led to similar correlation smiles.
- The stochastic correlation copula can also achieve a reasonable skew, close to that observed in the market.
- Non parametric measures of dependence, such as Kendall's τ or the tail dependence coefficient are of little help for explaining model quotes.

- The distribution of the conditional default probability is the key input when pricing CDO tranche premiums and when comparing different models.

We also refer to Cousin and Laurent [2008a,b] which are a follow-up of this review paper. We relied mainly on the supermodular order to study dependence effects between default dates in factor models. As we intend to show in upcoming research, the stochastic orders theory has a larger application field in credit modelling, even for models as simple as the one factor Gaussian copula.

The recent turmoil in capital markets, following the subprime crisis in the US, raises the issue of updating credit correlation models in the aftermath of the crisis. We already mentioned that, while rather complex products are still in trading books and need some risk-management, the structured credit market tends to move forward to simpler products, synthetic CDOs, either on standard indexes or bespoke portfolios being the most prominent example. This relaxes the incentives to shift to more sophisticated dynamic models, some of them being still in their infancy, others could not be easily calibrated in the extremely high correlation environment recently experimented. Some major market participants seemingly needed to step back to outmoded static copula models. Despite huge losses recently encountered in the derivatives markets, academic researchers should think in concrete rather than ideological terms and take into account some facts such: prominence of the one factor Gaussian copula and base correlation approaches, huge investments in systems according to the previous framework and the fact that correlation models do drive the cognitive processes as far as risk management is concerned. Therefore, one may question the usefulness of investigating alternative dependence structures that was partly the purpose of this paper. The main issue is that the one factor Gaussian copula provides a very poor fit to loss distributions. This can be seen through the steepness of the base correlation curve. This is actually a practical issue, since interpolation schemes of base correlation provide weird results in this context, such as negative thin tranchelets prices, shaky credit spread sensitivities and so on. Using a model that leads to a better fit to market tranche quotes with the same set of parameters, will result in a flattening of the implied parameter curve, which in turn will solve most of the issues discussed above.

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Appendix

1) Data for the Basket default swaps and CDO examples.

Basket default swaps and homogeneous CDO examples

1D	1W	1M	2M	3M	6M	9M	1Y	2Y	3Y	4Y	5Y
2.02	2.05	2.06	2.07	2.08	2.14	2.23	2.37	2.80	3.17	3.47	3.71

Table a: default free yield curve (continuous rates)

iTraxx example

9	14	17	20	21	23	25	28	31	34	37	45	68
10	14	18	20	21	23	25	28	31	35	37	45	72
10	15	18	20	21	23	26	28	32	35	37	46	73

10	15	18	20	21	23	26	28	33	35	38	47	106
10	15	18	20	22	24	26	29	33	35	38	48	120
10	15	18	20	22	24	26	30	33	36	40	51	
10	16	18	21	22	24	27	30	34	36	43	52	
10	17	18	21	22	45	27	30	34	36	44	53	
13	17	19	21	23	25	27	31	34	37	44	56	
13	17	19	21	23	25	27	31	34	37	44	58	

Table b: 5 year credit spreads iTraxx Europe

The default free rates were obtained from the swap market in Euros on the 08/02/2005.

1D	1W	1M	2M	3M	6M	9M	1Y	2Y	3Y	4Y	5Y
2.07	2.09	2.10	2.12	2.14	2.18	2.24	2.34	2.59	2.78	2.93	3.06

Table c: default free yield curve (continuous rates)

2) Supermodular order.

Let $f:\mathbb{R}^n \to \mathbb{R}$. We consider the difference operators $\Delta_i^\varepsilon f(x) = f(x+\varepsilon e_i) - f(x)$, where e_i is the *i*-th unit vector and $\varepsilon > 0$. *f* is said to be supermodular, if $\Delta_i^\varepsilon \Delta_j^\delta f(x) \geq 0$ holds for all $x \in \mathbb{R}^n, 1 \leq i \leq j \leq n$ and $\varepsilon, \delta > 0$. If f is smooth, supermodularity states that all non diagonal terms of the Hessian matrix are non negative. The concept of supermodularity was introduced in social sciences and game theory to analyse how one agent's decision affects the incentives of others. A random vector $X = (X_1, \ldots, X_n)$ is said to be smaller than the random vector $Y = (Y_1, \ldots, Y_n)$, with respect to the supermodular order, if $E[f(X)] \leq E[f(Y)]$ for all supermodular functions such that the expectation exists. This means that the coordinates of Y are more dependent in a mathematical sense than the coordinates of X.

3) Supermodular ordering and stochastic correlation model.

Let $p \leq p'$ and consider the following model:

$$V_i = \min(C_i, D_i)\left(\rho V + \sqrt{1-\rho^2}\bar{V}_i\right) + \left(1 - \min(C_i, D_i)\right)\left(\beta V + \sqrt{1-\beta^2}\bar{V}_i\right)$$

where $C_1, \ldots, C_n, D_1, \ldots, D_n, V, \bar{V}_1, \ldots, \bar{V}_n$ are all independent, $C_1, \ldots, C_n$ are Bernoulli variables with parameter $\frac{p}{p'}$ and $D_1, \ldots, D_n$ are Bernoulli variables with parameter p'. $\min(C_i, D_i)$ is a Bernoulli variable with parameter p. As a consequence, $(V_1, \ldots, V_n)$ follow a stochastic correlation model with parameters (ρ, β, p). We now compare with:

$$W_i = D_i\left(\rho' V + \sqrt{1-\rho'^2}\bar{V}_i\right) + (1 - D_i)\left(\beta' V + \sqrt{1-\beta'^2}\bar{V}_i\right),$$

where $\rho \leq \rho' \leq 1, \beta \leq \beta' \leq 1$. $(W_1, \ldots, W_n)$ follows a stochastic correlation model with parameters (ρ', β', p'). $(W_1, \ldots, W_n) | C_1, \ldots, C_n, D_1, \ldots, D_n$ is Gaussian with correlation parameter $\rho' D_i + \beta'(1 - D_i)$, while $(V_1, \ldots, V_n) | C_1, \ldots, C_n, D_1, \ldots, D_n$ is Gaussian with correlation parameter $\rho \min(C_i, D_i) + \beta(1 - \min(C_i, D_i))$. Since $\rho \min(C_i, D_i) + \beta(1 - \min(C_i, D_i)) \leq \rho' D_i + \beta'(1 - D_i)$, we have: $(V_1, \ldots, V_n) | C_1, \ldots, C_n, D_1, \ldots, D_n \leq_{\text{sm}} (W_1, \ldots, W_n) | C_1, \ldots, C_n, D_1, \ldots, D_n$. From the invariance of supermodular order under mixing: $(V_1, \ldots, V_n) \leq_{\text{sm}} (W_1, \ldots, W_n)$. Thus increasing the probability of being in the high correlation state p, or increasing any of the two correlation parameters ρ, β leads to an increase in dependence with respect to the supermodular order.

4) Supermodular ordering for general stochastic correlation models.

The two state correlation model can be easily generalized. Let us consider the following modelling of latent variables V_i :

$$V_i = \tilde{\rho}_i V + \sqrt{1-\tilde{\rho}_i^2}\,\overline{V}_i \ , \ i = 1,\ldots,n \ ,$$

where $\tilde{\rho}_1,\ldots,\tilde{\rho}_n$ are independent stochastic correlations with distribution function *F*. We still have independent default times conditionally on *V* and:

$$p_t^{i|V} = \int_0^1 \Phi\left(\frac{-\rho V + \Phi^{-1}\left(F_i(t)\right)}{\sqrt{1-\rho^2}} \right) dF(\rho)$$

We can compare stochastic correlation models in a fairly general framework. Let us consider another stochastic correlation model associated with distribution function *G*. We denote by $\tilde{\beta}_1,\ldots,\tilde{\beta}_n$ the corresponding stochastic correlation parameters:

$$W_i = \tilde{\beta}_i V + \sqrt{1-\tilde{\beta}_i^2}\,\overline{V}_i \ , \ i = 1,\ldots,n$$

Let us assume that $G(u) \le F(u), \forall u \in [0,1]$. This means that $\tilde{\rho}_1 \le \tilde{\beta}_1,\ldots,\tilde{\rho}_n \le \tilde{\beta}_n$ with respect to first order stochastic dominance. As a consequence, there exists non-negative random variables $v_1,\ldots,v_n$ independent from $V,\overline{V}_1,\ldots,\overline{V}_n$ such that: $\tilde{\beta}_1 = \tilde{\rho}_1 + v_1,\ldots,\tilde{\beta}_n = \tilde{\rho}_n + v_n$ [34], where the previous equalities hold in distribution. $(W_1,\ldots,W_n)\,|\,\tilde{\rho}_1,\ldots,\tilde{\rho}_n,v_1,\ldots,v_n$ and $(V_1,\ldots,V_n)\,|\,\tilde{\rho}_1,\ldots,\tilde{\rho}_n,v_1,\ldots,v_n$ are Gaussian with correlation parameter respectively equal to $\tilde{\beta}_1 = \tilde{\rho}_1 + v_1,\ldots,\tilde{\beta}_n = \tilde{\rho}_n + v_n$ and $\tilde{\rho}_1,\ldots,\tilde{\rho}_n$. This ensures that:

$$(V_1,\ldots,V_n)\,|\,\tilde{\rho}_1,\ldots\tilde{\rho}_n,v_1,\ldots,v_n \le_{\text{sm}} (W_1,\ldots,W_n)\,|\,\tilde{\rho}_1,\ldots\tilde{\rho}_n,v_1,\ldots,v_n \ ,$$

and eventually $(V_1,\ldots,V_n) \le_{\text{sm}} (W_1,\ldots,W_n)$. Ordering of stochastic correlation models is related to the first order stochastic dominance of the mixing correlation parameter.

5) Supermodular ordering and Marshall-Olkin copula.

Since the supermodular ordering is invariant under increasing transforms, we will consider the latent variables V_i. When $\alpha = 0$, these are independent and when $\alpha = 1$, there are comonotonic. We want to address the dependence of the vector of default times $(V_1,\ldots,V_n)$ with respect to α. Intuitively, increasing α gives more relative importance to the common shock V and should be associated with an increased dependence.

We set $\beta \ge \alpha$. We denote by $(V_1^{'},\ldots,V_n^{'})$ the latent variables associated with parameter β. In a distributional sense, we can equivalently write:

$$\left(V_1^{'},\ldots,V_n^{'}\right) \equiv \left(\min\left(V,\hat{V},\overline{V}_1^{'}\right),\ldots,\min\left(V,\hat{V},\overline{V}_n^{'}\right)\right),$$

Where $V,\hat{V},\overline{V}_1^{'},\ldots,\overline{V}_n^{'}$ are independent exponential random variables with parameters equal to: $\alpha,\beta-\alpha,1-\beta,\ldots,1-\beta$. Let us remark that $\left(\min\left(t,\hat{V},\overline{V}_1^{'}\right),\ldots,\min\left(t,\hat{V},\overline{V}_n^{'}\right)\right)$ and $\left(\min\left(t,\overline{V}_1\right),\ldots,\min\left(t,\overline{V}_n\right)\right)$ have the same marginal distributions for all *t*, since $\min\left(\hat{V},\overline{V}_i^{'}\right)$ are independent exponential random variables with parameter $1-\alpha$ and $\min\left(t,\hat{V},\overline{V}_i^{'}\right) = \min\left(t,\min\left(\hat{V},\overline{V}_i^{'}\right)\right)$. Morevover $\min\left(t,\hat{V},\overline{V}_i^{'}\right)$ is increasing in $\hat{V}$. Thus, this corresponds to model 3.2 in Bäuerle and Müller [1998]. We can then conclude that:

$$\left(\min\left(V,\overline{V}_1\right),\ldots,\min\left(V,\overline{V}_n\right)\right) \le_{sm} \left(\min\left(V,\hat{V},\overline{V}_1^{'}\right),\ldots,\min\left(V,\hat{V},\overline{V}_n^{'}\right)\right),$$

[34] We simply set $v_i = G^{-1}\left(F\left(\tilde{\rho}_i\right)\right) - \tilde{\rho}_i$, $i = 1,\ldots,n$.

which means that increasing the dependence parameter α does indeed lead to an increase in the dependence between default times with respect to the supermodular order.

6) Premium leg computation for a super senior tranche.

Let us first deal with the special case of a homogeneous portfolio: $E_1 = \cdots = E_n = \frac{1}{n}$, $\delta_1 = \cdots = \delta_n = \delta$. Then, the detachment point of the super-senior tranche equals $1 - \frac{\delta n}{1-\delta} L(t)$. However, in the general case, that detachment point is not collinear to the aggregate loss and thus the pricing methodology is slightly more involved than for other tranches. Let us thus deal with the heterogeneous case. The outstanding nominal of the super senior tranche is equal to $\left(\sum_{i=1}^{n} E_i \left(1 - \delta_i N_i(t)\right) - \max\left(K, L(t)\right) \right)^+$, where K is the attachment point of the tranche. Provided that $0 \le \delta_i \le 1$, $E_i \ge 0$, $\forall i \in \{1, \ldots, n\}$, it can readily be seen that:

$$\sum_{i=1}^{n} E_i \left(1 - \delta_i N_i(t)\right) \ge L(t) = \sum_{i=1}^{n} E_i \left(1 - \delta_i\right) N_i(t).$$

Given that, simple algebra allows to write the outstanding nominal as $\left(\sum_{i=1}^{n} E_i \left(1 - \delta_i N_i(t)\right) - K \right)^+ - \left(L(t) - K\right)^+$[35]. Given the conditional independence of the default indicators given the common factor V, the characteristic function of $\sum_{i=1}^{n} E_i \left(1 - \delta_i N_i(t)\right)$ can be easily derived along the same lines as the characteristic function of $L(t)$. One can then proceed through a Fourier inversion technique to compute the distribution of $\sum_{i=1}^{n} E_i \left(1 - \delta_i N_i(t)\right)$ as in Gregory and Laurent [2003] or some other inversion techniques as used for instance in finance by Carr and Madan [1999] and discussed in an insurance context by Dufresne, Garrido and Morales [2005] to directly compute $E\left[\left(\sum_{i=1}^{n} E_i \left(1 - \delta_i N_i(t)\right) - K \right)^+ \right]$. The well-known recursion techniques also apply quite well with small adaptation in this context.

7) Premiums computed under the Student *t* copula and under the independence assumption.

As noticed before, in the Student *t* copula, the case $\rho = 0$ is not associated with the independence case. From the stated results on stochastic orders, we just know that it acts as a lower bound on first to default swaps or equity tranche premiums. Comparing with the independence case is thus a bit more involved. Given that we have $\tau_i = F_i^{-1}\left(t_\nu \left(\bar{V}_i \sqrt{Z} \right) \right)$. We remark that $\tau_i \le t \Leftrightarrow \bar{V}_i \sqrt{Z} \le t_\nu^{-1}\left(F_i(t)\right) = z_i$, where F_i denotes the marginal distribution of τ_i. For notational simplicity we have omitted the dependence of z_i with respect to t. Let us remark that for practical purpose, in most cases $z_i < 0$, corresponds to default probabilities smaller than 0.5, which will further assumed for simplicity. We have $\sqrt{Z} \times \bar{V}_i \le z_i \Leftrightarrow \frac{-z_i}{\sqrt{Z}} + \bar{V}_i \le 0$. To conclude, we rely on Theorem 3.4 in Bäuerle and Müller [1998]. To make the connexion with their notations, we state $Z_i = U_i = \bar{V}_i$, $V = \sqrt{Z}$, $g_i\left(Z_i, W\right) = 1_{Z_i \le \Phi^{-1}\left(F_i(t)\right)}$ and

[35] One can notice that this simple decomposition of the outstanding nominal does not rely on the constancy of recovery rates.

$\tilde{g}_i(U_i,V,W)=1_{\frac{-z_i}{V}+U_i\le 0}$. g_i corresponds to the default indicator in the independence case while $\tilde{g}_i$ is the default indicator in the Student *t* copula case and $\rho=0$. A direct application of Theorem 3.4 in Bäuerle and Müller [1998] implies that the default indicators are greater, with respect to the supermodular order, in the Student *t* case than their independent counterparts.

8) Independence case and factor models.

Let us first introduce a rather general notion of factor models.

Definition: Let us consider a set of default times $(\tau_1,\ldots,\tau_n)$. These default times admit the *monotone unidimensional representation* if there exists a random variable V such that:

(i) The default times $\tau_1,\ldots,\tau_n$ are conditionally independent given V .

(ii) $Q(\tau_i > t|V)$ is non decreasing in V for all $i\in\{1,\ldots,n\}$ and all t .

The concept of *monotone unidimensional representation* has been widely used in various fields and is studied among others by Junker and Ellis [1997]. Let us remark that by considering $-V$ instead of V , we can replace "non decreasing" by "non increasing" in *(ii)*. It can readily be seen that the one factor Gaussian copula, the stochastic correlation model, the double *t* copula, the Clayton copula and the Marshall-Olkin copula studied in this paper all fall within that factor model framework. Since the Student *t* copula is associated with a two factor model, it does fall in the previous class.

Esary, Proschan and Walkup [1967] have introduced the following notion of positive dependence:

Definition: A random vector $(\tau_1,\ldots,\tau_n)$ is *positively associated*, if $Cov\big(f(\tau_1,\ldots,\tau_n),g(\tau_1,\ldots,\tau_n)\big)\ge 0$, for every pair of coordinatewise nondecreasing function f and g such that the above covariance exists.

It can be easily checked that when $\tau_1,\ldots,\tau_n$ are jointly independent, there are positively associated (see Esary, Proschan and Walkup [1967] for details). It can also be easily checked that if default times admit the *monotone unidimensional representation*, there are positively associated. A straightforward proof can be found in Rosenbaum [1984] or in Holland and Rosenbaum [1986]. They actually prove a stronger property named *conditional independence* which does not need to be detailed here.

Definition: A random vector $(\tau_1,\ldots,\tau_n)$ is *weakly positively associated*, if for every pair of disjoint subsets A_1,A_2 of $\{1,\ldots,n\}$,

$$Cov\big(f(\tau_i),i\in A_1,g(\tau_i),i\in A_2\big)\ge 0$$

for every pair of coordinatewise nondecreasing functions f,g such that the covariances are well-defined.

Definition: A random vector $(\tau_1,\ldots,\tau_n)$ is *weakly associated in sequence* if for all $t\in\mathbb{R}$, $1\le i\le n-1$ and non-decreasing function f , we have: $Cov\Big(1_{\{\tau_i>t\}},f\big(\tau_{(i+1)}\big)\Big)\ge 0$, where $\tau_{(i+1)}=(\tau_{i+1},\ldots,\tau_n)$.

Obviously, if $(\tau_1,\ldots,\tau_n)$ is associated, it is weakly associated and then it is weakly associated in sequence. As for the notion of weak association, we refer for example to Burton, Dabrowski and Dehling [1986], Christofides and Vaggelatou, [2004] and the book of Müller and Stoyan [2002]. The notion of weak association in sequence is used by Rüschendorf [2004].

Corollary 2.3 of Rüschendorf [2004] states that if $(\tau_1,\ldots,\tau_n)$ is weakly associated in sequence, then it has *positive supermodular dependence*. Positive supermodular dependence simply means that $(\tau_1,\ldots,\tau_n)$ is greater, with respect to the supermodular order than the random vector with same marginal distributions but with independent components. As a consequence, the independence case is associated with an upper bound on base tranches premiums in factor models that admit the monotone unidimensional representation.

In the case of an exchangeable sequence of default indicators, positive supermodular dependence (of the default indicators) is an almost direct consequence of Cousin and Laurent [2008a]. Using different techniques, Burton and Dabrowski [1992] also provide some results on the positive dependence of exchangeable sequences of default indicators. Regarding the latter point, additional results can also be found in Müller and Scarsini [2005]. Eventually, similar results on positive dependence can be proven in the case of the Student *t* copula, which is associated with a two factor model, using the literature on stochastic orders and factor models.

9) An arbitrage-free pricing of CDO tranches where no base correlation can be defined.

Let us consider the following counterexample involving three names with equal credit curves. We consider a Gaussian copula model such that the correlation between the first two names is equal to -100%. One could think of two competitors, only one could survive. Thus, $V_1 = V, V_2 = -V$. If we assume that marginal default probabilities $F_1(t), F_2(t)$ are less than 0.5, we can indeed check that only one of the first two names can default: $\tau_1 \le t \Leftrightarrow V \le \Phi^{-1}\left(F_1(t)\right) < 0$ and $\tau_2 \le t \Leftrightarrow -V \le \Phi^{-1}\left(F_2(t)\right) < 0$. This implies that $\{\tau_1 \le t\} \cap \{\tau_2 \le t\} = \varnothing$. The third name is uncorrelated with the first two names: $V_3 = \overline{V}_3$. The nominals are equal to 1 for the first two names and 0.5 for the third name. We assume zero recoveries. Let us consider a $[1.5-3]$ senior tranche. Since names 1 and 2 cannot default altogether, the maximal loss on the credit portfolio is equal to 1.5. Thus, the premium associated with the previous tranche is equal to zero. On the other hand, the lowest admissible flat correlation is -50%. For smaller values, the covariance matrix would not be semi-definite positive. Thanks to the previous ordering results on Gaussian vectors, such a correlation structure leads to the lowest senior tranche premium consistent with a flat correlation matrix. Let us remark that there is a positive probability that names 1 and 2 default altogether leading to a loss of at least 0.5 on the $[1.5-3]$ tranche. As a consequence, the senior tranche premium is positive for any base correlation. Since the arbitrage free premium of the senior tranche is equal to zero, it is not possible to find a base correlation (even allowing for negative base correlations) that matches this premium. Of course, this case is rather unlikely, but it shows that base correlation cannot be assimilated to implied volatility which is always defined.

10) Large portfolio approximations.

Let us denote by $L = Z_1 + \cdots + Z_n$. By linearity of conditional expectations, we have to show that $E\left[L|V\right] \le_{cx} L$ where V is some random vector. Thanks to conditional Jensen inequality, given any convex function f, $f\left(E\left[L|V\right]\right) \le E\left[f(L)|V\right]$, where the previous inequality holds almost surely. Taking expectations on both sides and using the law of iterated expectations, we get $E\left[f\left(E\left[L|V\right]\right)\right] \le E\left[f(L)\right]$, which means that $E\left[L|V\right] \le_{cx} L$. The intuition behind this result is straightforward. By projecting the loss L, we reduce risk. In the case, where the default times are independent given V, we wipe off idiosyncratic risk from the portfolio and only keep factor risk, which leads to a reduction in the riskiness of the portfolio.

11) Large homogeneous portfolio approximation.

If we assume that the $Z_i = M_i 1_{\{\tau_i \le t\}}$'s are independent conditionally upon V and identically distributed, we have: $\frac{1}{n}\sum_{i=1}^{n} Z_i \underset{Q-a.s.}{\rightarrow} E\left[Z_i \middle| V\right]$ as $n \to \infty$. A proof of this result can be found in Chabaane *et al*. [2004]. Convergence in mean can also easily be proved. If $M_i = 1-\delta$, where δ stands for the common recovery rate and $F(t) = Q(\tau_i \le t)$ is the common marginal default probability, then the limit writes $(1-\delta)Q(\tau_i \le t | V)$. The right-hand term is known as the large homogeneous portfolio approximation, where the portfolio notional is equal to one. Thus, for the one factor Gaussian copula case, the large homogeneous portfolio approximation is provided by $(1-\delta)\Phi\left(\frac{-\rho V + \Phi^{-1}(F(t))}{\sqrt{1-\rho^2}}\right)$, which obviously coincides with the other approximation in the homogeneous case. Comparisons between these two approximations can be achieved but this is out of the scope of the paper.

12) Monotonicity results for large portfolios.

Using Cousin and Laurent [2008a], we have:

$$0 \le \rho \le \rho^* \Rightarrow \tilde{p}_i = \Phi\left(\frac{-\rho V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho^2}}\right) \le_{cx} \tilde{p}_i^* = \Phi\left(\frac{-\rho^* V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho^{*2}}}\right),\ i = 1,\ldots,n.$$

Then, as a consequence of Ky Fan – Lorentz theorem, we have: $0 \le \rho \le \rho^* \Rightarrow (\tilde{p}_1,\ldots,\tilde{p}_n) \le_{dcx} (\tilde{p}_1^*,\ldots,\tilde{p}_n^*)$ where $\le_{dcx}$ holds for the directional convex order (see Müller and Stoyan [2002] or Rüschendorf [2004] for details). Now for positive losses given default, $M_i \ge 0, i = 1,\ldots,n$, we have $LH_\rho(t) = \sum_{i=1}^{n} M_i \tilde{p}_i \le_{cx} LH_{\rho^*}(t) = \sum_{i=1}^{n} M_i \tilde{p}_i^*$ since for any convex function $g:\mathbb{R} \to \mathbb{R}$, $f(x_1,\ldots,x_n) = g(M_1 x_1 + \cdots + M_n x_n)$ is directionally convex.

While the previous proof is straightforward, it only holds for the one factor case. It thus readily extends to the stochastic correlation, double *t*, Clayton and Marshall-Olkin cases, but not for the Student *t* case which is associated with a two factor model. For this purpose, one can use the following approach:

For $M \in \mathbb{N}$, define $V_i^m = \rho V + \sqrt{1-\rho^2}\bar{V}_i^m$, where $V, \bar{V}_i^m, m = 1,\ldots,M; i = 1,\ldots,n$ are independent standard Gaussian variables. Then, we define default times by $\tau_i^m = F_i^{-1}(\Phi(V_i^m))$ and $L_\rho^M(t) = \sum_{i=1}^{n} M_i \left(\frac{1}{M}\sum_{m=1}^{M} 1_{\{\tau_i^m \le t\}}\right)$. Let us remark that thanks to the previous stochastic ordering results, $0 \le \rho \le \rho^* \Rightarrow L_\rho^M(t) \le_{sl} L_{\rho^*}^M(t)$ for any $M \in \mathbb{N}$. As a consequence of de Finetti's theorem, $L_\rho^M(t) \underset{Q-a.s.}{\longrightarrow} LH_\rho(t)$ as $M \to \infty$. Since the aggregate loss is bounded by the portfolio nominal, using dominated convergence theorem, we conclude that $0 \le \rho \le \rho^* \Rightarrow LH_\rho(t) \le_{sl} LH_{\rho^*}(t)$ where $\le_{sl}$ refers to the stop-loss order.