Choice of copula and pricing of credit derivatives

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Choice of copula and pricing of credit derivatives

- Basket default swaps and CDO tranches
- Factors and conditional independence framework
- Model dependence for credit derivatives premiums
- Model dependence and sensitivity analysis

Basket default swaps and CDO tranches

- $i = 1, \ldots, n$ names.
- $\tau_1, \ldots, \tau_n$ default times.
- N_i nominal of credit *i*,
- δ_i recovery rate (between 0 and 1)
  - $N_i(1 - \delta_i)$ loss given default (of name i)
  - if $N_i(1 - \delta_i)$ does not depend on *i*: <u>homogeneous</u> case
  - otherwise, <u>heterogeneous</u> case.

Basket default swaps and CDO tranches

- Credit default swap (CDS) on name *i*:
- Default leg:
  - *payment of* $N_i(1-\delta_i)$ *at* τ_i *if* $\tau_i \leqslant T$
  - *Where T is the maturity of the CDS*
- Premium leg:
- constant periodic premium paid until $\min(\tau_i, T)$
  - *CDS premiums depend on maturity T*
  - *Liquid markets: CDS premiums, inputs of pricing models*

Basket default swaps and CDO tranches

- First to default swap:
- Default leg: payment of $N_i(1-\delta_i)$ at:

$$\tau^1 = \min(\tau_1, \ldots, \tau_n)$$

  - Where *i* is the name in default
  - If $\tau^1 \leqslant T$ maturity of First to default swap
- Premium leg:
  - *constant periodic premium until* $\min(\tau^1, T)$
  - *Remark: payment in case of simultaneous defaults ?*

Basket default swaps and CDO tranches

- General Basket default swaps
- $\tau^1, \ldots, \tau^n$ ordered default times
- *k*-th to default swap default leg:
  - Payment of $N_i(1-\delta_i)$ at τ^k
  - where *i* is the name in default,
  - If $\tau^k \leqslant T$ maturity of *k*-th to default swap
- Premium leg:
  - *constant periodic premium until* $\min(\tau^k, T)$

Basket default swaps and CDO tranches

- Payments are based on the accumulated losses on the pool of credits
- Accumulated loss at *t*:

$$L(t) = \sum_{1 \leq i \leq n} N_i(1-\delta_i)N_i(t)$$

  - where $N_i(t) = 1_{\tau_i \leq t}$, $N_i(1-\delta_i)$ loss given default.
  - $L(t)$ pure jump process

Basket default swaps and CDO tranches

- Tranches with thresholds $0 \leq A \leq B \leq \sum N_j$
  - *Mezzanine: losses are between A and B*
- Cumulated payments at time *t on mezzanine tranche*

$$M(t) = (L(t) - A))\, 1_{[A,B]}(L(t)) + (B - A) 1_{]B,\infty[}(L(t))$$

- *Payments on default leg:*

$\Delta M(t) = M(t) - M(t^-)$ at time $t \leq T$

- *Payments on premium leg:*
  - periodic premium,
  - proportional to outstanding nominal: $B - A - M(t)$

Factors and conditional independence

- Payoffs depend on default times and recovery rates
- Pricing rule : some « risk-neutral » probability Q
- From now on, recovery rates are independent variables
  - *More on recovery rates and default dates:*
  - *Double impact, credit risk assessment and collateral value*
- Default dates may be dependent
- Marginal distribution function: $F_i(t) = Q(\tau_i \leq t)$
- Marginal survival function: $S_i(t) = Q(\tau_i > t)$

Factors and conditional independence

- Joint survival function:

$$S(t_1, \ldots, t_n) = Q(\tau_1 > t_1, \ldots, \tau_n > t_n)$$

  - *Needs to be specified given marginals.*
  - $S_i(t) = Q(\tau_i > t)$ *given from CDS quotes.*

- (Survival) Copula of default times:

$$C(S_1(t_1), \ldots, S_n(t_n)) = S(t_1, \ldots, t_n)$$

  - *C characterizes the dependence between default times.*

Factors and conditional independence

- Factor approaches to joint distributions:
  - *V: low dimensional factor, not observed « latent factor ».*
  - *Conditionally on V, default times are independent.*
  - *Conditional default probabilities:*

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right), \quad q_t^{i|V} = Q\left(\tau_i > t \mid V\right).$$

  - *Conditional joint distribution:*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid V) = \prod_{1 \leq i \leq n} p_{t_i}^{i|V}$$

  - *Joint survival function (implies integration wrt V):*

$$Q(\tau_1 > t_1, \ldots, \tau_n > t_n) = E\left[\prod_{i=1}^{n} q_{t_i}^{i|V}\right]$$

Factors and conditional independence

- One factor Gaussian copula:
  - $V, \bar{V}_i, \ i = 1, \ldots, n$ *independent Gaussian,*

$$V_i = \rho_i V + \sqrt{1 - \rho_i^2}\bar{V}_i$$

  - *Default times:* $\tau_i = F_i^{-1}(\Phi(V_i))$
  - *Conditional default probabilities:* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \rho_i^2}}\right)$
- Joint survival function:

$$S(t_1, \ldots, t_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\rho_i v - \Phi^{-1}(F_i(t_i))}{\sqrt{1 - \rho_i^2}} \right) \right) \varphi(v) dv$$

- Can be extended to Student *t* copulas (two factors).

Factors and conditional independence

- Gaussian copula
  - *No tail dependence (if* $|\rho|<1$*)*
  - *Upper tail dependence*

$$\lim_{u\to 1} Q\left(\tau_i > F_i^{-1}(u)\,\middle|\,\tau_j > F_j^{-1}(u)\right) = \lim_{u\to 1}\frac{C(u,u)+1-2u}{1-u}$$

- Kendall's tau $\rho_K = \frac{2}{\pi}\arcsin\rho$

$$\rho_K = 4\iint_{[0,1]^2} C_\rho(u,v)\,dC_\rho(u,v) - 1$$

- Spearman rho $\rho_S = \frac{6}{\pi}\arcsin(\rho/2)$

$$\rho_S = 12\iint_{[0,1]^2} uv\,dC_\rho(u,v) - 3 = 12\iint_{[0,1]^2} C_\rho(u,v)\,du\,dv - 3$$

Factors and conditional independence

- Concordance ordering

$$\rho \leqslant \rho' \Rightarrow C_\rho(u_1, \ldots, u_n) \leqslant C_{\rho'}(u_1, \ldots, u_n)$$

- $\rho = 0$ independence case
  - $C(u_1, \ldots, u_n) = u_1 \times \ldots \times u_n$
  - *Product copula*
- $\rho = 1$ comonotonic case
  - $C(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n)$
  - *Upper Fréchet bound*

Factors and conditional independence

- *Clayton* copula (Schönbucher & Schubert)
- Conditional default probabilities

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

  - *V: Gamma distribution with parameter* θ
  - *Frailty model: multiplicative effect on default intensity*
- Joint survival function:

$$S(t_1, \ldots, t_n) = \int \prod_{i=1}^{n} \left(1 - p_{t_i}^{i|V}\right) \frac{1}{\Gamma(1/\theta)} e^{-V} V^{(1-\theta)/\theta} dV$$

  - *Copula:*

$$C(u_1, \ldots, u_n) = \left(u_1^{-\theta} + \ldots + u_n^{-\theta} - n + 1\right)^{-1/\theta}$$

Factors and conditional independence

- Clayton copula:
  - *Archimedean copula*
  - *lower tail dependence:* $\lambda_L = 2^{-1/\theta}$
  - *no upper tail dependence*

- Kendall tau $\rho_K = \dfrac{\theta}{\theta + 2}$
  - *Spearman rho has to be computed numerically*
- C_θ increasing with θ
- $\theta = 0$ independence case
- $\theta = +\infty$ comonotonic case

Factors and conditional independence

- Shock models (Duffie & Singleton, Wong)
- Modelling of defaut dates: $\tau_i = \min(\bar{\tau}_i, \tau)$
  - $Q(\tau_i = \tau_j) \geq Q\left(\tau \leq \min(\bar{\tau}_i, \bar{\tau}_j)\right) > 0$ *simultaneous defaults.*
  - *Conditionally on* τ, τ_i *are independent.*

$$Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid \tau) = \prod_{1 \leq i \leq n} Q(\tau_i \leq t_i \mid \tau)$$

- Conditional default probabilities:

$$p_t^{i|\tau} = 1_{\tau > t} Q(\bar{\tau}_i \leq t) + 1_{\tau \leq t}$$

Factors and conditional independence

- Shock models
- $\tau, \bar{\tau}_i$ exponential distributions with parameters $\lambda, \bar{\lambda}_i$
- Survival copula $\hat{C}$. $\alpha_i = \lambda/(\lambda + \bar{\lambda}_i)$
- *Marshall Olkin* copula

$$\hat{C}(u_i, u_j) = \min(u_i^{1-\alpha_i} u_j, u_i u_j^{1-\alpha_j})$$

- Tail dependence $\min(\alpha_i, \alpha_j)$
- Kendall tau: $\rho_K^{i,j} = \dfrac{\alpha_i \alpha_j}{\alpha_i + \alpha_j - \alpha_i \alpha_j}$
- Spearman rho $\rho_S^{i,j} = \dfrac{3\alpha_i \alpha_j}{2\alpha_i + 2\alpha_j - \alpha_i \alpha_j}$

Factors and conditional independence

- Marshall-Olkin copula (shock models)
- Symmetric case: $\alpha_i = \alpha$
- $\alpha = 0$ independence case
- $\alpha = 1$ comonotonic case
- Marshall-Olkin copula increasing with α

Factors and conditional independence

- AJD: Duffie, Pan & Singleton ;Duffie & Garleanu.
  - $n+1$ *independent affine jump diffusion processes:*

$$X_1, \ldots, X_n, X_c$$

- Conditional default probabilities:

$$Q(\tau_i > t \mid V) = q_t^{i|V} = V\alpha_i(t)$$

$$V = \exp\left(-\int_0^t X_c(s)ds\right), \quad \alpha_i(t) = E\left[\exp\left(-\int_0^t X_i(s)ds\right)\right].$$

- Survival function:

$$Q(\tau_1 > t, \ldots, \tau_n > t) = E\left[V^n\right] \times \prod_{i=1}^{n} \alpha_i(t).$$

  - *Explicitely known.*

Factors and conditional independence

- Why factor models ?
  - *Standard approach in finance and statistics*
  - *Tackle with large dimensions*
- We need tractable dependence between defaults:
  - *Parsimonious modelling*
    - One factor Gaussian copula: n parameters
  - *Semi-explicit computations for portfolio credit derivatives*
    - Premiums
    - Greeks
- Exchangability leads to one factor models
  - *De Finetti*

Model dependence for credit derivatives premiums

- First to default time $\tau^1 = \min(\tau_1, \ldots, \tau_n)$
- First to default swap:
  - *Credit protection at first to default time*
- Survival function of first to default time

$$Q(\tau^1 > t) = Q(\tau_1 > t, \ldots, \tau_n > t) = E\left[\prod_{i=1}^{n} q_t^{i|V}\right]$$

- Semi-analytical expressions of:
  - *First to default, second to default, … last to default swap premiums*

Model dependence for credit derivatives premiums

- Example: first to default swap
  - *Default leg* $\int_0^T \sum_{i=1}^{n} M_i B(t) E\left[\prod_{j\neq i}\left(1-p_t^{j|V}\right)\frac{dp_t^{i|V}}{dt}\right]dt$
  - *One factor Gaussian* $p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$
  - *Clayton* $p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$
  - *Marshall Olkin* $p_t^{i|\tau} = 1_{\tau>t} Q(\bar{\tau}_i \leq t) + 1_{\tau \leq t}$
    - « basket defaults swaps, CDO's and Factor Copulas » available on www.defaultrisk.com
    - « I will survive », RISK magazine, june 2003

Model dependence for credit derivatives premiums

- First to default swap premium vs number of names
  - *From n=1 to n=50 names*
  - *Unit nominal*
  - *Credit spreads = 80 bp*
  - *Recovery rates = 40 %*
  - *Maturity = 5 years*
  - *Basket premiums in bp*
- Comparison between Gaussian, Clayton and Marshall-Olkin copulas:
  - *Gaussian correlation parameter= 30%*

names	Gaussian	Clayton	MO
1	80	80	80
5	332	336	244
10	567	574	448
15	756	762	652
20	917	921	856
25	1060	1060	1060
30	1189	1183	1264
35	1307	1294	1468
40	1417	1397	1672
45	1521	1492	1875
50	1618	1580	2079
kendall	19%	8%	33%

Model dependence for credit derivatives premiums

- From first to last to default swap premiums
  - *10 names, unit nominal*
  - *Spreads of names uniformly distributed between 60 and 150 bp*
  - *Recovery rate = 40%*
  - *Maturity = 5 years*
  - *Gaussian correlation: 30%*
- Same FTD premiums imply consistent prices for protection at all ranks
- Model with simultaneous defaults provides very different results

Rank	Clayton	Gaussian	MO
1	723	723	723
2	277	274	160
3	122	123	53
4	55	56	37
5	24	25	36
6	10	11	36
7	3.6	4.3	36
8	1.2	1.5	36
9	0.28	0.39	36
10	0.04	0.06	36
kendall	9%	19%	NS

Model dependence for credit derivatives premiums

- *CDO tranche premiums*
- *Use of loss distributions over different time horizons*
- *Computation of loss distributions from FFT*
- *Explicit margin computations for tranches*

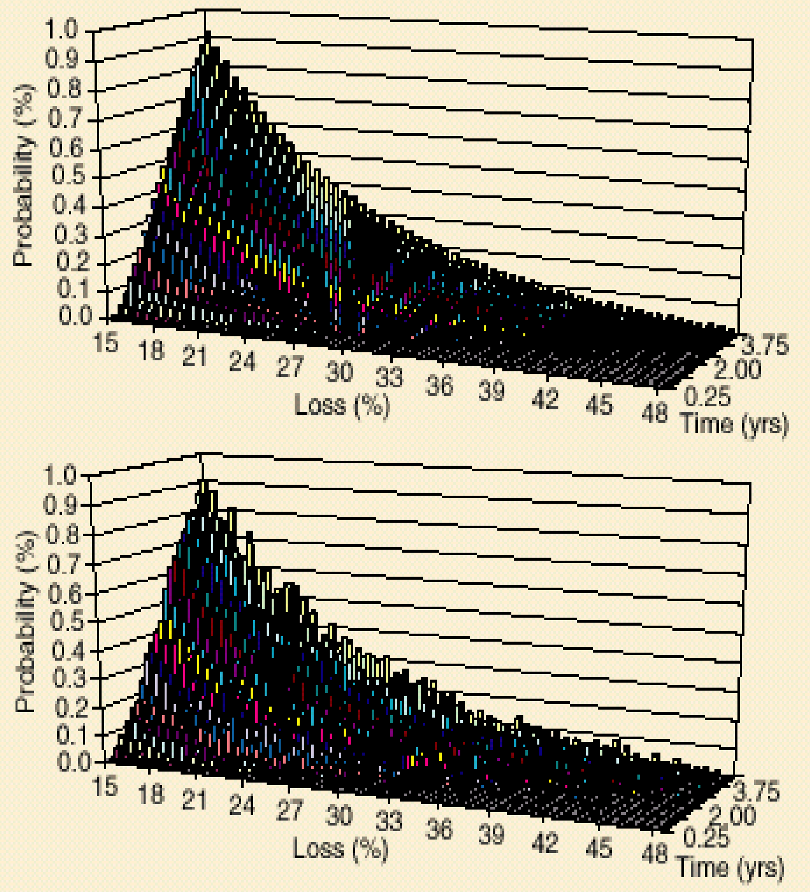
Model dependence for credit derivatives premiums

- Accumulated loss at *t*: $L(t) = \sum_{1 \leq i \leq n} N_i(1-\delta_i) N_i(t)$
  - Where $N_i(t) = 1_{\tau_i \leq t}$, $N_i(1-\delta_i)$ loss given default.
- Characteristic function: $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
- By conditioning: $\varphi_{L(t)}(u) = E\left[\prod_{1 \leq j \leq n} \left(1 - p_t^{j|V} + p_t^{j|V} \varphi_{1-\delta_j}(uN_j)\right)\right]$
- Distribution of $L(t)$ is obtained by FFT.

Model dependence for credit derivatives premiums

- One hundred names, same nominal.
- Recovery rates: 40%
- Credit spreads uniformly distributed between 60 and 250 bp.
- Gaussian copula, correlation: 50%
- 10^5 Monte Carlo simulations

3. Loss distribution



Loss distribution over time for the table B example with 50% correlation for the semi-explicit approach (top) and Monte Carlo simulation (bottom)

Model dependence for credit derivatives premiums

- Mezzanine: pays whenever losses are between *A* and *B*
- Cumulated payments at time *t* on mezzanine tranche

$$M(t) = (L(t) - A))\, 1_{[A,B]}(L(t)) + (B - A) 1_{]B,\infty[}(L(t))$$

- Explicit margin computations of different tranches
  - *Upfront premium:* $E\left[\int_0^T B(t) dM(t)\right]$
    - *B(t) discount factor, T maturity of CDO*
  - *Integration by parts* $B(T)E[M(T)] + \int_0^T E[M(t)]dB(t)$
  - *where* $E[M(t)] = (B - A)Q(L(t) > B) + \int_A^B (x - A) dF_{L(t)}(x)$

Model dependence for credit derivatives premiums

B. Pricing of five-year maturity CDO tranches

	Equity (0–3%)		Mezzanine (3–14%)		Senior (14–100%)	
	SE	MC	SE	MC	SE	MC
0%	8,219.4	8,228.5	816.2	814.3	0.0	0.0
20%	4,321.1	4,325.3	809.4	806.9	13.7	13.7
40%	2,698.8	2,696.7	734.3	731.4	33.4	33.2
60%	1,750.6	1,738.5	641.0	637.8	54.1	53.7
80%	1,077.5	1,067.9	529.5	526.9	77.0	76.6
100%	410.3	406.6	371.2	367.0	110.4	109.6

Premiums in basis points per annum as a function of correlation for 5-year maturity CDO tranches on a portfolio with credit spreads uniformly distributed between 60 and 250bp. The recovery rates are 40%

- *One factor Gaussian copula*
- *CDO tranches margins with respect to correlation parameter*

Model dependence for credit derivatives premiums

- CDO margins (bp)
  - *Gaussian copula*
  - *Attachement points: 3%, 10%*
  - *100 names*
  - *Unit nominal*
  - *Credit spreads uniformaly distributed between 60 and 150 bp*
  - *5 years maturity*

ρ	equity	mezzanine	senior
0 %	6176	694	0.05
10 %	4046	758	5.8
30 %	2303	698	23
50 %	1489	583	40
70 %	933	470	56

Model dependence for credit derivatives premiums

- CDO margins (bp)
  - *Gaussian correlation = 10%*
  - *Parameters of Clayton and Marshall Olkin copulas are set for matching of equity tranches.*
- For the pricing of CDO tranches, the Clayton and Gaussian copula models are close.
- Very different results with Marshall-Olkin copula

	Gaussian	Clayton	MO
equity	4060	4060	4060
mezzanine	786	785	314
senior	6	5	30
kendall	6%	3%	not constant

Model dependence for credit derivatives premiums

- *Credit spreads uniformly distributed between 80bp and 120bp*
- *100 names*

Equity tranche

ρ	0%	10%	30%	50%	70%
θ	0	0.054	0.196	0.406	0.748
λ	0	0.0052	0.0096	0.0119	0.0137
premium (bp)	6039	3854	2158	1367	862

Model dependence for credit derivatives premiums

Mezzanine tranche

ρ	0%	10%	30%	50%	70%
Gaussian	644	725	669	560	443
Clayton	644	724	663	556	448
MO	644	282	139	127	136

Senior tranche

ρ	0%	10%	30%	50%	70%
Gaussian	0	5	22	38	54
Clayton	0	4	21	37	53
MO	0	28	52	64	74

Model dependence and sensitivity analysis

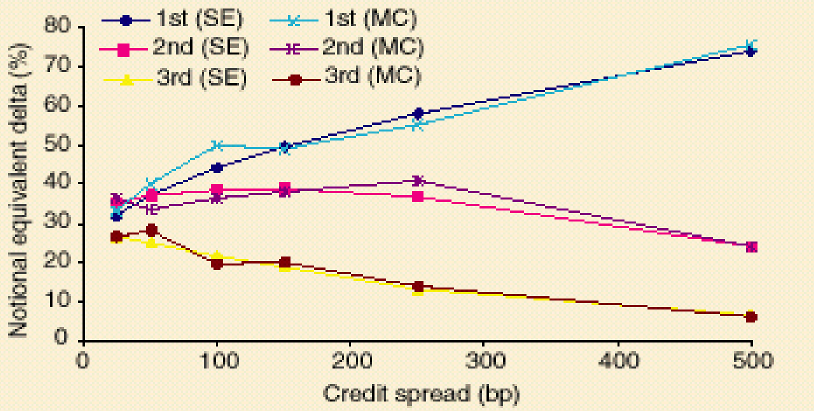
- Example: six names portfolio
- Changes in credit curves of individual names
- Amount of individual CDS to hedge the basket
- Semi-analytical more accurate than 10^5 Monte Carlo simulations.
- Much quicker: about 25 Monte Carlo simulations.

A. Comparison of the semi-explicit formulas with Monte Carlo simulations

	First to default		Second to default		Third to default	
	SE	MC	SE	MC	SE	MC
0%	1,075.1	1,075.9	214.8	214.7	28.2	27.7
20%	927.0	925.9	247.2	247.5	61.4	61.8
30%	859.9	857.9	256.8	257.6	77.6	78.0
40%	796.6	795.2	263.3	264.2	92.7	93.0
60%	679.6	678.0	268.8	268.9	119.5	119.8
80%	573.1	571.7	266.2	266.1	141.0	140.9
100%	500.0	500.0	250.0	250.0	150.0	150.0

Premiums in basis points per annum as a function of correlation for a five-year maturity basket with credit spreads of 25, 50, 100, 150, 250 and 500bp and equal recovery rates of 40%

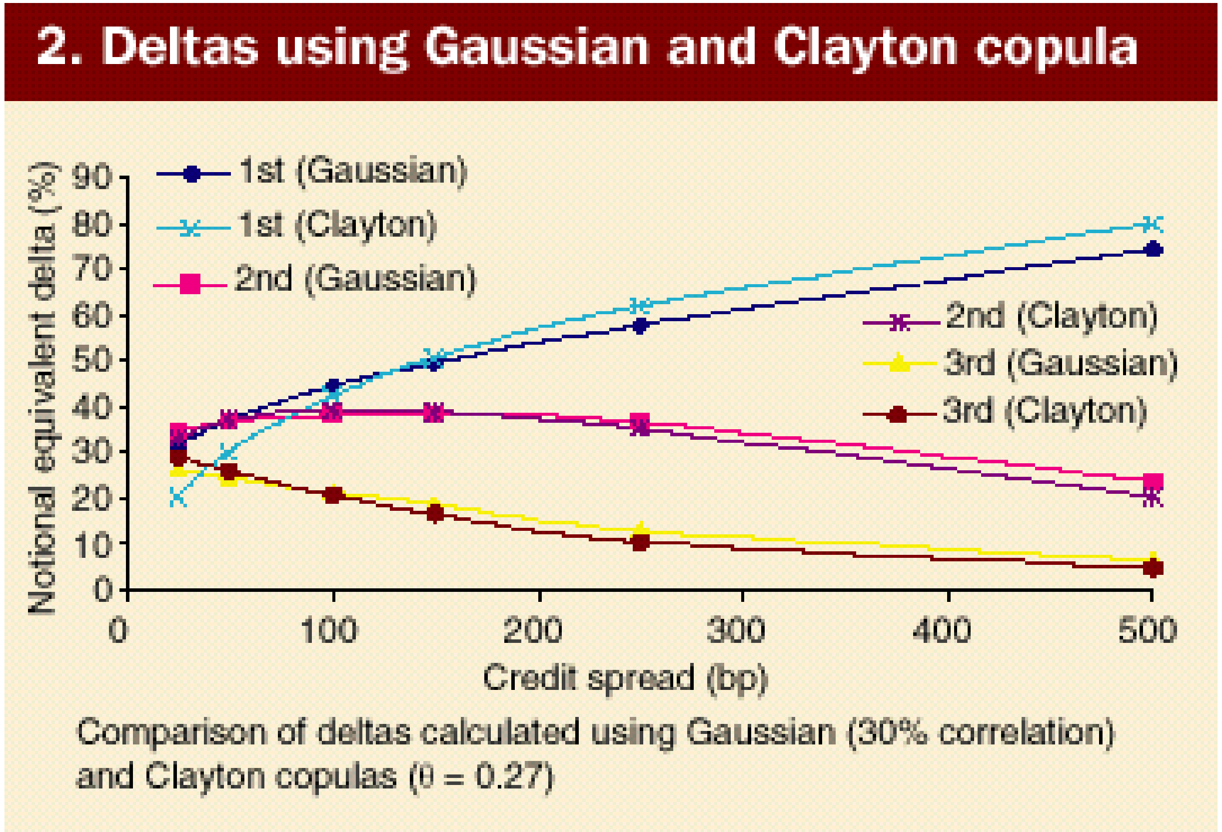
1. Deltas calculated using semi-explicit formulas and Monte Carlo approaches



Comparison of deltas calculated using the analytical formulas and 105 Monte Carlo simulations for the example given in table A. The Monte Carlo deltas are calculated by applying a 10bp parallel shift to each curve

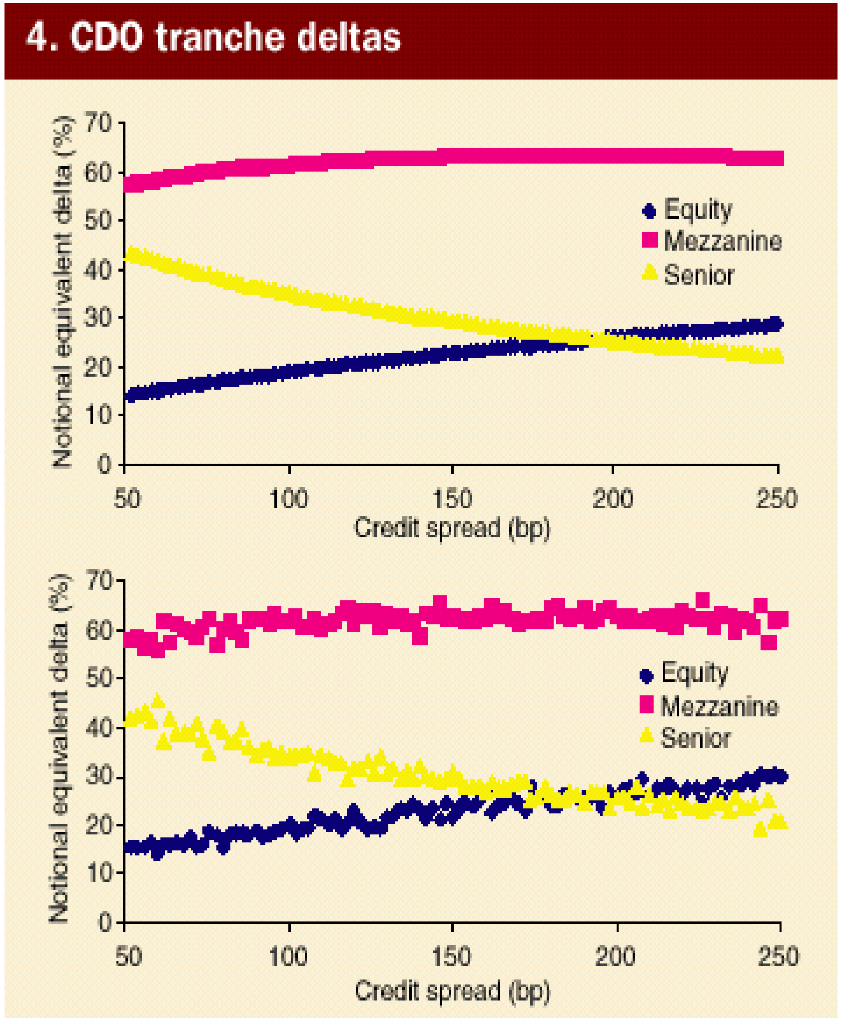
Model dependence and sensitivity analysis

- Changes in credit curves of individual names
  - *Dependence upon the choice of copula for defaults*



Model dependence and sensitivity analysis

- Hedging of CDO tranches with respect to credit curves of individual names
- Amount of individual CDS to hedge the CDO tranche
- Semi-analytic : some seconds
- Monte Carlo more than one hour and still shaky



CDO tranche deltas using the analytical method (top) and Monte Carlo (bottom) for a correlation of 50%

Conclusion

- Factor models of default times:
  - *Deal easily with a large range of names and dependence structures*
  - *Simple computation of basket credit derivatives and CDO's*
    - Prices and risk parameters
- Gaussian and Clayton copulas provide very similar patterns
  - *Rank correlation and tail dependence not meaningful*
  - *Student t needs to be investigated*
- Shock models quite different