



An overview of the valuation of collateralized derivatives contracts



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29 May 2013



An overview of the valuation of collateralized derivatives contracts

- Overview of the presentation
 - *Financial context*
 - *Variation margins paid on a collateral account*
 - *Settlement prices and collateralization schemes*
 - *Pricing equation : perfect collateralization*
 - OIS discounting
 - Futures pricing
 - Costless collateral
 - Posting bonds
 - *Overcollateralization, haircuts, run on repos*
 - *Pricing equation for unilateral collateral agreement*
 - CVA and funding cost adjustments
 - Trade contribution

An overview of the valuation of collateralized derivatives contracts

- Due to new regulations and counterparty risk management, a large amount of derivatives contracts is collateralized
- OTC transactions
 - *Swaps ISDA (CSA: Credit Support Annex)*
 - *2012 Margin Survey report 137,869 active collateral agreements*
- Central clearing
 - *LCH Clearnet, ICE, Eurex,*
 - *Futures exchanges (CME, ...)*
 - *SEF (Swap Exchange Facilities)*
- Need of a unifying pricing framework
 - *Variation margins, initial margins,*
 - *Bilateral, unilateral agreements*
 - *Collateral type: Cash, bonds, currencies*
 - *We remain in the standard mathematical finance approach*

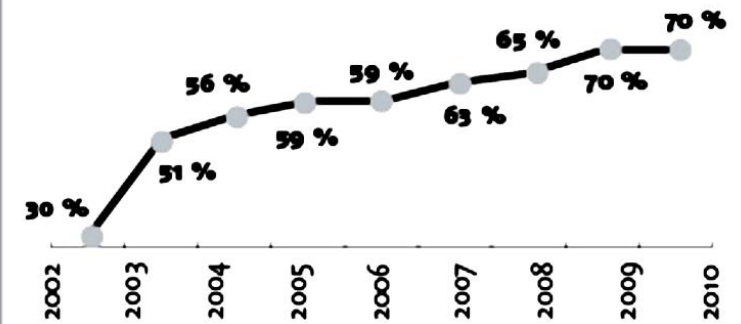


An overview of the valuation of collateralized derivatives contracts

- Most ISDA trades are associated with collateral agreements (right figure)
- Swapclear (LCH.Clearnet) prominent CCP for IRS outstanding notional US \$ 380 trillion (figure below)



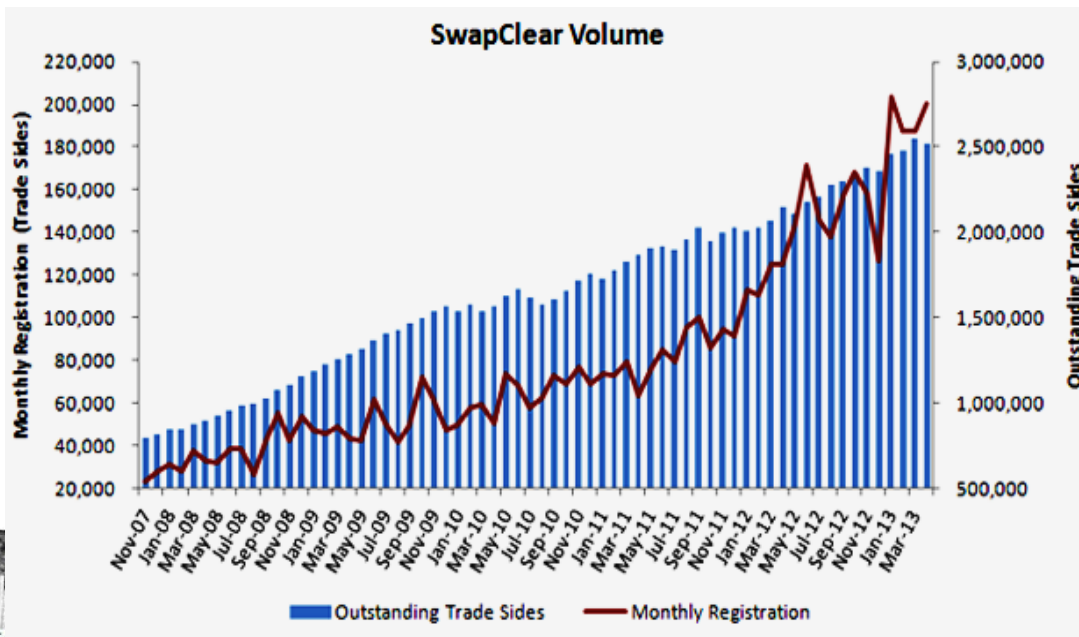
1. Part du volume des dérivés OTC échangés ayant fait l'objet d'un échange de collatéral



Par type de dérivés en 2011

Swap de taux	79 %
CDS	93 %
Swap de changes	58 %
Dérivés sur actions	72 %
Dérivés sur matières premières	60 %

Source : ISDA.



An overview of the valuation of collateralized derivatives contracts

- Variation margins paid on a **collateral account**
 - $V(s)$ amount on collateral account in a bilateral agreement
 - Collateral accounts recorded by ISDA evaluated at US \$ 3.6 trillion end of year 2011
 - $A(s)$ price of posted security
 - Approximately 80% cash, 20% government bonds
 - $V(s)/A(s)$ number of posted securities
- $dVM(s)$ **Variation margin**: net inflow on collateral account
 - $dVM(s) = V(s + ds) - \frac{V(s)}{A(s)} A(s + ds)$
 - Equivalently written as: $dVM(s) = dV(s) - V(s) \frac{dA(s)}{A(s)}$
 - $dA(s)/A(s)$ realized return on collateral

An overview of the valuation of collateralized derivatives contracts

- The dynamics of collateral account $V(s)$ can equivalently be written as:

- $$dV(s) = V(s) \times \frac{dA(s)}{A(s)} + dVM(s)$$

- $V(s) \times \frac{dA(s)}{A(s)}$ *self-financed part*

- $dVM(s)$ *inflow in collateral account*

- Case of cash-collateral $A \equiv C$

- $dC(s) = c(s)C(s)ds$

- $c(s)$: *EONIA, fed fund rate, ...*

- $dV(s) = c(s)V(s)ds + dVM(s)$

An overview of the valuation of collateralized derivatives contracts

- Collateralization schemes
 - $h(t)$ settlement price (credit risk exposure)
 - $V(t) = h(t)$ Bilateral collateral agreement
 - Interdealer contracts
 - Unilateral collateral agreement $V(t) = \min(h(t), 0)$
 - Unilateral: sovereign entities

Unilateral			Bilateral			Total Active
ISDA collateral agreements	Non-ISDA agreements	Total number Unilateral	ISDA collateral agreements	Non-ISDA agreements	Total number Bilateral	
14,212	8,001	22,213	103,398	12,258	115,656	137,869
10.3%	5.9%	16.1%	74.9%	9.0%	83.9%	100%

- $V(s) = \alpha_A \times h(s)$, $\alpha_A > 1$ overcollateralization
 - $1/\alpha_A$ haircut ratio

An overview of the valuation of collateralized derivatives contracts

- The basic pricing equation (bilateral, no haircut)
 - Does not account for possible price impact of initial margin
 - Bilateral case, no haircut $V(t) = h(t)$
 - Enter a collateralized derivatives contract at t
 - Buy collateralized security: outflow $h(t)$
 - Receive collateral to secure credit exposure : inflow $V(t)$
 - Since $V(t) = h(t)$ **net cash-flow at trade inception $t = 0$**
 - Exit trade at $t + dt$ after paying variation margin $dVM(t)$
 - $dVM(t) = dV(t) - V(t) \frac{dA(t)}{A(t)} = dh(t) - h(t) \frac{dA(t)}{A(t)}$
 - Since $V(t + dt) = h(t + dt)$ no extra cash-flow at $t + dt$
 - Q^β usual risk-neutral pricing measure, assumed to be given
- $E_t^{Q^\beta} [dVM(t)] = 0 \Rightarrow E_t^{Q^\beta} \left[\frac{dh(t)}{h(t)} \right] = E_t^{Q^\beta} \left[\frac{dA(t)}{A(t)} \right]$

An overview of the valuation of collateralized derivatives contracts

- $E_t^{Q^\beta} \left[\frac{dh(t)}{h(t)} \right] = E_t^{Q^\beta} \left[\frac{dA(t)}{A(t)} \right]$
- $E_t^{Q^\beta} \left[\frac{dA(t)}{A(t)} \right] = r_A(t)dt$ $r_A(t)$: expected rate of return on collateral
- $\Rightarrow E_t^{Q^\beta} \left[\frac{dh(t)}{h(t)} \right] = r_A(t)dt$ $h(t)$ settlement price
- Plus terminal condition: $h(T)$ payment at maturity date T
- $h(t) = E_t^{Q^\beta} \left[h(T) \exp \left(- \int_t^T r_A(s) ds \right) \right]$
 - Discount with expected rate of return on collateral
 - $t \rightarrow T$: liquidity assumptions
 - Assumes collateral is available from t to T
 - No haircut on collateral being introduced after t
 - No gap risks: possible exit at settlement and collateral price

An overview of the valuation of collateralized derivatives contracts

- Cash-collateral
 - $dC(t) = c(t)C(t)dt$, $c(t)$ EONIA, effective fed fund rate
 - $r_A(t) = c(t)$
 - $h(t) = E_t^{Q^\beta} \left[h(T) \exp \left(- \int_t^T c(s) ds \right) \right]$ “OIS discounting”
- $c(t) = 0$, futures market (Duffie, 1989)
 - $dVM(t) = dh(t) - c(t)h(t)dt = dh(t)$
 - variation margin $dVM(t) = \text{change in settlement price } dh(t)$
 - $h(t) = E_t^{Q^\beta} [h(T)]$, $h(t)$ Q^β -martingale,
- $c(t) = r(t)$, ($r(t)$ default-free short term rate)
 - “costless collateral”, Johannes & Sundaresan (2007)
 - Then, collateralized prices = uncollateralized prices
 - Usually $c(t) > r(t)$!

An overview of the valuation of collateralized derivatives contracts

■ Stylized collateralized OIS (Overnight Index Swap)

- *Payment of* $h(T) = \exp(r_f T) - \exp\left(\int_0^T c(s) ds\right)$
 - $c(s)$ EONIA, r_f fixed rate
 - Cash-collateral with collateral rate $c(s)$
- *Settlement price* $h(0) = E^{Q^\beta} \left[h(T) \exp\left(-\int_0^T c(s) ds\right) \right]$
- *Newly traded contract: $r_f(T)$ is such that $h(0) = 0$*
 - Thus, $\exp(-r_f(T)T) = E^{Q^\beta} \left[\exp\left(-\int_0^T c(s) ds\right) \right]$
 - Observable market input $r_f(T)$ directly provides collateralized discount factor $E^{Q^\beta} \left[\exp\left(-\int_0^T c(s) ds\right) \right]$
- *Market observables on collat. markets drive PV computations*
 - **This extends to collateralized Libor contracts**
 - PV of collateralized Libor = forward (collat.) Libor \times collateralized discount factor

An overview of the valuation of collateralized derivatives contracts

- Swapclear (LCH.Clearnet)
 - *OIS and Euribor swaps*
 - *Cash-collateral remunerated at Eonia*
 - *Different swap rates*

EUR

IRS	OIS
	Anl Mny / EONIA
3 month	0.06650
6 month	0.05720
1 year	0.05240
2 year	0.10770
3 year	0.23605
5 year	0.58408
10 year	1.40660
30 year	2.18460

EUR

IRS	OIS	
	Anl Bnd / 1MEURIBOR	Anl Bnd / 3MEURIBOR
2 year	0.18288	0.28688
3 year	0.29955	0.40705
5 year	0.64708	0.75308
10 year	1.45860	1.55260
30 year	2.21610	2.27860

An overview of the valuation of collateralized derivatives contracts

- Bonds posted as collateral: bond price $A(t)$
 - *Short-term repo contract $t \rightarrow t + dt$,*
 - no haircut
 - *At t , buy the bond, deliver it, receive cash-collateral*
 - no haircut \Rightarrow no net payment at t : $-A(t) + A(t) = 0$
 - *At $t + dt$ receive bond, sell it, reimburse cash + interest*
 - Net payment at $t + dt$: $A(t + dt) - (1 + \text{repo}_A(t)dt)A(t)$
 - $\text{repo}_A(t)$: repo rate, does not account for day count conventions
 - $E_t^{Q^\beta} [A(t + dt) - (1 + \text{repo}_A(t)dt)A(t)] = 0$
- $\Rightarrow E_t^{Q^\beta} \left[\frac{dA(t)}{A(t)} \right] = \text{repo}_A(t)dt$

An overview of the valuation of collateralized derivatives contracts

- Bonds posted as collateral: bond price $A(t)$
 - $E_t^{Q^\beta} \left[\frac{dA(t)}{A(t)} \right] = \text{repo}_A(t) dt$
 - Expected rate of return on posted bond = repo rate $\text{repo}_A(t)$
 - Repo rate: market observable
 - $h(t) = E_t^{Q^\beta} \left[h(T) \exp \left(- \int_t^T \text{repo}_A(s) ds \right) \right]$
 - Parallels OIS discounting
 - *Under perfect collateralization, no CVA/DVA counterparty risk is involved*
 - *Unobserved default free short rate $r(t)$ is not involved*
 - Deriving pricing equation only involves cash-flows at $t + dt$
 - No lending/borrowing between t and $t + dt$

An overview of the valuation of collateralized derivatives contracts

- Overcollateralization, haircuts, run on repos
 - $V(s) = \alpha_A \times h(s)$
 - $V(s)$ collateral account, $h(s)$ settlement price
 - $\alpha_A > 1$ overcollateralization, $\alpha_A < 1$ partial collateralization
 - $1/\alpha_A$ haircut ratio
 - Enter a collateralized derivatives contract at t
 - Buy collateralized security: outflow $h(t)$
 - Receive collateral to secure credit exposure : inflow $\alpha_A h(t)$
 - Net inflow at $t = (\alpha_A - 1)h(t)$ is not anymore equal to zero
 - Taking into account cash-flows at $t + dt$
 - Variation margin & unwinding collateralized contracts
 - Leads to $E_t^{Q^\beta} \left[\frac{dh(t)}{h(t)} \right] = \alpha_A \times r_A(t) + (1 - \alpha_A)r(t)$

An overview of the valuation of collateralized derivatives contracts

- Overcollateralization, haircuts, run on repos

- $E_t^{Q^\beta} \left[\frac{dh(t)}{h(t)} \right] = \alpha_A(t) \times r_A(t) + (1 - \alpha_A(t))r(t)$

- Perfect collateralization $\alpha_A = 1$, $E_t^{Q^\beta} \left[\frac{dh(t)}{h(t)} \right] = r_A(t)$

- No collateralization $\alpha_A = 0$, $E_t^{Q^\beta} \left[\frac{dh(t)}{h(t)} \right] = r(t)$

- Time dependent $\alpha_A(t)$ allows to account for increase of collateral requirement during periods of market stress

- **Departure from discounting at repo rate.**

- *Previous equation accounts for settlement price between two default-free counterparties*

- Does not account for counterparty risks and funding costs

- Only accounts for adjustments due to collateral cash-flows

- Involves unobserved default free short rate $r(t)$

An overview of the valuation of collateralized derivatives contracts

■ Unilateral collateral agreement

- $\alpha_A = 1$, if $h(t) \leq 0$, $\alpha_A = 0$, if $h(t) > 0$
- $V(t) = \min(0, h(t))$
- *Applicable discount rate: $r(t)1_{\{h(t)>0\}} + r_A(t)1_{\{h(t)\leq 0\}}$*
- *Settlement price $h(t)$ solves for*
- $E_t^{Q^\beta} [dh(t)] = \left(r(t)\max(0, h(t)) + r_A(t)\min(0, h(t)) \right) dt$
- *BSDE with generator $g(h) = rh^+ + r_A h^-$*
- *$h(t)$ conditional g – expectation of $h(T)$ (Peng, 2004)*
- *g degree-one homogeneous $\Rightarrow h(t)$ degree one homogeneous with respect to $h(T)$*
- $r \geq r_A \Rightarrow$ *concave price functional (portfolio effects)*

An overview of the valuation of collateralized derivatives contracts

- Previous equation can be extended: funding rate and CVA
 - $\alpha_A = 1$, if $h(t) \leq 0$ applicable discount rate is $r_A(t)$ expected rate of return on collateral
 - $c(t)$ usually EONIA in case of cash collateral
 - $\text{repo}_A(t)$ repo rate in case of posted bonds
 - No extra funding term (deal is funded through the collateral account)
 - No DVA term, counterparty is fully protected against own default
 - $\alpha_A = 0$, if $h(t) > 0$ applicable discount rate is $r_b + \lambda(1 - \delta)$
 - λ default intensity of counterparty, δ recovery rate
 - Recovery of Market Value, $\lambda(1 - \delta)$ CVA cost
 - r_b : default-free short term borrowing rate
 - No collateral is being posted, deal is funded on the market

$$E_t^{Q^B} [dh(t)] = \left((r_b + \lambda(1 - \delta))(s) \mathbf{1}_{\{h(s) > 0\}} + r_A(s) \mathbf{1}_{\{h(s) \leq 0\}} \right) h(t) dt$$

An overview of the valuation of collateralized derivatives contracts

- Previous equation can be extended: funding rate and CVA

- $E_t^{Q^\beta} [dh(t)] = \left((r_b + \lambda(1 - \delta))(s)1_{\{h(s)>0\}} + r_A(s)1_{\{h(s)\leq 0\}} \right) h(t)dt$

- r_b is free of DVA. r_b is not a market observable
- $\lambda(1 - \delta)$ related to short term CDS of counterparty
- Recovery of Market Value: proxy for “risky close-out convention”

- Trade contributions

- $\exp\left(-\int_t^T ((r_b + \lambda(1 - \delta))(s)1_{\{h(s)>0\}} + r_A(s)1_{\{h(s)\leq 0\}})ds\right)$ pricing kernel or Gâteaux derivative of price functional
- Trade contribution of deal $h_j(T)$ with $h(T) = h_1(T) + \dots + h_n(T)$
- $E_t^{Q^\beta} \left[h_j(T) \exp\left(-\int_t^T ((r_b + \lambda(1 - \delta))(s)1_{\{h(s)>0\}} + r_A(s)1_{\{h(s)\leq 0\}})ds\right) \right]$
- Complies with marginal pricing and Euler’s allocation rules
- Linear trade contributions, CSA/portfolio dependent change of measure