Alternative Risk Measures for Alternative Investments*

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Abstract

This paper deals with portfolio optimization under different risk constraints. We use a set of hedge funds where departure from normality are significant. We optimize the expected return under standard deviation, semi-variance, VaR and expected shortfall (or CVaR) constraints. As far as the VaR is concerned, we compare different estimators. While the optimization with respect to VaR constraints appears to be difficult and lengthy, very fast optimization algorithms exists for the other risk constraints. We find that the choice of a particular VaR estimator is less discriminant than the choice of the risk constraint itself. We provide financial interpretations of the optimal portfolios associated with a decomposition of risk measures.

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Introduction

In spite of several deficiencies, the mean-variance analysis introduced by the pioneering work of Markowitz [1952] still remains a popular tool in quantitative portfolio management. Many extensions of the original setting have been proposed in order to broaden the practical use of the method (e.g. Black & Litterman [1992]) and the main drawbacks of this approach, such as the uncertainty of the estimation of the covariances and expected returns, are now assessed (Bouchaud & Potters [2000], Jorion [1985], Ledoit & Wolf [2003]). Considering other risk measures and extending the classical framework to account for skewness and kurtosis of asset returns has also been discussed for long (see Kaplanski & Kroll [2002] for a survey). More recently, starting from Artzner et al [1999], theoretical properties of a series of risk measures, such as VaR (Value at Risk) have been investigated. Alexander & Baptista [2001] compare the use of VaR and variance as a tool to compute efficient frontiers. They show that for a risk-adverse investor, the use of VaR can lead to select portfolios with higher variance returns compared to mean-variance analysis. While VaR lacks the sub-additivity property, some coherent alternatives, such as expected shortfall (or conditional VaR, see Pflug [2000], Acerbi et al [2001], Acerbi & Tasche [2001, 2002], Rockafellar & Uryasev [2000, 2002]), the absolute deviation studied by Denneberg [2000] or the semi-variance based risk measure of Fischer [2001], have been proposed and new computational algorithms have been studied.

We want here to evaluate the consequences of the choice of a risk measure for portfolio management. Therefore, we compare mean-VaR, mean-expected shortfall and mean-semi-variance efficient frontiers with a benchmark mean-variance frontier. Whenever returns are Gaussian all mean risk efficient frontiers are equivalent. Typical investments that exhibit non Gaussian features are hedge funds. Thus, we consider a database of such hedge funds where we might expect some significant differences upon the chosen risk measure.

We firstly investigate mean-VaR efficient frontiers and we focus on estimation techniques. VaR estimators do depend on the chosen methodology. Consigli [2002] considers several VaR estimators taking into account the asymmetry of the returns and fat tail effects. Even under the assumption of iid returns, the estimation error for VaR is quite significant for low probability levels and realistic sample sizes. We compare different estimators of VaR, either historical or using different kernel estimators of the quantiles and study the resulting efficient frontiers. Computational issues are not straightforward either. For instance, the set of portfolios that fulfill a VaR constraint is not necessarily convex. We thus rely on a genetic algorithm.

Some authors have recently investigated the use of alternative risk measures for portfolio management. Krokhmal, Uryasev & Zrazhevsky [2002] look for hedge fund portfolio optimization under different risk measures, CVaR, conditional draw down at risk, mean absolute deviation, maximum loss. They show that the resulting efficient frontiers are close and that combining several risk measures allows for a better risk management. They also consider the consequences of relaxing the constraints. Based on our hedge fund database, we compare efficient frontiers based on VaR, expected shortfall and semi-variance constraints. We study how portfolio allocations depend upon the structure of the risk measure. The computations of efficient frontiers under an empirical expected shortfall constraint are easier thanks to the linear programming approach of ROCKAFELLAR & URYASEV [2000]. Similarly, DE ATHYADE [2001], KONNO et al [2003] provide optimization algorithms under semi-variance constraints that are easy to implement.

The paper is organized as follows. In the first section, we briefly describe our data set and provide its main descriptive statistics. In a second section we consider portfolio optimization under VaR constraints. We investigate the use of different VaR estimators. The third section deals with alternative risk constraints, expected shortfall and semi-variance. We study how optimal portfolio allocations depend upon the chosen risk measure. Proofs are gathered in appendix.

1 Hedge fund database

We have been considering a database of sixteen hedge funds spanning the most representative styles one can encounter in the alternative investment industry. Focusing on the monthly returns of the considered funds over the time period from January 1990 to July 2001, which represents 139 data points per fund, we report on table 1 their main descriptive statistics. Symbols m, σ, s and κ refer respectively to the mean, the standard deviation, the skewness and the kurtosis of the returns. As some statistics suggest, most returns exhibit significant departure from Normality, which is confirmed by Jarque-Bera statistics.

[Insert table 1 about here]

In table 2, we report some basic dependence properties of the funds. The Betas with respect to the Standard & Poor's 500 index are given in the second column. A t-test (third column) shows that most of them are significant at the 5% level. More precisely, most funds have a significantly positive exposure to the market risk. A noticeable exception is provided by AXA Rosenberg, which is negatively correlated with the market, while it declares a market neutral strategy. However on the overall, the Betas remain relatively small, which testifies to a good diverfication potential with respect to the market risk.

[Insert table 2 about here]

The last column of table 2 presents the values of the (linear) correlation coefficients between each fund and the CSFB/Tremont index corresponding to its declared style. A wide range of correlations, spanning from -28% to 88%, can be observed. Large anti-correlations are not a surprise. Indeed, this result is in line with the many previous studies which have underline the existence of important desagreements between the declared and the actual investment styles (DIBARTOLOMEO & WITKOWSKI [1997] and BROWN & GOETZMANN [1997]). We also report the correlation matrix of the returns (table 3). Let us stress, that unlike typical mutual funds, there are a large number of fairly negative correlation parameters, which is again the sign of a good potential for diversification.

[Insert table 3 about here]

2 Optimizing under VaR constraints

Popularized by books like the one by Jorion [2000] and dubbed by the BIS, the Value-at-Risk has become one of the most prominent tools for risk management. This section will discuss the impact of the way VaR is assessed on the optimal allocation of a portfolio.

2.1 Risk Definitions

We firstly recall some useful definitions and properties concerning the higher and lower quantiles of X:

Definition 2.1 (higher and lower quantile)
Let $(\Omega, \mathcal{A}, P)$ be a probabilistic space and $\alpha \in]0, 1[$. For X being a real random variable defined on $(\Omega, \mathcal{A}, P)$,

$$q_\alpha^+(X) = \sup\{x \in \mathbb{R}, P(X < x) \leq \alpha\} \tag{2.1}$$

is the higher quantile of order α of X.

$$q_\alpha^-(X) = \inf\{x \in \mathbb{R}, P(X \leq x) \geq \alpha\} \tag{2.2}$$

is the lower quantile of order α of X,

we then get $Q_\alpha(X) = [q_\alpha^-(X), q_\alpha^+(X)]$. Let us remark that the quantiles associated with some random variable X only depends on the distribution of X (invariance in distribution). For instance, let F be the distribution function of X ; then we get $q_\alpha^-(X) = \inf\{x \in \mathbb{R}, F(x) \geq \alpha\}$. We can thus equivalently define quantiles for real distributions. We also recall that the following relations hold:

$$q_\alpha^-(-X) = -q_{1-\alpha}^+(X), \quad q_\alpha^+(-X) = -q_{1-\alpha}^-(X). \tag{2.3}$$

Following Acerbi & Tasche [2002] (see also Delbaen [2000], Pflug [2000], Acerbi, Nordio & Sirtori [2001], Tasche [2002] for related references), we now define the Value-at-Risk of X:

Definition 2.2 (Value-at-Risk)
Let X be a random variable defined on a probabilistic space $(\Omega, \mathcal{A}, P)$ and $\alpha \in]0, 1[$. We define the Value-at-Risk of X at the level α and we denote $\mathrm{VaR}_\alpha(X)$:

$$\mathrm{VaR}_\alpha(X) = q_{1-\alpha}^-(-X) = -q_\alpha^+(X). \tag{2.4}$$

In the following, R, taking values in $\mathbb{R}^p$, with $p \in \mathbb{N}$ will represent the random vector of (monthly) asset returns defined on a probabilistic space $(\Omega, \mathcal{A}, P)$. We will denote by $a \in \mathbb{R}^p$ the portfolio allocation. Thus, the portfolio return will be given by $a'R$ where $'$ denotes the transpose operator. The Value-at-Risk associated with portfolio allocation a is then given by $\mathrm{VaR}_\alpha(a'R) = -q_\alpha^+(a'R)$. For practical applications, one needs some estimator of the higher α quantile of a portfolio return distribution, $q_\alpha^+(a'R)$ to compute the VaR.

2.2 Different estimators of the historical VaR

We will thereafter describe different nonparametric estimators of the VaR in a portfolio context. All these estimators are based on the empirical distribution of portfolio returns. They share asymptotic normality and consistency, provided that the portfolio returns are independent and identically distributed (iid)[1].

Let us denote by $r_1, \dots, r_n$ the set of historical asset returns. It can be easily checked that the higher quantile of order α associated with the empirical distribution can be written as:

$$q_{n,\alpha,a}^+ = \sup \left\{ a'r_i, i = 1, \dots, n, \sum_{j=1}^{n} 1_{a'r_j < a'r_i} \leq \alpha \right\}. \tag{2.5}$$

and corresponds to one of the historical portfolio returns associated with portfolio allocation a.

For $z \in \mathbb{R}$, we denote by $[z] = \max\{n \in \mathbb{N}, n \leqslant z\}$, the integer part of z. We denote by $(a'r)_{1:n} \leqslant \dots \leqslant (a'r)_{n:n}$, the ordered statistics of portfolio returns. Let us remark that we may have $(a'r)_{i:n} = (a'r)_{j:n}$ for $i \neq j$. In this case, we talk of multiple scenarios and we then choose an arbitrary ordering. If there are no $j \in \{1, \dots, n\}$, $j \neq i$, such that $(a'r)_{i:n} = (a'r)_{j:n}$, we will say that i is an isolated scenario (for portfolio allocation a). We can now state the following:

Proposition 2.1 (Empirical higher portfolio quantile)
Let us consider some portfolio allocation a and risk level $\alpha \in]0,1[$. Then, the empirical higher quantile of order α can be written as:

$$q_{n,\alpha,a}^+ = (a'r)_{[n\alpha]+1:n}. \tag{2.6}$$

Thus, in accordance with the definition 2.2, the *empirical VaR* can be readily obtained from the empirical higher quantile:

Definition 2.3 (Empirical VaR)
Let $r_1, \dots, r_n$ be the set of historical returns, a portfolio allocation $a \in \mathbb{R}^p$ and $0 < \alpha < 1$. Then, we define the empirical VaR *as:*

$$\mathrm{VaR}_{n,\alpha,a}^E = -(a'r)_{[n\alpha]+1:n}. \tag{2.7}$$

[1] Such an assumption is not fulfilled for our dataset. In fact, it is well-known that returns on hedge funds exhibit significant serial correlations due to the illiquidity of this kind of investment vehicles (see Getmansky, Lo & Makarov [2003], for instance.). In the sequel, for simplicity and due to the lack of econometric literature about the accuracy of quantiles estimates in the presence of time dependence between the sample realizations, we will neglected the influence of these serial correlations.

Silvapulle & Granger [2001] have used some other quantile estimation techniques based on previous work by Sheather & Marron [1990]. These quantile estimators are weighted averages of empirical quantiles and are known in the statistical literature as L estimators or kernel quantile estimators. Let us denote by $K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ the Gaussian density function. We will consider the following estimator of $q_\alpha^+(a'R)$:

$$q_{n,\alpha,a}^+ = \frac{\sum_{i=1}^n K\left(h^{-1} \cdot \left(\frac{i-1/2}{n} - \alpha\right)\right)(a'r)_{i:n}}{\sum_{i=1}^n K\left(h^{-1} \cdot \left(\frac{i-1/2}{n} - \alpha\right)\right)}, \tag{2.8}$$

where $h = \sigma n^{-1/5}$ and $\sigma = \sqrt{\frac{n^2-1}{12n^2}}$ is the standard deviation of $\frac{1}{n}, \ldots, \frac{n}{n}$. We then define the Kernel VaR as $\mathrm{VaR}_{n,\alpha,a} = -q_{n,\alpha,a}^+$. Let us remark that this Kernel VaR is a weighted average of ordered portfolio returns. So, the definition of the *kernel VaR* follows:

Definition 2.4 (Kernel VaR)
Let $r_1, \ldots, r_n$ be the set of historical returns, a portfolio allocation $a \in \mathbb{R}^p$ and $0 < \alpha < 1$. Then, we define the kernel VaR *as:*

$$\mathrm{VaR}_{n,\alpha,a}^K = -\frac{\sum_{i=1}^n K\left(h^{-1} \cdot \left(\frac{i-1/2}{n} - \alpha\right)\right)(a'r)_{i:n}}{\sum_{i=1}^n K\left(h^{-1} \cdot \left(\frac{i-1/2}{n} - \alpha\right)\right)}. \tag{2.9}$$

Gouriéroux, Laurent & Scaillet [2000] have proposed another kernel based approach to the estimation of VaR in terms of quantile of the kernel estimate of the empirical cdf of portfolio returns. We also refer to Azzalini [1981] for a study of the asymptotic properties of such an estimator. The α-quantile estimator $q_{n,\alpha,a}^+$ is given by the unique solution of:

$$\frac{1}{n}\sum_{i=1}^n \Phi\left(\frac{-(a'r)_{i:n} + q_{n,\alpha,a}^+}{h}\right) = \alpha, \tag{2.10}$$

where Φ is the Gaussian cdf. As usual, the bandwidth is chosen such that $h = (4/3)^{1/5}\sigma_{n,a} n^{-1/5}$ where:

$$\sigma_{n,a} = \left(\frac{1}{n}\sum_{i=1}^n (a'r)_{i:n}^2 - \left(\frac{1}{n}\sum_{i=1}^n (a'r)_{i:n}\right)^2\right)^{1/2}, \tag{2.11}$$

is the empirical standard deviation of portfolio returns. As above, the VaR estimator is then $\mathrm{VaR}_{n,\alpha,a} = -q_{n,\alpha,a}^+$, so that:

Definition 2.5 (GLS VaR)
Let $r_1, \ldots, r_n$ be the set of historical returns, a portfolio allocation $a \in \mathbb{R}^p$ and $0 < \alpha < 1$. Then, we define the GLS VaR*, denoted $\mathrm{VaR}_{n,\alpha,a}^{GLS}$, as the solution of:*

$$\frac{1}{n}\sum_{i=1}^n \Phi\left(-\frac{(a'r)_{i:n} + \mathrm{VaR}_{n,\alpha,a}^{GLS}}{h}\right) = \alpha. \tag{2.12}$$

Eventually, let us recall that when returns $a'R$ are Gaussian, we have:

$$\mathrm{VaR}_\alpha(a'R) = -E[a'R] - \Phi^{-1}(\alpha)\sigma(a'R), \tag{2.13}$$

where $E[a'R]$ and $\sigma(a'R)$ denote respectively the mean and standard deviation of portfolio returns. This leads to the following VaR estimator (denoted further *Gaussian VaR*) based on the empirical counterparts of the mean and standard deviation of the portfolio returns:

Definition 2.6 (Gaussian VaR)
Let $r_1, \dots, r_n$ be the set of historical returns, a portfolio allocation $a \in \mathbb{R}^p$ and $0 < \alpha < 1$. Then, we define the Gaussian VaR *as:*

$$\mathrm{VaR}^{\Phi}_{n,\alpha,a} = -\frac{1}{n}\sum_{i=1}^{n}(a'r)_{i:n} - \Phi^{-1}(\alpha)\left(\frac{1}{n}\sum_{i=1}^{n}(a'r)^2_{i:n} - \left(\frac{1}{n}\sum_{i=1}^{n}(a'r)_{i:n}\right)^2\right)^{1/2}. \tag{2.14}$$

Of course, this latter VaR estimator is consistent only under the assumption of Gaussian returns which is unlikely in our case, as recalled in section 1.

Let us remark that the four proposed VaR estimators only depend on the ordered portfolio returns $(a'r)_{i:n}$, $i = 1, \dots, n$. Moreover, the four VaR estimators are differentiable and positively homogeneous of degree one, with respect to $(a'r)$. Thus applying Euler's identity, we can write the following risk measure decomposition, for any of the four previously defined VaR estimator:

$$\mathrm{VaR}^{\cdot}_{n,\alpha,a} = \sum_{i=1}^{n} \frac{\partial \mathrm{VaR}^{\cdot}_{n,\alpha,a}}{\partial (a'r)_{i:n}} \times (a'r)_{i:n} \tag{2.15}$$

[Insert figure 1 about here]

Figure 1 depicts the relative weights $\frac{(a'r)_{i:n}}{\mathrm{VaR}_{n,\alpha,a}}\frac{\partial \mathrm{VaR}_{n,\alpha,a}}{\partial (a'r)_{i:n}}$ associated with the different VaR estimators[2], which sum up to one (due to equation 2.15). It can be seen that the empirical and the Kernel VaRs are rather similar, though of course the latter is smoother. Indeed, as expected from definition 2.3, the empirical VaR puts all the weight on a single realization – here the seventh smallest one (since $[139 \times 0.05] + 1 = 7$) – while the Kernel VaR involves several ordered statistics $(a'r)_{i:n}$, even if one can observe that the main contributions come from the values in the direct neighborhood of the seventh smallest observation. Let us remark that the Gaussian VaR puts an almost equal weight to any observation. It only very slightly overweights the smallest ones, therefore, it is unable to reliably account of extreme risks. The GLS VaR stands in between the Gaussian and the Kernel VaR. This is due to bandwidth effects since the optimal bandwidth selection involves the standard deviation of the returns.

[2]the weights are not constant for the GLS and Gaussian VaRs. However the overall shape does not depend too much on the chosen portfolio. Here $\alpha = 5\%$.

2.3 Optimization under VaR constraints

We now address the issue of computing mean-VaR efficient frontiers, that is we want to solve the following optimization problem:

$$\max_{a\in\mathbb{R}_+^p} E[a'R], \tag{2.16}$$

under the constraint $\mathrm{VaR}_\alpha(a'R) \leq v$ for different risk levels v. Optimization is carried out over $\mathbb{R}_+^P$ since short-sells are assume not to be allowed, with is very reasonable for a fund of hedge funds. Contrarily to other risk measures like the Expected-shortfall (which will be discussed in the next section) or more generally coherent measures of risk, the Value-at-Risk is usually non-convex, so that it does not provide a very suitable objective function for an optimization problem. Indeed, such a problem can exhibit several local minima and thus, usual algorithms cannot be applied since they typically fail to reach the global optimum of the problem. In the following, we will provide some estimated mean-VaR efficient frontiers by solving:

$$\max_{a\in\mathbb{R}_+^p} \sum_{i=1}^{n} a'r_i, \tag{2.17}$$

under the constraint $\mathrm{VaR}_{n,\alpha,a} \leq v$ for different risk levels v. Let us remark that we need to solve different optimization problems depending on the chosen VaR estimator. These optimization problems are no more convex, since the mapping $a \longmapsto \mathrm{VaR}_{n,\alpha,a}$ may be non-convex. Moreover, the empirical VaR and the kernel VaR are not even differentiable with respect to the portfolio allocation a, which again forbids us to use standard algorithms based upon differentiation like the gradient method for instance.

2.3.1 Genetic Algorithms

Two alternative approaches can be proposed: the simulated annealing algorithm or genetic algorithms. Both of them are able to deal with multiple local minima optimization problems, preventing from being trapped in a local minimum, and do not require a differentiable objective function. Thus, they appear to be well adapted to our present concern. The choice of genetic algorithms has been retained as the most relevant since this class of algorithms embeds the simulated annealing algorithms.

The idea underlying genetic algorithms is based on the mimicry of the natural selection process and genetic principles. The genetic algorithm starts with a population of trial vectors – called *genes* – containing the parameters to optimize, namely the portfolio allocation vector a, and unfolds as follows:

- The first step consists in the *replication* (or reproduction) of the initial trial vectors according to their fitness, that is the genes whose VaR is the smallest have the highest probability to reproduce. Thus, on the average, the new population has a lower VaR than the initial one, but its diversification is also smaller since the fittest genes obviously appear twice or more in the new population.

- The second step is the *crossover* (or recombination) which leads to combine the different parameters from several vectors drawn from the new population in order to mix their characteristics.

- The third and last step is the *mutation*, where some genes undergo random changes, *i.e.*, some parameters of the vectors born of the crossover are randomly modified. This step is essential to maintain the diversity of the population which in turn ensures the exploration of the whole optimization space.

The vectors obtained after this third step are then used as initial population and the process is reiterated in order to get a new generation of genes and so on. The convergence of this algorithm to the global minimum of the problem is ensured by the fundamental theorem of genetic algorithms, stated in GOLDBERG [1989]. An example of particularly efficient genetic algorithms is the Differential Evolutionary Genetic Algorithm by PRICE & STORN [1997]. We have thereafter used the DORSEY & MAYER [1995] algorithm.

2.3.2 Analysis of the mean-VaR efficient frontiers and of mean-VaR efficient portfolios

[Insert figure 2 about here]

Let us depict (see figure 2) the different mean-VaR efficient frontiers obtained from the four VaR estimators defined in paragraph 2.2 with a probability level α set to 5%. Let us remark that the three non parametric VaR-efficient frontiers (*i.e.*, computed under Empirical, Kernel and GLS VaR estimators) are rather close. For example, efficient portfolios for a 0.4% level of Value-at-Risk have an expected return varying between 1.05% and 1.25%, thus a relative variation of 20%. As will be seen below, other choices of risk measures such as expected shortfall or semi-variance will lead to larger differences. Not surprisingly, due to the asymmetry and the heavy tails of the returns distributions, the Gaussian VaR efficient frontier[3] also differs from the non parametric VaR efficient frontiers.

Figure 2 also represents the various individual funds. We clearly observe a wide scattering of the representative points, showing that some funds are very far from efficiency. Among them, the five dark triangles point out the funds which do not enter into the compositions of any efficient portfolio. Figure 3 and table 4 provide a more accurate description of the evolution of the composition of efficient portfolios with respect to the considered VaR estimator.

[Insert figure 3 about here]

Figure 3 represents the evolution of optimal allocations with respect to the level of expected return for the four different VaR estimators. We notice the similarity between the portfolio allocations

[3]Let us remark that it also corresponds to the mean-variance frontier.

obtained for the Gaussian and the GLS VaRs on the one hand and between the empirical and the Kernel VaRs on the other hand. We also remark that for these two latter estimators, portfolio allocation can change very quickly with the considered level of return, which can be related to the fact that Kernel and empirical VaRs actually involve a weak number of realizations of the portfolio return (see figure 1 on VaR decomposition). Eventually, we observe that the smoother the VaR estimator, the smoother the variations of the optimal allocation.

Focusing on a given level of expected return (arbitrarily set to 1.2%), table 4 provides the optimal allocation according to the four VaR estimators. Even if the optimal weights are significantly different, we can remark that the preponderant funds remain the same, namely: Bennet Restructuring, Genesis Emerging Markets, Blue Rock Capital and GAMut Investments. One can remark (see table 1) that all these funds have a Sharpe Ratio larger than one, excepted Genesis Emerging Markets, which could *a priori* suggest the existence of a relation between the Sharpe ratios and the weights of the funds in the optimal portfolios. However, the assessment of the rank correlation between these two quantities shows the absence of such a relation, expected for the Gaussian VaR obviously.

[Insert table 4 about here]

Now, in order to compare the similarity of two portfolios in terms of asset allocation, we define the distance between two portfolio allocations a and b as: $d(a,b) = \sum_{j=1}^{p} \mid a_j - b_j \mid$, where p is the number of assets (here $p = 16$), a is the p dimensional vector of the first portfolio weights and b the second portfolio allocation. Let us remark that $d(a,b)$ takes values between 0 and 2. Let us also remark that $\| a'R - b'R \|_1 = E\left[\mid a'R - b'R \mid\right]$ can be small even if $d(a,b)$ is rather large due to correlation effects between assets.

Table 5 compares portfolio allocations for different efficient frontiers associated with the same level of expected return. We notice that the mean-variance optimal portfolios are far from the optimal portfolios under non parametric VaR constraints. Within the non parametric approaches, Kernel VaR and empirical VaR lead to very similar portfolios while the portfolios computed under GLS VaR constraints differ slightly more. This is in line with the sensitivity analysis of the different VaR estimators with respect to the distribution of returns. One might think to use bootstrap techniques to compute confidence intervals on the distance between portfolios. However, due to the slowness of the optimization process (by use of genetic algorithms) this has not been done.

[Insert table 5 about here]

To end with the investigation of the impact of the VaR estimators on the VaR-efficient portfolios, let us try to quantify diversification effects of our different VaR estimates. To this aim, we introduce the *participation ratio* defined as $\frac{1}{\sum_{j=1}^{p} a_j^2}$ associated with a portfolio allocation $a = (a_1, \ldots, a_p)$. If the portfolio is based on a single asset, the value of the participation ratio is equal to 1 whereas with an equally weighted portfolio, the value is n. We remark that the level of diversification tends to be lower for higher levels of expected returns: the search for high expected returns requires investments in the funds with higher historical returns, thus lessening the participation ratio. We notice that once again, the patterns are close for Kernel and empirical VaR constraints. Moreover, the overall degree of diversification is higher when using these constraints than when using GLS VaR estimators or when considering mean-variance efficient frontiers. We emphasize that though empirical VaR only involves a single rank statistics, this does not preclude portfolio diversification.

[Insert figure 4 about here]

Eventually, let us stress that introducing the Standard & Poor's 500 index into the portfolio does not significantly change the shape of efficient frontiers (see figure 5) nor the optimal allocations. It simply means that some hedge-funds replicate sufficiently well the behavior of the market index. Genesis Emerging Markets, with a Beta equal to 78% (see table 2), is such an example.

[Insert figure 5 about here]

3 Optimizing under alternative risk constraints

We will now deal with portfolio optimization under alternative risk constraints, such as expected shortfall and semi-variance, which enjoy coherence properties and therefore represent suitable constraints for our optimal allocation problem.

3.1 Optimization under expected shortfall constraints

Let us first turn to another quantile based risk measure, namely the Expected-Shortfall:

Definition 3.7 (Expected-Shortfall)
Let X be a random variable defined on a probabilistic space $(\Omega, \mathcal{A}, P)$ with finite expectation[4]*. Let $\alpha \in]0,1[$. We define the Expected-Shortfall of X at the level α, denoted by $\mathrm{ES}_\alpha(X)$, as the solution of:*

$$\inf_{\zeta \in \mathbb{R}} \frac{E^P[(X-\zeta)^-]}{\alpha} - \zeta. \tag{3.18}$$

In order to obtain a closed form expression of the Expected-Shortfall, let us now recall the following basic property shared by any quantile:

Proposition 3.2 (Quantile characterization)
Let X be a real random variable defined on a probabilistic space $(\Omega, \mathcal{A}, P)$ with finite expectation. Let $\alpha \in]0,1[$ and $\zeta \in \mathbb{R}$. Let us consider the real function H_α taking values in $[0, \infty[$ defined by:

$$H_\alpha(\zeta) = \alpha E\left[(X-\zeta)^+\right] + (1-\alpha) E\left[(X-\zeta)^-\right]. \tag{3.19}$$

H_α is minimal on $Q_\alpha(X) = [q_\alpha^-(X), q_\alpha^+(X)]$.

As a consequence, we get:

[4]We could use the weaker assumption $E[X^-] < \infty$ where $X^- = \max(0, -X)$.

Proposition 3.3 (Expected-Shortfall characterization)
The criteria $\frac{E^P[(X-\zeta)^-]}{\alpha} - \zeta$ *is minimal on the quantile set* $Q_\alpha(X)$, *so that a closed form expression for the Expected-Shortfall is given by*

$$\mathrm{ES}_\alpha(X) = -\frac{1}{\alpha}\left(E^P\left[X1_{\{X\leqslant q_\alpha(X)\}}\right] + q_\alpha(X)\left(\alpha - P\left(X\leqslant q_\alpha(X)\right)\right)\right), \tag{3.20}$$

for any quantile $q_\alpha \in [q_\alpha^-(X), q_\alpha^+(X)]$.

We can also state a well known property of the Expected-Shortfall:

Corollary 3.1 (Sub-additivity of Expected-Shortfall)
Let X, Y *be two random variables with finite expectation defined on a probabilistic space* $(\Omega, \mathcal{A}, P)$ *and* $\alpha \in]0,1[$. *Then:*

$$\mathrm{ES}_\alpha(X+Y) \leqslant \mathrm{ES}_\alpha(X) + \mathrm{ES}_\alpha(Y). \tag{3.21}$$

Since the expected shortfall is also invariant in law, positively homogeneous, and invariant with respect to translations, it is thus a coherent measure of risk. It is also possible to relate Expected shortfall and VaR through the following proposition, which also shows that the expected shortfall is a spectral measure of risk (see Acerbi [2002] for a study of spectral measures of risk):

Proposition 3.4 (Quantile representation of ES)
Let X *be a random variable defined on a probabilistic space* $(\Omega, \mathcal{A}, P)$ *with finite expectation and* $\alpha \in]0,1[$. *We can write:*

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X)du. \tag{3.22}$$

In the following, we will consider the mean-expected shortfall optimization problem:

$$\max_{a\in\mathbb{R}^p_+} E[a'R], \tag{3.23}$$

under the constraint $\mathrm{ES}_\alpha(a'R) \leq v$ for different risk levels v. We estimate the expected shortfall as in Rockafellar and Uryasev [2000], namely as the expected shortfall corresponding to the empirical measure (we refer to Scaillet [2001] for another nonparametric approach). This is provided by (see Rockafellar & Uryasev [2002] for a similar result):

Proposition 3.5 (Expected-Shortfall, empirical measure)
Let us consider some portfolio allocation a *and risk level* $\alpha \in]0,1[$. *Then, the empirical expected shortfall can be written as:*

$$\mathrm{ES}_{n,\alpha,a} = -\frac{1}{n\alpha}\left(\sum_{i=1}^{[n\alpha]}(a'r)_{i:n} + (n\alpha - [n\alpha])\,(a'r)_{[n\alpha]+1:n}\right). \tag{3.24}$$

We have implemented the algorithm proposed by Rockafellar and Uryasev [2000] for the estimation of mean-expected shortfall frontiers:

$$\max_{a\in\mathbb{R}^p_+} \sum_{i=1}^n a'r_i, \tag{3.25}$$

under the constraint $\mathrm{ES}_{n,\alpha,a} \leq v$ for different risk levels v. This approach is based on the previous characterizations of the expected shortfall and the following proposition (see theorems 14 and 15 in Rockafellar and Uryasev [2002]):

Proposition 3.6 (Expected-Shortfall minimization)
Let $\alpha \in]0,1[$ and r_i, $i = 1, \ldots, n$ be the historical returns. We then have:

$$\min_{a \in \mathbb{R}_+^p} \mathrm{ES}_{n,\alpha,a} = \min_{(a,\zeta) \in \mathbb{R}_+^p \times \mathbb{R}} \frac{1}{n\alpha} \times \left(\sum_{i=1}^{n} (a' r_i - \zeta)^- \right) - \zeta. \tag{3.26}$$

Moreover,

$$(a^*, \zeta^*) \in \underset{(a,\zeta)}{\operatorname{argmin}} \frac{1}{n\alpha} \times \left(\sum_{i=1}^{n} (a' r_i - \zeta)^- \right) - \zeta \Longleftrightarrow \begin{cases} a^* \in \underset{a}{\operatorname{argmin}} \, \mathrm{ES}_{n,\alpha,a}, \\ \zeta^* \in \underset{\zeta}{\operatorname{argmin}} \frac{1}{n\alpha} \times \left(\sum_{i=1}^{n} (a^{*\prime} r_i - \zeta)^- \right) - \zeta \end{cases}$$

As a consequence of this result, Rockafellar & Uryasev [2000, 2002] transform an optimization of expected return under expected shortfall constraints into a linear program.

3.2 Optimization under semi-variance constraints

One sided moments lead to popular risk measures in portfolio management such as the semi-variance. The semi-variance is invariant in law, sub-additive and positively homogeneous of degree one. We thereafter use the following definition (see Fischer [2001]):

Definition 3.8 (Coherent risk-measure based on semi-variance)
Let X be a square integrable random variable defined on a probabilistic space $(\Omega, \mathcal{A}, P)$. We define:

$$\mathrm{SV}(X) = -E[X] + \| (X - E[X])^- \|_2, \tag{3.27}$$

where $(x)- = \max(-x, 0)$ and $\| X \|_2 = \left(E[X^2]\right)^{1/2}$.

Let us remark that thanks to the expectation term, SV is translation invariant and monotonic. Thus, it is a coherent measure, but however it fails to be comonotonically additive and then to be a spectral or distortion risk measure, as the Expected-Shortfall was.

As for the case of the expected shortfall, we consider the empirical counterpart of the risk measure:

Definition 3.9 (Coherent semi-variance, empirical measure)
Let us consider some portfolio allocation a and risk level $\alpha \in]0,1[$. Then, the empirical counterpart of the coherent form of the semi-variance can be written as:

$$\mathrm{SV}_{n,a} = -\frac{1}{n} \sum_{i=1}^{n} a' r_i + \left(\frac{1}{n} \sum_{i=1}^{n} \left(\max\left(0, a' r_i - \frac{1}{n} \sum_{j=1}^{n} a' r_j \right) \right)^2 \right)^{1/2}. \tag{3.28}$$

The estimated mean-semi variance frontier is then provided by solving:

$$\max_{a \in \mathbb{R}^p_+} \sum_{i=1}^{n} a' r_i, \tag{3.29}$$

under the constraint $\mathrm{SV}_{n,a} \leq v$ for different risk levels v. We will use here the recursive algorithm of DE ATHAYDE [2001].

3.3 Analysis of individual funds

Let us first emphasize that $\mathrm{ES}_{n,\alpha,a}$ and $\mathrm{SV}_{n,a}$ only depend on the ordered portfolio returns $(a'r)_{i:n}$, $i = 1, \ldots, n$. Moreover $\mathrm{ES}_{n,\alpha,a}$ and $\mathrm{SV}_{n,a}$ are differentiable[5] and positively homogeneous of degree one with respect to portfolio values. Therefore, we can decompose the risk measures as in the case of VaR by use of Euler's equality. As for the VaR constraints, figure 6 represents the weights associated with the ordered returns for the risk measures under assessement (with $\alpha = 5\%$). We clearly observe that the Value-at-Risk and Expected-Shortfall weights are concentrated on extreme rank statistics while the weights involved in the calculation of the variance and the semi-variance exhibit a much smoother pattern.

[Insert figure 6 about here]

Now, in order to study the relative content of each risk measures, let us assess the rank correlation between these various risk measures for the sixteen hedge-funds of our dataset. The results are given in table 6. We observe that the four different kind of risk measures considered until now are very strongly correlated (the rank correlation coefficients are all larger than 90%), which simply means that, on the overall, the riskier fund according to one of the risk measure is also the riskier according to any of the three other risk measures. We can also remark that the VaR and the Expected-Shortfall at the 5% probability level remain very similar to the variance, involve only a little fraction of the information contained in the skewness and are almost totally uncorrelated with the kurtosis. It means that, at the 5% probability level, the VaR and the Expected-Shortall are still very sensitive to the bulk of the distribution of returns (*i.e.* the smaller risks) and almost insensitive to the far tail of the distribution (*i.e.* the larger risks). It would thus have been interesting to compute the VaR and the Expected-Shortfall at the 1% probability level, but due to the small number of historical observations one has at its disposal when dealing with hedge-funds, it has appeared to be too much noisy.

3.4 Efficient portfolios for alternative risk measures

We firstly report the efficient frontiers, corresponding to VaR, expected shortfall, variance and semi-variance constraints. To ease the reading, we have only plot the efficient frontiers under the Kernel VaR constraints. The upper panel of figure 7 depicts the four efficient frontiers in the

[5]This holds almost surely for $\mathrm{SV}_{n,a}$.

Mean-Kernel VaR diagram. We remark that the VaR efficient frontier is far from the others (even from the Expected Shortfall efficient frontier). It may be ascribed to the fact that VaR estimates only involve a few rank statistics. In addition, we do not observe strong differences between Semi-variance, Variance, Expected shortfall efficient frontier. It is however worth noticing that, in this diagram, the Semi-variance optimal portfolios appear more efficient than the Expected-Shortfall optimal portfolios. Indeed, for any value of the level of expected return, their VaR is lower than the VaR of the ES-optimal portfolios at the same level of expected return. The lower panel shows the same efficient frontiers in the Mean-Expected Shortfall diagram. We remark the same kind of behavior, and again, the Semi-variance optimal portfolios appear as the second most efficient ones, *ceteris paribus.*

[Insert figure 7 about here]

Focusing again on a given level of expected return (still equal to 1.2%), table 7 provides the optimal allocation according to the four risk measures. As in the previous case, where we have only considered VaR estimators, we remark that, on the overall, the dominant funds remain the same: Bennet Restructuring, Genesis Emerging Markets, Arrowsmith Funds, Blue Rock Capital and GAMut Investments. Nevertheless, we have to grant that the Expected shortfall leads to much more diversified portofolios than the three others risk measures. Clearly, the VaR, the standard deviation and the semi-variance yield efficient portfolios involving a significant contribution of only five or six funds, while the efficient Expected-Shortfall portfolio is made of eight funds.

[Insert table 7 about here]

This observation is confirmed by figures 8 and 9 which depict the composition and the participation ratio of each optimal portfolio versus its level of expected return. On the overall, the Expected-Shortfall is the risk measure yielding the more diversified portfolios. However, for levels of expected return larger than 1.4%, the Kernel VaR compares with it. For values of expected returns less than 1.4%, the Kernel VaR and the Semi-variance leads to the same level of diversification and on the whole, the Variance (or Gaussian VaR) is the risk measure which yields the most concentrated portfolios.

[Insert figures 8 and 9 about here]

We now compare, on table 8, the distance between efficient portfolios associated with different risk measures and different levels of expected returns. We remark that optimal portfolios tend to be closer for high levels of expected return, which simply means that the risk constraints are less binding. These tables highlight a relative proximity between VaR and semi-variance on the one hand and between variance and semi-variance on the other hand. The optimal portfolios with respect to the expected shortfall constraint stand apart, as already underlined.

[Insert table 8 about here]

Eventually, let us notice that the individual risk borne by each fund does not explain its proportion into optimal portfolios. For instance the rank correlation between the risk of individual funds and their weight in the optimal allocation is equal to 43% according to the VaR, to 37% according to Semi-variance and only 16% according to the Expected-Shortfall. To sum up, a large amount of marginal risk is not synonimous with a low weight in the optimal allocation. Conversely, a low level of marginal individual risk does not amount to a hight weight in optimal portfolios. It clearly highlights the importance of the dependence between risks in the tails, as already stressed by Geman & Kharoubi [2003] or Malevergne & Sornette [2002, 2004].

4 Conclusion

We have seen that the way VaR is computed is particularly important. The use of historical VaR even at a 95% level leads to portfolio allocations that change quickly with the return objectives. Under the standard choices of bandwidth, the kernel VaR of Silvapulle & Granger usually provides results that are close to those under the empirical VaR constraint, while Gouriéroux et al kernel VaR is associated with a smoother weighting scheme of returns.

As can be seen on the risk decomposition of risk measures, the empirical expected shortfall constraints is not so far away from the Gouriéroux et al kernel VaR constraint. We also remark that the variance and semi-variance constraints depend quite a lot on extreme returns due to the squared returns in the computations. The risk decomposition of risk measures allows to understand the structure of optimal portfolios.

Regarding the implementation issues, optimizing under variance, semi-variance and expected shortfall constraints can be done very quickly, while optimization under VaR constraints is extremely lengthy[6]. Since it is computationally easy to deal with expected shortfall constraints, one can think of relying on Larsen, Mausser & Uryasev [2001] algorithm which provides an approximation approach for VaR efficient frontiers based on the Rockafellar & Uryasev [2000] framework and on a fine management of confidence intervals and extreme scenarios. This results in very quick computations unlike the use of genetic algorithms. However, there is no guarantee that the resulting portfolio is optimal and no way to check the degree of approximation. In order to investigate the accuracy of the method, we have implemented it. Based on our hedge fund dataset, we found big departures from the mean-VaR frontier[7]. This is consistent with our finding that VaR and expected shortfall optimal portfolios are quite different in our examples.

A question that has not been dealt with concerns the actual choice of the risk measure from the point of view of the portfolio manager. In fact, and even if this problem deserves much more attention than we can paid to it here, our results bring some good news. We have seen that, on the overall, the composition of optimal portfolios remains dominated by the same few funds. Therefore, the choice of a particular risk measure does not seem so important. Even VaR, which lacks sub-additivity does not yield so different optimal allocations. Obviously, this conclusion could turn out to be wrong when one focuses on more extreme risks. Recall that our study has only relie on risks

[6]About one week for a single efficient frontier.
[7]Detailed results can be asked to the authors.

of moderate sizes since the Value-at-Risk and the Expected-Shortfall have only be assessed at the 5% level. Such a problem is worth beeing investigated in deepth and is let for future research.

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A Proofs

Proof of proposition (2.1): while the proof is rather straightforward when there are only isolated scenarios, the previous result appears to be true in the general case where multiple scenarios can occur. We remark that:

$$P\left(a'R \leq (a'r)_{i:n}\right) \geq \frac{i}{n},$$

for $i = 1, \ldots, n$ (strict inequality can occur when i is associated with a multiple scenario) and:

$$P\left(a'R < (a'r)_{i:n}\right) \leq \frac{i-1}{n},$$

where strict inequality can occur in the case i is associated with multiple scenarios. Let $\alpha \in \left[\frac{i-1}{n}, \frac{i}{n}\right[$. Then,

$$P\left(a'R < (a'r)_{i:n}\right) \leq \frac{i-1}{n} \leq \alpha < \frac{i}{n} \leq P\left(a'R \leq (a'r)_{i:n}\right). \tag{A.30}$$

In Laurent [2003], the order α higher quantile for discrete distributions of X taking values in $x_1, \ldots, x_n$ is characterized by: $P(X < x_i) \leq \alpha < P(X \leq x_i)$. Together with equation (A.30), this shows that $(a'r)_{i:n}$ is the higher α quantile of $a'R$. Since $\alpha \in \left[\frac{i-1}{n}, \frac{i}{n}\right[\Leftrightarrow i = [n\alpha] + 1$, we obtain the stated result.

Proof of proposition (3.2): we denote $(X-\zeta)^+ = \max(X-\zeta, 0)$ and $(X-\zeta)^- = \max(\zeta - X, 0)$. Given X and α, we denote $Z(\zeta) = \alpha(X-\zeta)^+ + (1-\alpha)(X-\zeta)^-$.

- Let $\zeta \in \mathbb{R}$ and x an α quantile of X. Let us firstly assume that $\zeta > x$.

  $Z(\zeta) - Z(x) = (1-\alpha)(\zeta - x) \times 1_{]-\infty, x]}(X) + ((1-\alpha)\zeta + \alpha x - X)\, 1_{]x,\zeta[}(X) + \alpha(x-\zeta)1_{[\zeta,\infty[}(X)$.

  On the other hand, $((1-\alpha)\zeta + \alpha x - X)\, 1_{]x,\zeta[}(X) \geq \alpha(x-\zeta)1_{]x,\zeta[}(X)$. Then:

  $$H_\alpha(\zeta) - H_\alpha(x) \geq (1-\alpha)(\zeta - x)P(X \leq x) + \alpha(x-\zeta)P(X > x),$$

  or equivalently:

  $$H_\alpha(\zeta) - H_\alpha(x) \geq (\zeta - x) \times ((1-\alpha)\, P(X \leq x) - \alpha P(X > x)).$$

  $(1-\alpha)P(X \leq x) - \alpha P(X > x) = P(X \leq x) - \alpha \geq 0$ since x is an α quantile. This shows that $H_\alpha(\zeta) \geq H_\alpha(x)$.

- Let us now assume that $\zeta < x$.

  $Z(\zeta) - Z(x) = (1-\alpha)(\zeta - x) \times 1_{]-\infty, \zeta]}(X) + (X - \alpha\zeta - (1-\alpha)x)\, 1_{]\zeta,x[}(X) + \alpha(x-\zeta)1_{[x,\infty[}(X)$.

  On the other hand, $(X - \alpha\zeta - (1-\alpha)x)\, 1_{]\zeta,x[}(X) \geq (1-\alpha)(\zeta - x)1_{]\zeta,x[}(X)$. Thus:

  $$H_\alpha(\zeta) - H_\alpha(x) \geq (1-\alpha)(\zeta - x)P(X < x) + \alpha(x-\zeta)P(X \geq x),$$

or equivalently:

$$H_\alpha(\zeta) - H_\alpha(x) \geq (x-\zeta) \times (\alpha P(X \geq x) - (1-\alpha) P(X < x)),$$

$\alpha P(X \geq x) - (1-\alpha)P(X < x) = \alpha - P(X < x) \geq 0$ since x is an α quantile. This shows that $H_\alpha(\zeta) \geq H_\alpha(x)$.

- Let us eventually check that H_α is constant over $Q_\alpha(X)$. Let ζ be an interior point of $Q_\alpha(X)$. Let ζ' be another quantile with $zeta' > \zeta$. From the first point (ζ is a quantile and $\zeta' > \zeta$), $H_\alpha(\zeta') \geq H_\alpha(\zeta)$. From the second point (ζ' is a quantile and $\zeta < \zeta'$), $H_\alpha(\zeta) \geq H_\alpha(\zeta')$; Thus, $H_\alpha(\zeta) = H_\alpha(\zeta')$. We can notice that H_α is continuous[8]. As a consequence H_α takes the same values on the boundary of $Q_\alpha(X)$ and on its interior.

Proof of proposition (3.3): $\dfrac{E^P[(X-\zeta)^-]}{\alpha} - \zeta = \dfrac{1}{\alpha} H_\alpha(\zeta) - E^P[X]$, where $H_\alpha(\zeta) = \alpha E^P\left[(X-\zeta)^+\right] + (1-\alpha)E^P\left[(X-\zeta)^-\right]$. The minimum is thus attained for $\zeta \in Q_\alpha(X)$.

Now, since the criteria $\dfrac{E^P[(X-\zeta)^-]}{\alpha} - \zeta$ is minimal on the quantile set $Q_\alpha(X)$. We can write $\dfrac{1}{\alpha}\left((X-\zeta)^- - \zeta\alpha\right) = \dfrac{1}{\alpha}\left(-X 1_{\{X \leqslant \zeta\}} + \zeta\left(1_{\{X \leqslant \zeta\}} - \alpha\right)\right)$ for any $\xi \in Q_\alpha(X)$. Thus, we can write the minimum of $\dfrac{E^P[(X-\zeta)^-]}{\alpha} - \zeta$ as:

$$-\frac{1}{\alpha}\left(E^P\left[X 1_{\{X \leqslant q_\alpha(X)\}}\right] + q_\alpha(X)\left(\alpha - P(X \leqslant q_\alpha(X))\right)\right) = \mathrm{ES}_\alpha(X), \quad \forall q_\alpha(X) \in Q_\alpha(X).$$

Proof of corollary (3.1): let $a, b \in \mathbb{R}$. Then, $a^- + b^- \geqslant (a+b)^-$[9]. For a and b being some numbers, we then have:

$$\frac{E^P[(X-a)^-]}{\alpha} - a + \frac{E^P[(Y-b)^-]}{\alpha} - b \geqslant \frac{E^P[(X+Y-(a+b))^-]}{\alpha} - (a+b).$$

Since $\dfrac{E^P[(X+Y-(a+b))^-]}{\alpha} - (a+b) \geqslant \inf\limits_{s\in\mathbb{R}} \dfrac{E^P[(X+Y-s)^-]}{\alpha} - s = -ES_\alpha(X+Y)$, we get $\forall a, b \in \mathbb{R}$,

$$\frac{E^P[(X-a)^-]}{\alpha} - a + \frac{E^P[(Y-b)^-]}{\alpha} - b \geqslant -ES_\alpha(X+Y),$$

and we can then write $-ES_\alpha(X) - ES_\alpha(Y) \geqslant -ES_\alpha(X+Y)$ which shows the sub-additivity of the expected shortfall.

Proof of proposition (3.4): we adapt the proof by Acerbi and Tasche [2002]. Since $VaR_\alpha(X) = -q_\alpha^+(X)$, we want to show that $ES_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha q_u^+(X)du$. Using:

$$ES_\alpha(X) = -\frac{1}{\alpha}\left(E^P\left[X 1_{\{X \leqslant q_\alpha^+(X)\}}\right] + q_\alpha^+(X)\left(\alpha - P(X \leqslant q_\alpha^+(X))\right)\right),$$

[8] The random variables $Z(\zeta)$ continuously depend on ζ. They can be bounded by an integrable random variable in a compact neighbourhood of ζ_0. From dominated convergence theorem, we obtain continuity in ζ_0.

[9] Indeed, $a^- \geqslant -a$, $b^- \geqslant -b$, thus $a^- + b^- \geqslant -(a+b)$. On the other hand $a^- + b^- \geqslant 0$ provides $a^- + b^- \geqslant \max(0, -(a+b)) = (a+b)^-$.

we need to show that:

$$\int_0^\alpha q_u^+(X)du = E^P\left[X1_{\{X\leqslant q_\alpha^+(X)\}}\right] + q_\alpha^+(X)\left(\alpha - P(X\leqslant q_\alpha^+(X))\right).$$

Let U be a uniform $[0,1]$ random variable defined on some probabilistic space. We define $Z = q_U^+(X)$. We remark that since $u \to q_u^+(X)$ is not decreasing, we have $\{U \le \alpha\} \subset \{Z \le q_\alpha^+(X)\}$ and $\{U > \alpha\} \cap \{Z \le q_\alpha^+(X)\} \subset \{Z = q_\alpha^+(X)\}$. From transfer theorem, we have $\int_0^\alpha q_u^+(X)du = E\left[Z1_{\{U\le\alpha\}}\right]$. Since $\{U \le \alpha\} \subset \{Z \le q_\alpha^+(X)\}$, we have $E\left[Z1_{\{U\le\alpha\}}\right] = E\left[Z1_{\{U\le\alpha\}}1_{\{Z\le q_\alpha^+(X)\}}\right]$, which we can rewrite as $E\left[Z1_{\{Z\le q_\alpha^+(X)\}}\right] - E\left[Z1_{\{U>\alpha\}\cap\{Z\le q_\alpha^+(X)\}}\right]$. Since $q_U^+(X)$ is distributed as X, the first expectation equals $E\left[X1_{\{X\le q_\alpha^+(X)\}}\right]$. By using the second set relation, we write the second expectation as $q_\alpha^+(X)E\left[1_{\{U>\alpha\}}1_{\{Z\le q_\alpha^+(X)\}}\right]$. Since $1_{\{U>\alpha\}}1_{\{Z\le q_\alpha^+(X)\}} = 1_{\{Z\le q_\alpha^+(X)\}} - 1_{\{Z\le q_\alpha^+(X)\}}1_{\{U\le\alpha\}} = 1_{\{Z\le q_\alpha^+(X)\}} - 1_{\{U\le\alpha\}}$, where the latter equality comes from the first set relation, and since Z is distributed as X, we can write the second expectation as $P(X \le q_\alpha^+(X)) - \alpha$. This proves the stated result.

Proof of proposition (3.5): we start from the quantile representation of the expected shortfall, $ES_\alpha(a'R) = -\frac{1}{\alpha}\int_0^\alpha q_u^+(a'R)du$. By, splitting the integral over $\left[\frac{i-1}{n}, \frac{i}{n}\right[$ intervals, we can write the expected shortfall as: $ES_\alpha(a'R) = -\frac{1}{\alpha}\sum_{i=1}^n \int_0^1 1_{\left[\frac{i-1}{n},\frac{i}{n}\right[}(u) q_u^+(a'R)1_{[0,\alpha]}du$. For $u \in \left[\frac{i-1}{n}, \frac{i}{n}\right[$, we have $q_u^+(a'R) = (a'r)_{i:n}$ which allows to write: $ES_\alpha(a'R) = -\frac{1}{\alpha}\sum_{i=1}^n (a'r)_{i:n}\int_0^1 1_{\left[\frac{i-1}{n},\frac{i}{n}\right[}(u)1_{[0,\alpha]}du$. The different integral terms appear to be equal to $\frac{1}{n}$ if $i \le [n\alpha]$, where $[z]$ is the integer part of z, $\alpha - \frac{i-1}{n}$ if $i = [n\alpha]+1$ and 0 otherwise. We then get the stated result $ES_\alpha(a'R) = -\frac{1}{\alpha}\left(\sum_{i=1}^{[n\alpha]}\frac{(a'r)_{i:n}}{n} + \left(\alpha - \frac{[n\alpha]}{n}\right)(a'r)_{[n\alpha]+1:n}\right)$.

Proof of proposition (3.6): let us prove the first point. We recall that $Q_\alpha(a'R) = \operatorname{argmin}_\zeta \frac{E\left[(a'R-\zeta)^-\right]}{\alpha} - \zeta$ and the corresponding value of the minimum equals $ES_\alpha(a'R)$. Let us consider a series (a_n, z_n) such that $\frac{E\left[(a_n'R-\zeta_n)^-\right]}{\alpha} - \zeta_n$ is converging to $\min_{(a,\zeta)}\frac{E\left[(a'R-\zeta)^-\right]}{\alpha} - \zeta$. From the stated optimization result, we have the inequalities:

$$\frac{E\left[(a_n'R-\zeta_n)^-\right]}{\alpha} - \zeta_n \ge \frac{E\left[(a_n'R-q_\alpha^+(a_n'R))^-\right]}{\alpha} - q_\alpha^+(a_n'R) \ge \min_{(a,\zeta)}\frac{E\left[(a'R-\zeta)^-\right]}{\alpha} - \zeta.$$

This shows that $\lim_{n\to\infty}\frac{E\left[(a_n'R-q_\alpha^+(a_n'R))^-\right]}{\alpha} - q_\alpha^+(a_n'R) = \min_{(a,\zeta)}\frac{E\left[(a'R-\zeta)^-\right]}{\alpha} - \zeta$ or equivalently that $\lim_{n\to\infty} ES_\alpha(a_n'R) = \min_{(a,\zeta)}\frac{E\left[(a'R-\zeta)^-\right]}{\alpha} - \zeta$. As a consequence $\min_a ES_\alpha(a'R) \le \min_{(a,\zeta)}\frac{E\left[(a'R-\zeta)^-\right]}{\alpha} - \zeta$. On the other hand, for $a \in \mathbb{R}^p$, $ES_\alpha(a'R) = \frac{E\left[(a'R-q_\alpha^+(a'R))^-\right]}{\alpha} - q_\alpha^+(a'R) \ge \min_{(a,\zeta)}\frac{E\left[(a'R-\zeta)^-\right]}{\alpha} - \zeta$ shows that $\min_a ES_\alpha(a'R) \ge \min_{(a,\zeta)}\frac{E\left[(a'R-\zeta)^-\right]}{\alpha} - \zeta$. This shows that:

$$\min_{a\in\mathbb{R}^p} ES_\alpha(a'R) = \min_{(a,\zeta)\in\mathbb{R}^p\times\mathbb{R}}\frac{E\left[(a'R-\zeta)^-\right]}{\alpha} - \zeta.$$

This holds for any distribution of returns. By using the empirical measure, we get the stated result.

Hedge funds summary statistics

Funds	Style	m	σ	s	κ	JB-test	p-value
Axa Rosenberg	Equity Market Neutral	5.61%	8.01%	0.82	13.65	186	0.00%
Discovery MasterFund	Equity Market Neutral	6.24%	14.91%	-0.27	0.25	0.1	93.65%
Aetos Corp	Event Driven	12.52%	8.13%	-1.69	7.78	63	0.00%
Bennet Restructuring	Event Driven	16.02%	7.48%	-0.74	7.37	55	0.00%
Calamos Convertible	Convertible Arbitrage	10.72%	8.09%	0.71	2.59	7	2.72%
Sage Capital	Convertible Arbitrage	9.81%	2.45%	-3.19	3	19	0.01%
Genesis Emerging Markets	Emerging Markets	10.54%	20.03%	-3.34	6.4	52	0.00%
RXR Secured Note	Fixed Income Arbitrage	12.29%	6.45%	2.33	4.84	29	0.00%
Arrowsmith Fund	Funds of Funds	26.91%	27.08%	14.51	60.7	3895	0.00%
Blue Rock Capital	Funds of Funds	8.65%	3.47%	1.66	7.51	59	0.00%
Dean Witter Cornerstone	Global Macro	13.95%	23.19%	7.42	9.17	139	0.00%
GAMut Investments	Global Macro	24.73%	14.43%	3.38	4.61	33	0.00%
Aquila International	Long Short Equity	9.86%	16.88%	-1.22	2.32	7	3.18%
Bay Capital Management	Long Short Equity	10.12%	19.31%	1.94	0.7	4	11.80%
Blenheim Investments LP	Managed Futures	16.51%	29.59%	3.07	10.25	114	0.00%
Red Oak Commodity	Managed Futures	19.80%	29.08%	1.94	3.52	16	0.03%

Table 1: Summary statistics for the sixteen hedge funds composing our dataset. The two first columns identify each fund and recall its investment style. The third to sixth columns provide the mean, the standard deviation, the skewness and the kurtosis of the monthly returns for each fund. The two last columns correspond to the well-known Jarque-Bera Normality test based upon the values of the skewness and the kurtosis.

Hedge funds dependence properties

Funds	Beta	t	Correlation
Axa Rosenberg	-0.14	3.08	-28.36%
Discovery MasterFund	0.01	0.17*	3.27%
Aetos Corp	0.25	5.17	34.05%
Bennet Restructuring	0.16	3.33	64.15%
Calamos Convertible	0.37	9.22	32.75%
Sage Capital	0.07	3.55	52.30%
Genesis Emerging Markets	0.78	7.79	88.06%
RXR Secured Note	0.21	5.21	1.14%
Arrowsmith Fund	0.37	2.28	-
Blue Rock Capital	0.09	3.78	-
Dean Witter Cornerstone	-0.03	0.22*	31.62%
GAMut Investments	0.06	0.67*	57.58%
Aquila International	0.7	8.42	72.07%
Bay Capital Management	0.24	2.1	27.85%
Blenheim Investments LP	0.1	0.56*	22.77%
Red Oak Commodity	0.7	4.23	21.60%

Table 2: This table summarizes some basic properties of dependence of the hedge funds under consideration in this article. The second column provides the value of the β between each fund and the Standard & Poor's 500 index, while the third column gives the value of the t-statistic allowing to assess the significance of the estimated β. Figures decorated with a star indicates a β not significantly different from zero, at the 5% level. The fourth column shows the linear correlation coefficient between each fund and the CSFB/Tremont index corresponding to its style.

Correlation matrix of hedge funds monthly returns

100	14	-3	0	-20	-13	-33	-4	-15	5	7	8	-24	-7	-2	-28
14	100	-1	-13	-5	-2	-5	6	-10	28	11	-2	3	2	-5	7
-3	-1	100	35	35	21	31	12	16	11	1	-13	30	-5	0	12
0	-13	35	100	25	31	40	-3	16	-4	3	-8	31	9	-4	11
-20	-5	35	25	100	40	50	21	18	30	-15	5	49	12	1	16
-13	-2	21	31	40	100	40	8	14	7	2	1	34	-5	-2	2
-33	-5	31	40	50	40	100	15	22	10	-8	-9	64	14	16	31
-4	6	12	-3	21	8	15	100	8	25	37	41	25	10	11	31
-15	-10	16	16	18	14	22	8	100	1	-9	-1	17	7	-1	11
5	28	11	-4	30	7	10	25	1	100	8	1	21	13	-5	13
7	11	1	3	-15	2	-8	37	-9	8	100	37	-3	7	6	17
8	-2	-13	-8	5	1	-9	41	-1	1	37	100	-2	10	22	29
-24	3	30	31	49	34	64	25	17	21	-3	-2	100	14	6	24
-7	2	-5	9	12	-5	14	10	7	13	7	10	14	100	10	9
-2	-5	0	-4	1	-2	16	11	-1	-5	6	22	6	10	100	50
-28	7	12	11	16	2	31	31	11	13	17	29	24	9	50	100

Table 3: Correlation matrix of the monthly returns of the sixteen hedge funds. Rows (and columns) follow the same order as in tables 1, 2.

Efficient portfolios for a 1.2% level of expected return

Funds	Gaussian VaR	Empirical VaR	GLS VaR	Kernel VaR
Axa Rosenberg	0.00%	**4%**	0%	0.90%
Discovery MasterFund	0.50%	1.10%	1.70%	2.20%
Aetos Corp	**6.90%**	0.20%	0.10%	0.10%
Bennet Restructuring	**30.50%**	**35.20%**	**41.20%**	**37.10%**
Calamos Convertible	0.00%	0.40%	0.00%	0%
Sage Capital	**27.20%**	**5.90%**	**12.50%**	**15.50%**
Genesis Emerging Markets	0.00%	0.50%	0.00%	0.70%
RXR Secured Note	2.90%	1.10%	2.50%	0.40%
Arrowsmith Fund	**4.10%**	**6.20%**	3.00%	**4.80%**
Blue Rock Capital	**7.20%**	**23.50%**	**19.20%**	**16.10%**
Dean Witter Cornerstone	0.00%	0.70%	0.00%	0.80%
GAMut Investments	**20.10%**	**19.40%**	**19.60%**	**19.10%**
Aquila International	0.00%	0.80%	0.00%	1.40%
Bay Capital Management	0.00%	0.10%	0.00%	0.00%
Blenheim Investments LP	0.50%	0.00%	0.00%	0.00%
Red Oak Commodity	0.00%	0.90%	0.00%	0.80%

Table 4: The table provides the optimal allocation according to the four VaR estimators. Figures in bold emphasize the funds whose weight is larger than 4%.

Distance between efficient portfolios for a level r of expected return

	$r = 1.75\%$			$r = 1.15\%$		
	Gaussian VaR	Kernel VaR	GLS VaR	Gaussian VaR	Kernel VaR	GLS VaR
Kernel VaR	0.26			0.61		
GLS VaR	0.21	0.15		0.46	0.23	
Empirical VaR	0.28	0.08	0.12	0.57	0.08	0.24

Table 5: Distance between two optimal portfolios computed under VaR constraints given by different estimators for a level of expected return r.

Rank correlation of risk measures

	Kernel VaR	Expected Shortfall	Semi-Variance	Variance	Skewness
Expected Shortfall	96%				
Semi-Variance	95%	98%			
Variance	93%	95%	99%		
Skewness	23%	25%	32%	40%	
Kurtosis	-3%	6%	15%	15%	38%

Table 6: Rank correlation between various risk measures and descriptive statistics for the sixteen hedge funds listed in table 1.

Efficient portfolios for a 1.2% level of expected return

Funds	Expected-Shortfall	Kernel VaR	Semi-variance	Variance
Axa Rosenberg	**6.30%**	0.90%	0.00%	0.00%
Discovery MasterFund	**4.70%**	2.20%	1.30%	0.50%
Aetos Corp	0.00%	0.10%	3.50%	**6.90%**
Bennet Restructuring	**12.00%**	**37.10%**	**29.90%**	**30.50%**
Calamos Convertible	0.00%	0.00%	0.00%	0.00%
Sage Capital	**26.00%**	**15.50%**	**14.00%**	**27.20%**
Genesis Emerging Markets	0.00%	0.70%	0.00%	0.00%
RXR Secured Note	0.00%	0.40%	**8.90%**	2.90%
Arrowsmith Fund	**8.70%**	**4.80%**	**4.30%**	**4.10%**
Blue Rock Capital	**8.00%**	**16.10%**	**16.50%**	**7.20%**
Dean Witter Cornerstone	3.00%	0.80%	0.00%	0.00%
GAMut Investments	**27.00%**	**19.10%**	**20.90%**	**20.10%**
Aquila International	0.00%	1.40%	0.00%	0.00%
Bay Capital Management	**4.30%**	0.00%	0.70%	0.00%
Blenheim Investments LP	0.00%	0.00%	0.10%	0.50%
Red Oak Commodity	0.00%	0.80%	0.00%	0.00%

Table 7: The table provides the optimal allocation according to the four risk measures. Figures in bold emphasize the funds whose weight is larger than 4%.

Distance between efficient portfolios for a level r of expected return

	$r = 1.75\%$			$r = 1.15\%$			$r = 1.75\%$		
	VaR	ES	Stdv	VaR	ES	Stdv	VaR	ES	Stdv
ES	0.34			0.85			0.16		
Stdv	0.48	0.63		0.61	0.61		0.26	0.26	
SV	0.30	0.40	0.28	0.36	0.67	0.34	0.21	0.21	0.05

Table 8: Distance between two optimal portfolios computed under different risk measures (VaR: Value-at-Risk, ES: Expected-Shortfall, Stdv: Standard deviation, SV: Semi-variance) for a level of expected return r.

Risk decomposition of VaR constraints

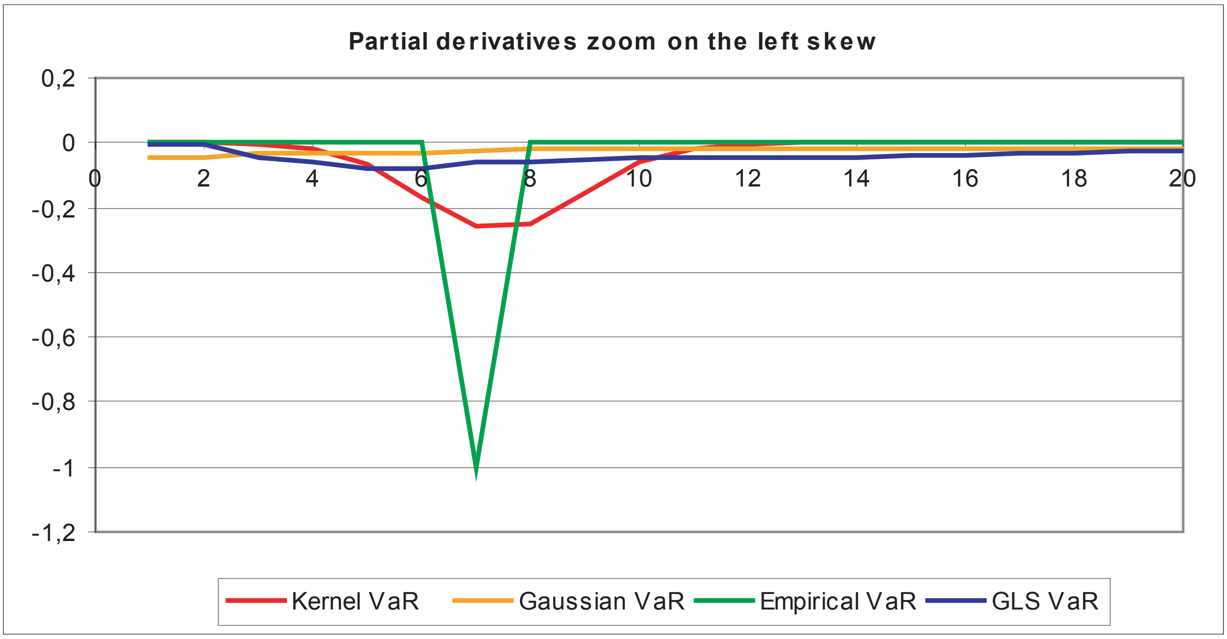

Figure 1: The figure provides the weights involved in the calculation of the different VaR estimators for each ordered statistics $(a'r)_{i:n}$

Efficient frontiers in the mean - Empirical VaR diagram

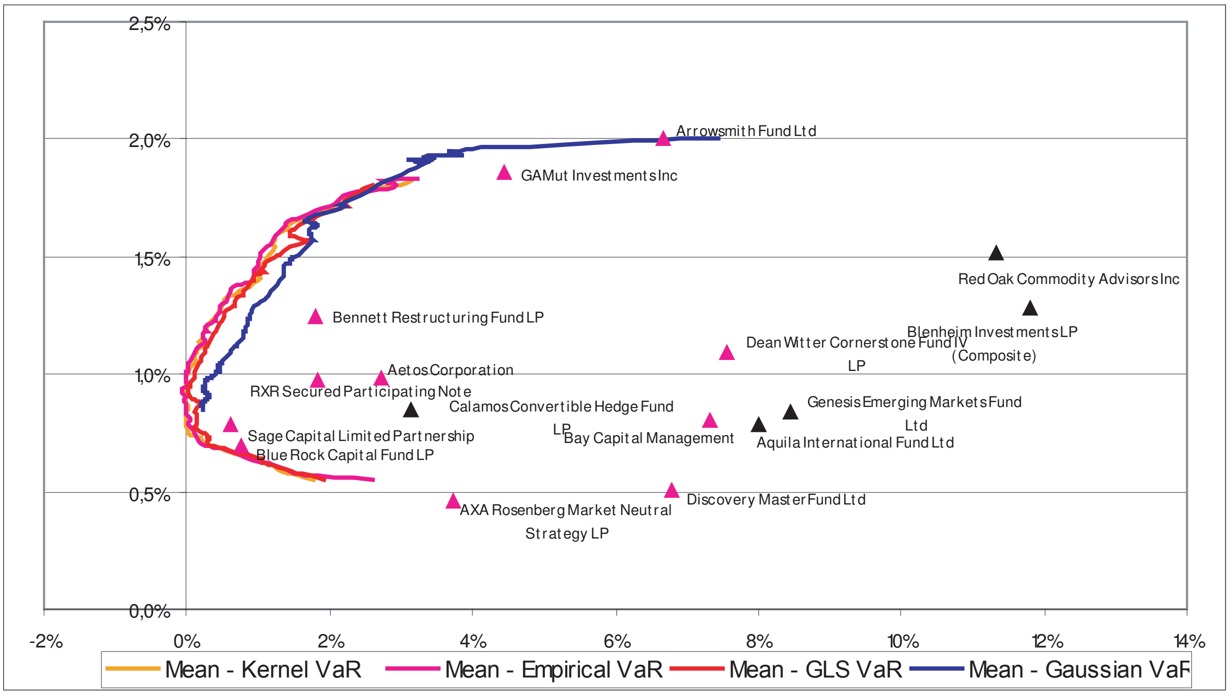

Figure 2: Mean-VaR efficient frontiers obtained from the four VaR estimators defined in paragraph 2.2 at the 5% probability level. Efficient frontiers are graphed in the mean-empirical VaR diagram

Efficient allocations

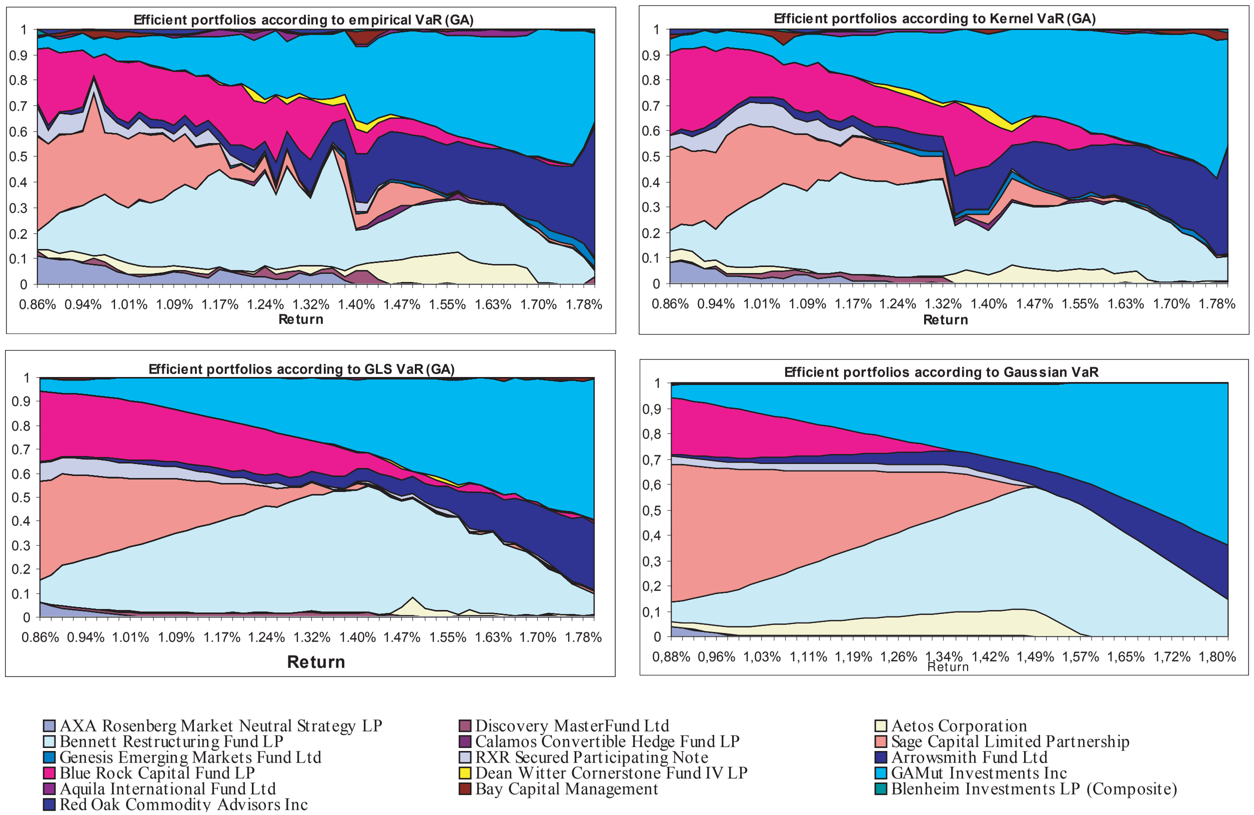

Figure 3: The four graphs represent the evolution of the optimal allocation with respect to the level of expected return for the four different VaR estimators.

Participation ratio

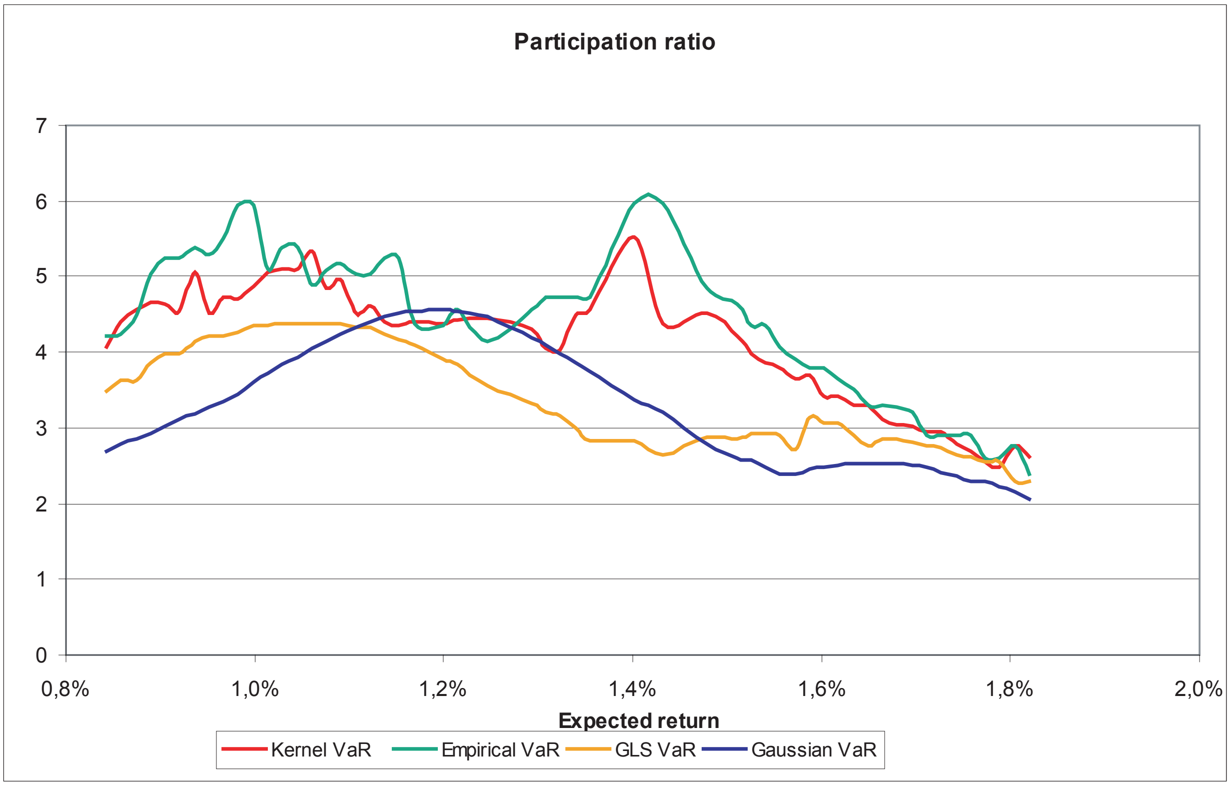

Figure 4:

Efficient frontiers in the mean - Empirical VaR diagram with S&P 500

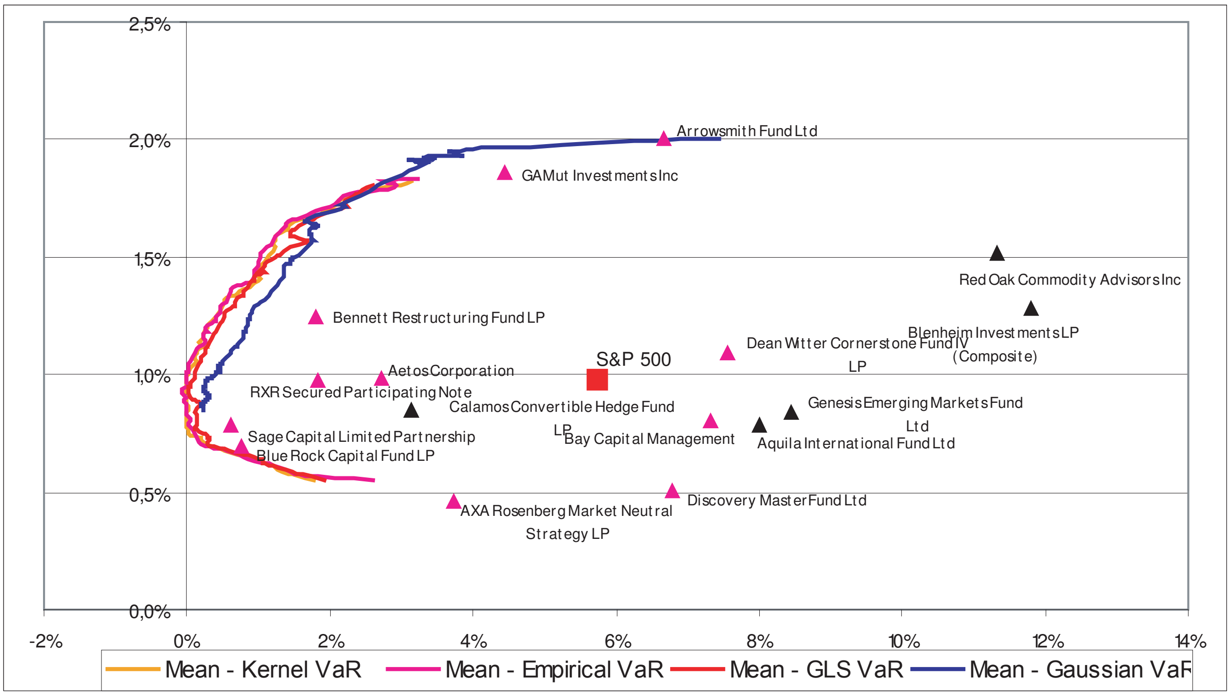

Figure 5: Mean-VaR efficient frontiers obtained from the four VaR estimators defined in paragraph 2.2 at the 5% probability level, when the Standard & Poor's 500 index it added to the set of hedge-funds. Efficient frontiers are graphed in the mean-empirical VaR diagram

Risk weights

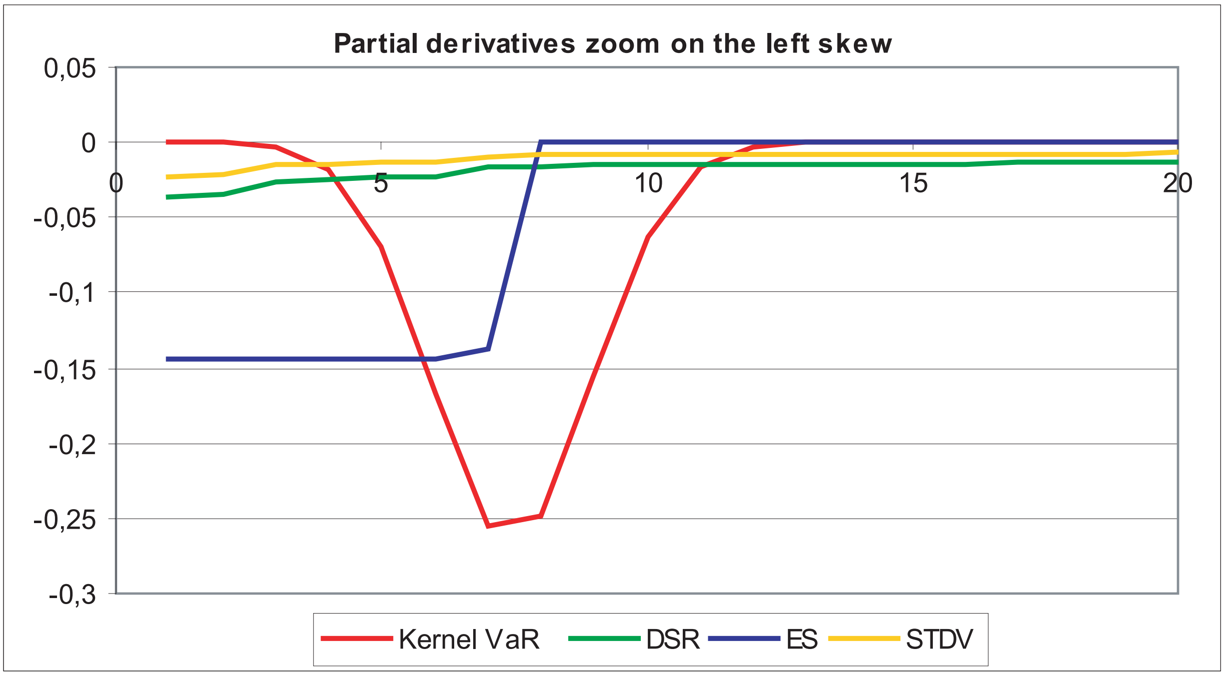

Figure 6: The figure provides the weights involved in the calculation of the Kernel VaR estimators, the semi-variance estimator (DSR), the Expected-Shortfall estimator (ES) and the standard deviation (STDV) for each ordered statistics $(a'r)_{i:n}$.

Efficient frontiers

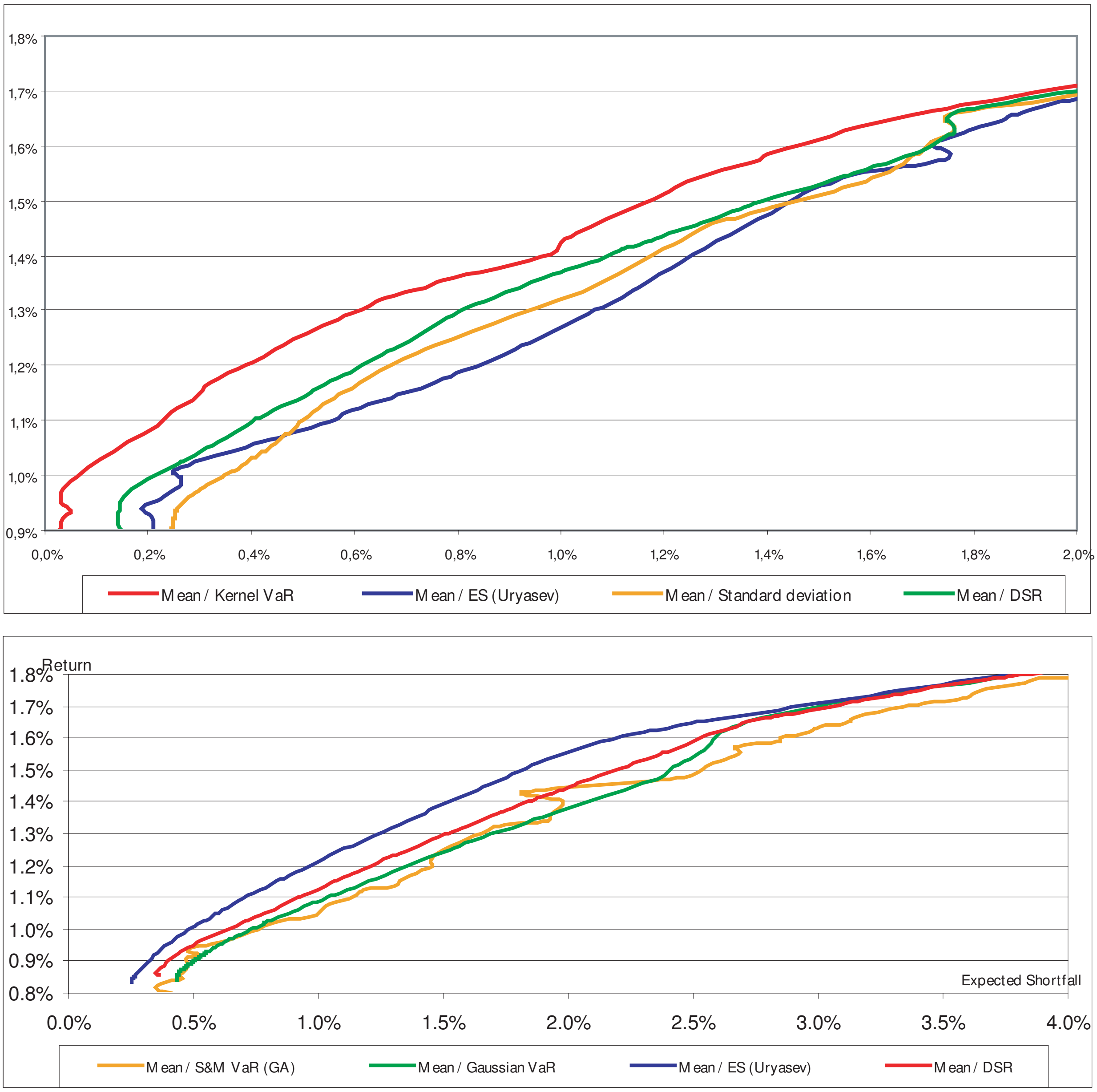

Figure 7: Mean-VaR efficient frontiers in the Mean-Empirical VaR diagram (upper panel) and in the Mean-ES diagram (lower panel).

Efficient allocations

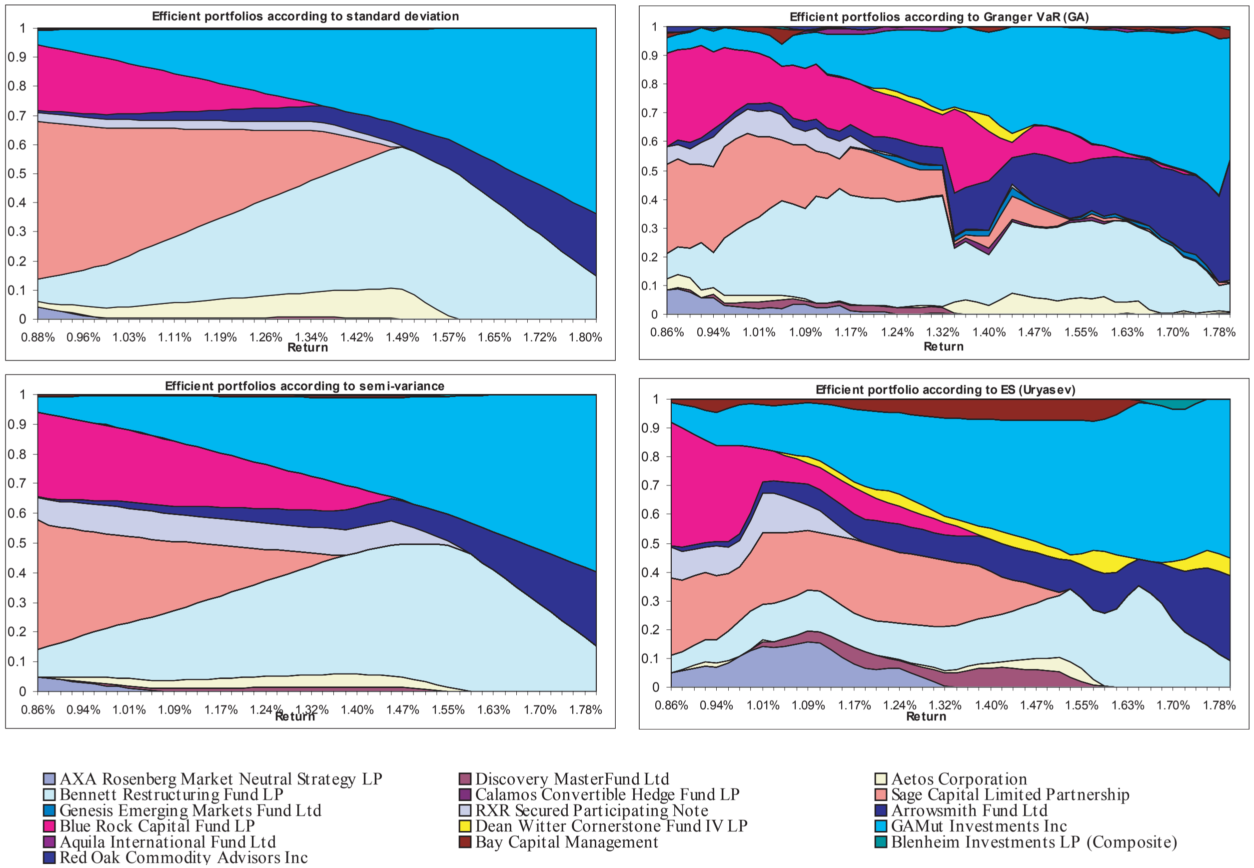

Figure 8: The four graphs represent the evolution of the optimal allocation with respect to the level of expected return for the four different risk estimators.

Participation ratio

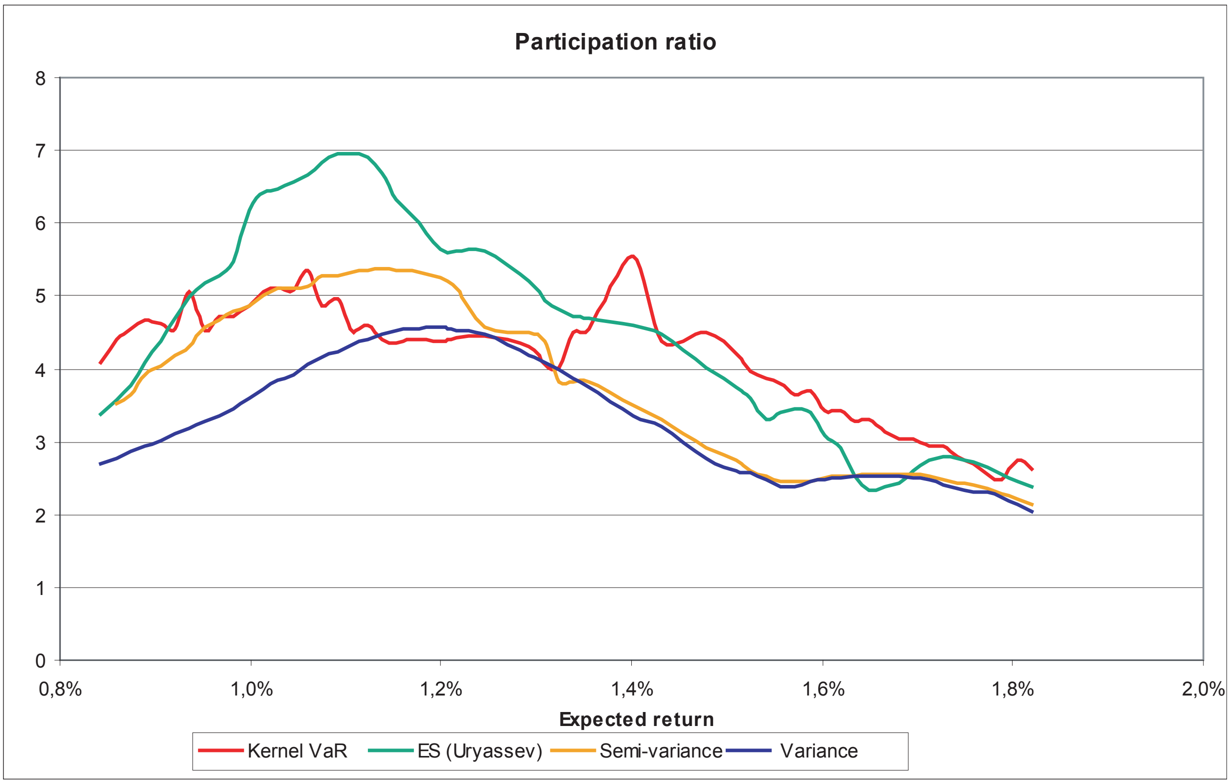

Figure 9: