CALIBRATION AND IMPLEMENTATION OF THE GENERALIZED VASICEK MODEL *

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Abstract

We present the problem of extracting implied parameters of the spot rate process under the risk-neutral probability, given a finite set of observed pure discount bond, cap and swaption prices. The calibration process is done analytically, assuming that the spot rate process follows a generalized Ornstein-Uhlenbeck process and takes into account the usual date treatments in the financial markets. We give necessary and sufficient conditions for the calibration process to perform well. We show that the constraints determine the prices of a wide range of interest rate options. This allows some kind of interpolation between interest rate derivative prices with no arbitrage opportunity and to make practical use of a misspecified yield curve model.

Résumé

Nous examinons le problème de l'extraction de paramètres non observés du processus du taux court sous la probabilité risque-neutre, étant donné un ensemble fini de prix observés de zéro-coupons, de caps et d'options de swaps. La calibration est faite de manière explicite, en supposant que le taux court suit un processus d'Ornstein-Uhlenbeck généralisé et prend en compte les conventions usuelles en matière de dates de valeur. Nous donnons des conditions nécessaires et suffisantes pour que l'ensemble des paramètres compatibles avec les contraintes soit non vide. Nous montrons que les contraintes déterminent les prix d'une large gamme d'options de taux d'intérêt. La procédure présentée permet de faire des interpolations entre des prix de produits dérivés des taux d'intérêt de manière cohérente et d'utiliser en pratique un modèle de courbe des taux mal spécifié.

keywords : Linear Gauss-Markov model, calibration, interpolation, volatility matrix, index amortizing swaps, change of numeraire, Monte Carlo simulations.

1 INTRODUCTION.

A yield curve model can be seen as a system whose outputs are prices and hedging strategies of interest rate derivatives. The outputs depend on some unobserved functional parameters in the drift and the diffusion coefficient of the spot rate process, under the risk-neutral probability. On another hand, a set of prices of frequently traded assets is observed and this induces some constraints on the functional parameters. The process that leads to express the set of the functional parameters given the set of constraints is called calibration. Once

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this is done, then the yield curve model can be effectively used, and the observed prices become the true inputs of the system.

One important example is given with the calibration to the initial yield curve in the Heath-Jarrow-Morton (1992) framework of "evolutionnary models". Unlike previous equilibrium or "partial equilibrium" models with Markovian state variables (Cox, Ingersoll and Ross (1985), Brennan and Schwartz (1979)), fitting the initial yield curve is straightforward and the initial yield curve indeed becomes an input of the model. But this framework does not take as exogeneous the prices of interest rate options, such as caps and swaptions that are now frequently traded assets with deep markets and low transaction costs.

When using time homogeneous volatilities for bond prices, one cannot perfectly reproduce a large number of cap and swaption prices. One way, illustrated by Brace and Musiela (1994), is to increase the number of factors until the errors become small enough, compared to the bid-ask spreads on the observed option prices. This can lead to time intensive computations when the number of factors is large.

Another approach illustrated by Jamshidian (1991), Hull and White (1990),(1993), Black and Karasinski (1991), Cherbonnier and Laurent (1993) consists in using non time homogeneous volatilities to widen the space of attainable option prices. The first papers consider as an input some "initial volatility curve" which is not observed and can only be approximated using prices of short-term options. Moreover, most of these papers give some procedures to build up binomial trees or other kinds of discrete approximations (Ho and Lee (1986), Black, Derman and Toy (1990), Black and Karasinski (1991), Hull and White (1990), Heath, Jarrow and Morton (1990), Jamshidian (1991)) which generate prices consistent with the observed prices but do not address the question whether these procedures will give some parameters or not. Cherbonnier and Laurent (1993) give necessary and sufficient conditions for identification of the parameters of a generalized Vasicek model provided that a continuum of pure discount bonds and cap prices is observed.

There is a consistency problem when using a continuous set of prices to identify the implied parameters. Since the number of really observed prices is always finite, some kind of interpolation is required to complete the set of prices. For instance, one may interpolate with spline functions the Black-Scholes term structure of implied cap or swaption volatilities. In some cases, the set of functional parameters consistent with the continuous set of generated prices will be empty, while the set of functional parameters consistent with the true set of observed prices is not empty, due to the particular specification for interpolating observed prices.

In this paper, we rather consider that a finite set of observed pure discount bond, cap and swaption prices is given, we remain in the framework of the one factor linear Gauss Markov model[2] and we allow for the three parameters of the spot rate process to be time-dependent. Since the implied parameters are functional, there is in general underidentification, but nevertheless, it may be sometimes impossible to fit the model to consistent option prices. This means that the set of attainable cap and swaption prices is strictly smaller than the set of consistent cap and swaption prices.

Since our framework is the one factor linear Gauss-Markov (LGM thereafter) model, the calibration process is explicit and simple, but this model does not allow the calibration of the strike structure of implied volatilities. We rather focus on the calibration of the term structure of implied volatilities[3].

We show that the observed prices completely determine the prices of wide range of path dependent options, provided that the set of payment and fixing dates remain the same as the one used for calibration.

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[2]This model in its one and multifactor version is already well documented ; see for instance, Vasicek (1977), Langetieg (1980), El Karoui, Mynéni and Viswanathan (1992) and Jamhshidian (1993).

[3]We can notice that in yield curve modelling, the term structure of implied volatilities is represented by a matrix since the implied volatility depends both on the maturity of the option and of the underlying bond.

2 CASH-FLOWS.

At each current period n, three dates will be used : f_n, the fixing date, d_{n-1}, the beginning of the application period, and d_n, the payment date. We denote $P(t,T)$, the price at time t of a discount bond paying 1 at time T.

We will then work with the following increasing sequences of dates :

$$f_1 \quad d_0 \quad f_2 \quad d_1 \quad . \quad . \quad . \quad f_n \quad d_{n-1} \quad f_{n+1} \quad d_n \quad . \quad . \quad d_N, \quad f_1 \geq 0,\ d_N \leq T_f. \tag{1}$$

In most cases f_1 will be the current date and will be set to 0. Let us consider the Pibor rate negociated at current date f_n ; the payment will occur at some date, d_{n-1} (it may be two business days after f_n) and the borrower will redeem at date d_n. We then get :

$$P(f_n, d_{n-1}) = \left(1 + \frac{J(d_{n-1}, d_n)}{360} Pibor(f_n)\right) P(f_n, d_n), \tag{2}$$

where J measures the number of days between two dates. We can notice that the Pibor rate is indeed a forward rate, because of the time interval between the fixing date and the payment date.

2.1 CAPLET PAYOFF.

The payoff at time d_n of the current caplet will then be :

$$POC^*(n,c) = [Pibor(f_n) - c]^+ \frac{J(d_{n-1}, d_n)}{360}. \tag{3}$$

We can notice that the payoff of the caplet is known until the fixing date f_n. In a frictionless market, this is equivalent to receiving the following payment at time d_{n-1} :

$$POC(n,c) = \left[1 - (1 + c\frac{J(d_{n-1}, d_n)}{360}) P(f_n, d_{n-1}, d_n))\right]^+, \tag{4}$$

where $P(f_n, d_{n-1}, d_n) = \dfrac{P(f_n, d_n)}{P(f_n, d_{n-1})}$ is the forward price at time d_{n-1} seen from date f_n of the discount bond maturing at d_n. It is well known (see for instance Briys, Crouhy and Schobel (1991)) that without date treatments a caplet is put on a discount bond. With usual date treatments, it is a put on the forward price of a discount bond.

2.2 SWAPTION PAYOFF.

Let us denote by r the strike of the swaption and d_{exe}, the "exercise date". The exercise date is assumed to be a trading day and to be one of the dates d_n, let's say d_i. This is not important for pricing but for calibration reasons. When the exercise date is one of the dates d_n, then it is quite easy to calibrate the LGM model with parsimonious notations. Moreover, this case will be the most common. When pricing swaptions where the exercise date is not one of the d_n, the pricing formulas remain valid but cannot be linked to payment dates of standard Libor caps.

$$d_i = d_{exe}. \tag{5}$$

The maturity of the underlying swap is assumed to be $4N$, where N is the maturity in years. The swaption is in practice exercised (the exercise is known) at the first trading day before $d_{exe} = d_i$, that is, f_{i+1}. d_i is a central date since it allows computations of further $d_n,\ n > i$. In three months Libor swaptions, the floating leg has quarterly payments, at times d_n, while the fixed leg is paid yearly, at times d_{4n}. The payoff of the swaption is the positive part of the net present value of the underlying swap at time f_{i+1}, that is :

$$\text{payoff}(f_{i+1}) = \left[r \left\{ \sum_{n=1}^{N} P(f_{i+1}, d_{i+4n}) \right\} + P(f_{i+1}, d_{i+4N}) - P(f_{i+1}, d_i) \right]^+ . \tag{6}$$

For computing explicit pricing formulas, we will choose as a numeraire the discount bond maturing at time d_i. In units of the numeraire, the payoff becomes :

$$\text{payoff}(d_i) = \left[r \left\{ \sum_{n=1}^{N} P(f_{i+1}, d_i, d_{i+4n}) \right\} + P(f_{i+1}, d_i, d_{i+4N}) - 1 \right]^+ . \tag{7}$$

This payoff corresponds to a payment at time d_i, which is the usual practice in the market. We can notice that, with usual date treatments, this is a call on the forward price of a coupon bond.

3 REPARAMETRIZATION OF A MARKOVIAN VOLATILITY STRUCTURE.

We assume that the process of the discount bond price is given by :

$$dP(t,T) = r(t)P(t,T)dt + \sigma(t,T)P(t,T)dW_t, \quad \forall\, t \in [0, T_f], \tag{8}$$

where $\sigma(t,T)$ is a deterministic volatility function of discount bond prices and $[0, T_f]$ is the time interval over which prices are studied. We know from El Karoui et al. (1991), Hull and White (1990), Caverhill (1994), Ritchken and Sankarasubramaniam (1995), that the spot rate $r(t)$ is markovian if and only the volatility function can be written as :

$$\boxed{\sigma(t,T) = \sigma(t) \int_t^T \exp\left(- \int_t^v \lambda(u)du \right) dv, \quad \forall\, t, T \text{ such that} : 0 \le t \le T \le T_f.} \tag{9}$$

$\sigma(t)$ and $\lambda(t)$ are directly linked to the functional parameters of the spot rate process. Knowledge of $\sigma(t,T)$ allows the pricing of options on general coupon bonds (i.e caps, floors, swaptions, amortizing swaptions...). For the volatility of the discount bond to be correctly defined, we can notice that λ must be integrable over any subinterval of $[0, T_f]$.

In the calibration of the LGM model through cap and swaption prices, the following functions are introduced :

Definition 1 *:*

$$\boxed{G_{a,b}(t) = \int_a^t \exp\left(- \int_b^v \lambda(u)du \right) dv,} \tag{10}$$

$$\boxed{H(t) = \int_0^t \frac{\sigma^2(s)}{G'^2(s)} ds.} \tag{11}$$

It can be easily seen that :

$$G_{a,b}(t) = \frac{G_{0,0}(t) - G_{0,0}(a)}{G'_{0,0}(b)}. \tag{12}$$

The functions G and H allow a reparametrization of the volatility function $\sigma(t,T)$ under the markovian assumption :

Property 1 *:*

(i) Let G be a real increasing and twice differentiable function defined over $[0, T_f]$, then we can define the mean-reversion parameter by :

$$\lambda(t) = -\frac{G''(t)}{G'(t)}. \tag{13}$$

(ii) Let H be a real positive, increasing and differentiable function defined over $[0, T_f]$, then we can define an instantaneous volatility of the spot rate by :

$$\sigma(t) = G'(t)\sqrt{H'(t)}. \tag{14}$$

(iii) The volatility structure of the discount bond prices given by :

$$\sigma(t, T) = \sqrt{H'(t)}\,[G(T) - G(t)], \tag{15}$$

implies the Markov property for the spot rate process.

The previous property simply states that given suitable functions G and H, it is possible to derive a markovian structure for the volatility function $\sigma(t, T)$ (i.e mean-reversion and instantaneous volatility parameters). We can notice that the functions $AG + B$ and H where A, B are two real numbers, A positive, lead to the same volatility structure $\sigma(t, T)$.

4 DISCRETE SEQUENCES OF VOLATILITIES.

It is convenient to introduce some volatility functions for the pricing and the calibration of options. Let us assume that functions G and H are known. Then, we can define the basis volatility functions $R(i, j, k)$ and $S(i, k)$ by :

Definition 2 *: Let us define the real functions R and S on $I_R \subset \mathbf{N}^3$ and $I_S \subset \mathbf{N}^2$ by :*

$$\begin{aligned} R(i, j, k) &= [G(d_k) - G(d_j)]^2 H(f_i),\ \forall (i, j, k) \in I_R, & (16)\\ S(i, k) &= R(i, i-1, k),\ \forall (i, k) \in I_S. & (17) \end{aligned}$$

We can say that R and S are generated by the increasing sequences of real numbers $\bar{G} = \{G(d_i), i \in I_G \subset \mathbf{N}\}$ and $\bar{H} = \{H(f_j), j \in I_H \subset \mathbf{N}\}$.

We can notice that : $R(i, j, j) = 0, \quad R(i, j, k) = R(i, k, j), \quad S(i, i-1) = 0.$

Definition 3 *: Let us denote by $C_{\bar{G},\bar{H}}$ the space of increasing functions G, H defined on $[0, T_f]$, such that G is twice differentiable, H is positive and differentiable, the first derivatives of G and H being uniformly bounded by positive numbers a_G and a_H and such that G and H are compatible with the increasing sequences $\bar{G} = \{G_i,\ i \in I_G \subset \mathbf{N}\}$ and $\bar{H} = \{H_j,\ j \in I_H \subset \mathbf{N}\}$.*

Property 2 *:*

(i) $C_{\bar{G},\bar{H}}$ is a convex closed subset of $C^2_{[0,T_f]} \times C^1_{[0,T_f]}$ under the uniform convergence topology.

(ii) $C_{\bar{G},\bar{H}}$ is non empty if and only if :

$$a_G < \frac{G_i - G_j}{d_i - d_j},\ \forall i, j \in I_G, \tag{18}$$

$$a_H < \frac{H_i - H_j}{f_i - f_j}, \ \forall i, j \in I_H. \tag{19}$$

The previous property simply states that given increasing sequences of real numbers $\bar{G}$ and $\bar{H}$, any functions G, H in $C_{\bar{G},\bar{H}}$ might be taken to build the volatility function $\sigma(t,T)$.

In practice, the price functions will only use $R(i,j,k)$ with $k > j \geq i-1$. The volatility function $S(i,k)$ is used in the price formulas of three month caps, floors and swaptions (plain vanilla or with a given predetermined amortizing rule). We show further that $\{S(i,j)\}$ is the matrix of volatilities of options of all maturities f_i on discount bonds of all maturities d_j. In practice, a lot of price functions will only use $S(i,j)$ with $k \geq i$.

Property 3 *: Let the functions R and S, defined on I_R and I_S, be generated by two increasing sequences $\bar{G}$ and $\bar{H}$; then we get :*

$$R(i,j,k) = \left(S(i,k)^{\frac{1}{2}} - S(i,j)^{\frac{1}{2}}\right)^2, \tag{20}$$

$$S(i,j) = S(i,i)\frac{\left(S(\mu,j)^{\frac{1}{2}} - S(\mu,i-1)^{\frac{1}{2}}\right)^2}{\left(S(\mu,i)^{\frac{1}{2}} - S(\mu,i-1)^{\frac{1}{2}}\right)^2}, \tag{21}$$

$\forall i, j, k, \mu$ such that S is defined.

Property 3 states that the sets of positive real numbers $\bar{S} = \{S(i,i), i \in \mathbf{N}\}$ and $S_\mu = \{S(\mu,j), j \in \mathbf{N}\}$ generate the functions S and R (i.e give the complete knowledge of function $S(i,j)$) :

$$S = \begin{pmatrix} . & S(1,i-1) & S(1,i) & . & . & S(1,j) \\ . & . & . & . & . & . \\ . & . & S(i,i) & . & . & S(i,j) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}. \tag{22}$$

The knowledge of the first line and of the diagonal of the matrix $\{S(i,j)\}$ completely specifies the volatility matrix. We show further that $\{S(i,i)\}$ is the vector of volatilities of caplets of all maturities and that $\{S(1,k)\}$ is the vector of volatilities of options of short maturity f_1 on discount bonds of all maturities d_k.

We now have to look at the derivation of functions H and G from the volatility function S.

Property 4 *: Let R and S two volatility functions generated by the series of positive real numbers $\bar{S}$ and S_μ. Then, there exist some generators $\bar{G}$ and $\bar{H}$ and they are defined by :*

$$G(d_j) = AS(\mu,j)^{\frac{1}{2}} + B, \tag{23}$$

$$H(f_i) = \frac{S(i,i)}{A^2\left(S(\mu,i)^{\frac{1}{2}} - S(\mu,i-1)^{\frac{1}{2}}\right)^2}, \tag{24}$$

where A and B are arbitrary real numbers.

Property 4 states that given the two vectors of volatilities (for caplets and swaptions with fixed exercise date), it is possible to derive the set of generators $\bar{G}$ and $\bar{H}$. Now, for $\bar{G}$ and $\bar{H}$ to generate some volatility structure $\sigma(t,T)$, the sequences must be increasing.

Property 5 : *Let R and S two volatility functions generated by the series of positive real numbers* $\bar{S} = \{S(i,i),\ i \in \mathbf{N}\}$ *and* $S_\mu = \{S(\mu, j),\ j \in \mathbf{N}\}$. *Then, there exist some increasing sequences* $\bar{G}$ *and* $\bar{H}$ *generating R and S if and only if the following properties are fulfilled :*

(i) $S(\mu, j)$ *is an increasing function of* j, *for all* $j \geq \mu$ *and a decreasing function of* j, *for all* $j \leq \mu$.

(ii) The sequence $\dfrac{S(i,i)}{\left(S(\mu,i)^{\frac{1}{2}} - S(\mu,i-1)^{\frac{1}{2}}\right)^2}$ *is increasing or equivalently the following property is fulfilled :* $S(i,i)^{\frac{1}{2}} + S(i+1,i) + 1^{\frac{1}{2}} > S(i,i+1)^{\frac{1}{2}},\ \forall i \in \mathbf{N}$.

We can notice that condition (i) is necessary and sufficient for the existence of increasing sequences $\bar{G}$. The first part of condition (ii) is in itself necessary and sufficient for the sequence $\bar{H}$ to be increasing. The second part of condition (ii) implies that the sequence $\bar{H}$ is increasing only if condition (i) is satisfied. We can notice that (i) and (ii) can be examined using only the sequences $\bar{S}$ and S_μ. With property 5, we can tell if a sequence of observed volatilities $\bar{S}$ and S_μ is compatible with a markovian volatility structure $\sigma(t,T)$.

Usually, we do not know the complete sequences $\bar{S}$ and S_μ. In that case, there might be several functions R and S compatible with the observed volatilities. The following property gives some results when the caplet volatilities are observed, but when the volatility vector of options on discount bonds with a fixed exercise date is partially observed.

Property 6 *Let R and S be two volatility functions generated by the series of positive real numbers* $\bar{S} = \{S(i,i),\ i \in \mathbf{N}\}$ *and* $S_\mu = \{S(\mu, n_j),\ j \in \mathbf{N},\ j \leq J\}$ *where* n_j *is an increasing sequence of date indices, with* $n_0 = \mu$. *Then, there exist some increasing sequences* $\bar{G}$ *and* $\bar{H}$ *generating R and S if and only if the following properties are fulfilled :*

(i) $S(\mu, n_j)$ *is an increasing function of* j, *for all* $j \leq J$.

(ii) The sequence $k_j = \dfrac{\sum_{i=n_j+1}^{n_{j+1}} S(i,i)^{\frac{1}{2}}}{S(\mu, n_{j+1})^{\frac{1}{2}} - S(\mu, n_j)^{\frac{1}{2}}}$ *is increasing in* j *and* k_0 *is greater than one.*

Proof : condition (i) is clearly necessary for the existence of an increasing sequence $\bar{G}$. It can be easily shown that if S is generated by increasing sequences $\bar{G}$ and $\bar{H}$, then : $k_j < \dfrac{H(f_{n_{j+1}})^{\frac{1}{2}}}{H(f_\mu)^{\frac{1}{2}}} < k_{j+1}$. Thus, k_j must be increasing.

Now, let us assume that conditions (i) and (ii) are satisfied. Let us define recursively the sequence S_i by :

$$S_\mu = S(\mu, \mu),$$

$$k_j = \frac{S(i,i)^{\frac{1}{2}}}{S_i^{\frac{1}{2}} - S_{i-1}^{\frac{1}{2}}},$$

if $i \in [n_j + 1, n_{j+1}]$. It can be seen recursively that : $S_{n_j} = S(\mu, n_j)$. Series $\{S_i,\ i \geq \mu\}$ and $\{S(i,i)\}$ satisfy the conditions (i) and (ii) of Property 5. Then there exist some increasing generating sequences $\bar{G}$ and $\bar{H}$.

To prove the previous property, we have exhibited some special simple generator with maybe some practical unlikely properties ; for instance, since $H(f_i)$ is piecewise constant, the volatility of the spot rate will stay at zero over some finite intervals. It can then be interesting to look at the set of generators in a more general way.

We assume that conditions (i) and (ii) of Property 6 are satisfied. Let us note by l_j, $l_j = \frac{1}{k_j}$. l_j is a decreasing sequence. Let us give another arbitrary decreasing sequence v_j such that : $1 > v_0 > l_0 > v_1 > l_2 > \ldots > v_j > l_j > v_{j+1} \ldots$. Now, we are going to build decreasing sequences :

$$w_i = \frac{S_i^{\frac{1}{2}} - S_{i-1}^{\frac{1}{2}}}{S(i,i)^{\frac{1}{2}}},$$

such that :

$$v_j > w_i > v_{j+1}, \ \forall i \in [n_j + 1, n_{j+1}],$$

and that S_i generated by w_i ($S_\mu = S(\mu, \mu)$) is compatible with the observed $S(\mu, n_j)$. Let us take an arbitrary decreasing sequence $\{\varepsilon_i\}$. We are going to look at sequences $\{w_i\}$ defined by :

$$w_i = l_j + a_j \varepsilon_i - b_j, \ \forall i \in [n_j + 1, n_{j+1}],$$

where $\{a_j\}$ and $\{b_j\}$ are two real sequences, $\{a_j\}$ positive. The compatibility condition may be written as :

$$\sum_{i=n_j+1}^{n_{j+1}} w_i S(i,i)^{\frac{1}{2}} = S(\mu, n_{j+1})^{\frac{1}{2}} - S(\mu, n_j)^{\frac{1}{2}}, \quad \forall j < J.$$

It can be written as : $\displaystyle\sum_{i=n_j+1}^{n_{j+1}} (a_j \varepsilon_i - b_j) S(i,i)^{\frac{1}{2}} = 0 \quad \forall j < J.$

w_i can now be written as : $w_i = l_j + a_j \left(\varepsilon_i - \displaystyle\sum_{i=n_j+1}^{n_{j+1}} p_i \varepsilon_i \right)$, $\forall i \in [n_j + 1, n_{j+1}]$, where p_i are some weights defined by :

$$p_i = \frac{S(i,i)^{\frac{1}{2}}}{\displaystyle\sum_{i=n_j+1}^{n_{j+1}} S(i,i)^{\frac{1}{2}}}, \quad \forall i \in [n_j + 1, n_{j+1}],$$

Eventually, a_j must be bounded so that w_i remain in the interval $]v_j, v_{j+1}[$. Any positive a_j such that :

$$a_j < \frac{v_j - l_j}{\varepsilon_{n_j+1} - \displaystyle\sum_{i=n_j+1}^{n_{j+1}} p_i \varepsilon_i}, \quad a_j < \frac{l_j - v_{j+1}}{\displaystyle\sum_{i=n_j+1}^{n_{j+1}} p_i \varepsilon_i - \varepsilon_{n_{j+1}}},$$

can generate some admissible w_i and thus some admissible generators $\bar{H}$ and $\bar{G}$.

5 PRICING FORMULAS WITH DATE TREATMENTS.

The pricing of caps, swaptions, options on discount bonds in the case of the generalized Vasicek model is already well known (see for instance El Karoui et al. (1991), Jamshidian (1989, 1993). In the following, we extend the classical results to take into account the "date treatments", i.e the fact that the payment of the strike and the delivery of the underlying asset (in case of delivery, payment of the value of the underlying asset in case of cash settlement) are delayed by some days ($d_{n-1} - f_n$ in our notations). Though theoretically quite simple and minor, this correction is practically important especially since it may change substantially some implied volatilities and lead to biases in the implied determination of the volatility structure $\sigma(t,T)$.

5.1 PRICING FORMULAS FOR THREE MONTH LIBOR CAPLETS.

Let us introduce the following notations : $K_n = 1 + r\frac{J(d_{n-1}, d_n)}{360}$. The price of the caplet is then given by :

$$\boxed{\begin{aligned} C_n(r, P(0, d_{n-1}), P(0, d_n), S(n, n)) &= P(0, d_{n-1})\phi(-V_n^*) - K_n P(0, d_n)\phi(-V_n^* - S(n, n)^{\frac{1}{2}}), \\ V_n^* &= \left(\log[K_n P(0, d_{n-1}, d_n)] - \frac{S(n, n)}{2}\right) S(n, n)^{-\frac{1}{2}}. \end{aligned}} \tag{25}$$

Proof : see the Appendix.

Now we look at the implicit volatility derivation. No static arbitrage opportunities imply that the price of the caplet C must be bounded :

$$\max[0, P(0, d_{n-1}) - K_n P(0, d_n)] \leq C \leq P(0, d_{n-1}).$$

Moreover, the following properties can be stated :

$$\lim_{S(n,n) \to 0} C_n(r, P(0, d_{n-1}), P(0, d_n), S(n, n)) = \max[0, P(0, d_{n-1}) - K_n P(0, d_n)],$$

$$\lim_{S(n,n) \to \infty} C_n(r, P(0, d_{n-1}), P(0, d_n), S(n, n)) = P(0, d_{n-1}),$$

$$\frac{\partial C}{\partial S(n, n)^{\frac{1}{2}}} > 0.$$

Property 7 *: Given an admissible caplet price, there exists a unique implied caplet volatility $S(n, n)$ in $[0, \infty[$.*

5.2 FORWARD CAPS.

The price of a forward cap is simply given by a sum of prices of caplets :

$$C_{n_1,n_2}(r, P(0, d_{n_1-1}), \ldots, P(0, d_{n_2}), S(n_1, n_1), \ldots, S(n_2, n_2)) = \sum_{n=n_1}^{n_2} C_n(r, P(0, d_{n-1}), P(0, d_n), S(n, n)).$$

The previous properties on caplets can be immediately extended and we can deduce :

Property 8 *: Given an admissible forward cap price, C_{n_1,n_2}, there exists a unique implied volatility $H(n_1, n_2)$ in $[0, \infty[$, i.e such that :*

$$C_{n_1,n_2} = \sum_{n=n_1}^{n_2} C_n(r, P(0, d_{n-1}), P(0, d_n), H(n_1, n_2)).$$

Moreover, we get : $\inf_{n \in [n_1, n_2]} S(n, n) \leq H(n_1, n_2) \leq \sup_{n \in [n_1, n_2]} S(n, n).$

$H(n_1, n_2)$ is some kind of average volatility for the caplets.

5.3 PRICES OF THREE MONTH LIBOR SWAPTIONS.

The price of the call swap is given by :

$$
\boxed{\begin{aligned}
CS_{i,N} &= P(0,d_{i+4N})\phi\left(V_N^* + S(i+1,i+4N)^{\frac{1}{2}}\right) \\
&\quad + r\sum_{n=1}^{N} P(0,d_{i+4n})\phi\left(V_N^* + S(i+1,i+4n)^{\frac{1}{2}}\right) - P(0,d_i)\phi(V_N^*), \\
1 &= r\sum_{n=1}^{N} P(0,d_i,d_{i+4n})\exp -\left\{S(i+1,i+4n)^{\frac{1}{2}}V_N^* + \frac{1}{2}S(i+1,i+4n)\right\} \\
&\quad + P(0,d_i,d_{i+4N})\exp -\left\{S(i+1,i+4N)^{\frac{1}{2}}V_N^* + \tfrac{1}{2}S(i+1,i+4N)\right\}.
\end{aligned}} \tag{26}
$$

Proof : see appendix. We can notice that in the standard option products that we have studied, we have only considered function $S(i,j)$. This is due to the fact that the first cash flow is three months after the exercise date. When looking at options on forward swaps, the volatility coming up in the price formula is $R(i,j,k)$ where $j \geq i$.

Property 9 *The price of the call swap is an increasing function of the volatilities* $S(i+1,i+4n)^{\frac{1}{2}}$ *:*

$$\frac{\partial CS_{i,N}}{\partial S(i+1,i+4n)^{\frac{1}{2}}} > 0, \quad \forall n \in [1,N].$$

Proof : The equation giving V_N^* can be rewritten as

$$
\begin{aligned}
\exp -\frac{V_N^*}{2} &= r\sum_{n=1}^{N} P(0,d_i,d_{i+4n})\exp -\frac{1}{2}\left\{V_N^* + S(i+1,i+4n)^{\frac{1}{2}}\right\}^2 \\
&\quad + P(0,d_i,d_{i+4N})\exp -\tfrac{1}{2}\left\{V_N^* + S(i+1,i+4N)^{\frac{1}{2}}V_N^*\right\}^2.
\end{aligned} \tag{27}
$$

This implies that $\dfrac{\partial CS_{i,N}}{\partial V_N^*} = 0$ and proves the property.

No static arbitrage opportunities implies the following bounds on swaption prices :

$$\left[r\left\{\sum_{n=1}^{N} P(0,d_{i+4n})\right\} + P(0,d_{i+4N}) - P(0,d_i)\right]^+ \leq CS_{i,N} \leq r\left\{\sum_{n=1}^{N} P(0,d_{i+4n})\right\} + P(0,d_{i+4N}). \tag{28}$$

It can be shown that :

$$\lim_{S(i+1,i+4N)\to\infty} CS_{i,N} = MCS_{i,N}, \tag{29}$$

with $MCS_{i,N}$ being given by :

$$
\boxed{\begin{aligned}
MCS_{i,N} &= (1+r)P(0,d_{i+4N}) + r\sum_{n=1}^{N-1} P(0,d_{i+4n})\phi\left(\bar{V}_N + S(i+1,i+4n)^{\frac{1}{2}}\right) - P(0,d_i)\phi(\bar{V}_N), \\
1 &= r\sum_{n=1}^{N-1} P(0,d_i,d_{i+4n})\exp -\left\{S(i+1,i+4n)^{\frac{1}{2}}\bar{V}_N + \frac{1}{2}S(i+1,i+4n)\right\}.
\end{aligned}} \tag{30}
$$

$MCS_{I,N}$ is the price of the sum of two simple payoffs : $1+r$ discount bonds maturing at d_{i+4N} and a call on an annuity paying r each year from d_{i+4} to $d_{i+4(N-1)}$ with strike 1.

$$\lim_{S(i+1,i+4N)\to 0} CS_{i,N} = mCS_{i,N}, \tag{31}$$

with $mCS_{i,N}$ being given by :

a) if $P(0,d_i,d_{i+4N}) < \dfrac{1}{1+r}$, then :

$$\boxed{\begin{aligned} mCS_{i,N} &= r\sum_{n=1}^{N-1} P(0,d_{i+4n})\phi\left(\bar{V}_N + S(i+1,i+4n)^{\frac{1}{2}}\right) \\ &\quad -(P(0,d_i)-(1+r)P(0,d_{i+4N}))\phi(\bar{V}_N), \\ 1-(1+r)P(0,d_i,d_{i+4N}) &= r\sum_{n=1}^{N-1} P(0,d_i,d_{i+4n})\exp-\left\{S(i+1,i+4n)^{\frac{1}{2}}\bar{V}_N + \frac{1}{2}S(i+1,i+4n)\right\}. \end{aligned}} \tag{32}$$

b) if $P(0,d_i,d_{i+4N}) \geq \dfrac{1}{1+r}$, then :

$$\boxed{mCS_{i,N} = r\sum_{n=1}^{N-1} P(0,d_{i+4n}) - (P(0,d_i)-(1+r)P(0,d_{i+4N})).} \tag{33}$$

$mCS_{i,N}$ is the price of a call on the previously defined annuity with a strike equal to $P(0,d_i)-(1+r)P(0,d_{i+4N})$.

6 CALIBRATION IN PRACTICE.

We have given some general conditions on option prices and implied volatilities, that must be fulfilled for the model to be calibrated. Let us now discuss how these conditions can be checked given observed prices of options.

We can notice that in the LGM model model, swaption (or bond option) prices depend only on the volatility structure $\sigma(t,T)$ and thus on the mean reversion parameter $\lambda(t)$ and the diffusion coefficient $\sigma(t)$. The calibration process goes then firstly through an identification of these two parameters using option prices. The third parameter may be identified with the knowledge of the initial yield curve. This is unlike the Cox, Ingersoll and Ross model where even with constant parameters, the three parameters of the spot rate process under the risk-neutral probability appear in option prices.

We calibrate some matrix term structure of volatilities, i.e we look at swaptions with different exercise dates and different underlying maturities. This later dimension is specific to yield curve modelling. We do not calibrate the strike structure of volatilities. Indeed, the LGM model is not well suited for such a calibration since the implied volatility of a caplet with a given strike determines the implied volatility of parent caplets for all strikes. This occurs whatever the number of factors may be. In some cases, the primary concern will be fitting the term structure of volatilities and in other cases fitting the smile curve will be more important. Of course, on a theoretical level, we would rather consider yield curve models with a wider set of attainable prices that would be able to handle both cases. But practically, this is at the expense of increasing the model complexity (at least a third functional parameter is required in the volatility function), and this does not guarantee that the more complex model will really be able to handle the observed implied volatilities.

The usual inputs will be cap prices of a given strike, for different maturities and some swaption prices with a given exercise date, different underlying maturities and may be different exercise rates. From a theoretical point of vue, the exercise rates of the caps and the swaptions may be chosen arbitrarily ; In practice, they will depend on the kind of interest rate derivative that has to be priced.

We can take advantage of the explicit pricing formulas for caps and swaptions to derive the parameters of the discount bond volatility function. We will work "forward", starting with short-term caps and short swaptions on short swaps and then increase the maturity of the caps and of the underlying swap in the swaptions. Induction here has not the same meaning as in Jamshidian (1991) since the evolution equation (or Fokker-Plank equation) is not used. Using non time homogeneous volatility functions allows an exact ajustment to the set of observed cap and swaption prices with a single factor model. This approach is different from Brace and Musiela (1994) where the volatilities are time homegeneous (thus depending on real parameters (λ_i, σ_i)), in a Gaussian multifactor framework.

6.1 CALIBRATION WITH A COMPLETE SET OF OPTION PRICES.

Let us first consider that we observe swaption prices with exercise date d_i and with all yearly underlying swap maturities, and that we also observe caplet prices corresponding to all current periods n. By property 7, the three month caplet prices will provide a set of well defined $S(j, j),\ j > 0$. To get the $S(i, i + 4n)$, we must proceed recursively : it is easily seen that $S(i, i + 4)$ is correctly defined. Let us assume that we know $S(i, i+4k), \quad k \leq n-1$. $S(i, i+4n)$ should then be known by solving equation (27). Unfortunately, this equation might not have any solution. Even if there is $S(i, i+4n)$ solving (27), we may have $S(i, i+4n) < S(i, i+4(n-1))$. In that case, the calibrating procedure fails. This means that the observed swaption price with underlying swap maturity of n years conditional is too small when compared to swaption with shorter underlying swaps ; in this case, the term structure of swaptions volatilities is too decreasing or too increasing to be explained by any one factor LGM model. This is due to restrictions on interest rate distributions in one factor models, that prevent large twists in the yield curve.

Once the $S(i, i + 4n)$ and $S(j, j)$ have been computed, we have to check compatibility conditions of property 6 and find some generating sequences $\bar{G}$ and $\bar{H}$.

In practice, not all relevant caplet prices and swaption prices are observed ; only prices of forward caps can easily be observed and some underlying swap maturities are not quoted on screens. The "missing prices" can be obtained by some interpolation method. The problem with this technique is the same as mentionned when generating a continuous set of prices. If there is a solution to the calibrating problem, it will have good properties (i.e it will consistent with the truly observed prices). If there is no solution, then this may due to the interpolation technique.

If compatible cap and swaption implied volatilities are extracted, then, by property 3, all the required functions for pricing complex options, mentionned in sections 7 and 8, would be known from the observation of standard options.

6.2 PIECEWISE CONSTANT MEAN REVERSION PARAMETER AND INCOMPLETE SET OF PRICES

The complete knowledge of function $S(k, l)$ from an incomplete set of observed caps and swaptions prices requires further assumptions. It can be achieved by assuming for instance that the parameter function $\lambda(t)$ is piecewise constant on intervals $[d_{4n}, d_{4(n+1)}[$ (the first interval being $[0, d_{i+4}[$). We will denote λ_n the value of $\lambda(t)$ on interval $[d_{i+4n}, d_{i+4(n+1)}[$ and λ_0, the value of $\lambda(t)$ on the first interval $[0, d_{i+4}[$. We assume here that we have all underlying swap maturities ; for instance, we have a set of underlying swap maturities equal to one year, two years, three years and so on up to ten years.

6.2.1 Calibration on the first interval $[0, d_{i+4}[$.

Let us choose $G(t) = G_{0,0}(t)$. We can notice that :

$$\left(\frac{S(i+1,i+4)}{S(i+1,i+1}\right)^{\frac{1}{2}} = \frac{G(d_{i+4}) - G(d_i)}{G(d_{i+1}) - G(d_i)}. \tag{34}$$

By integrating G', we get :

$$G(d_i) = \frac{1 - \exp -\lambda_0 d_i}{\lambda_0}, \tag{35}$$

$$G(d_{i+1}) = \frac{1 - \exp -\lambda_0 d_{i+1}}{\lambda_0}, \tag{36}$$

$$G(d_{i+4}) = \frac{1 - \exp -\lambda_0 d_{i+4}}{\lambda_0}, \tag{37}$$

$$\left(\frac{S(i+1,i+4)}{S(i+1,i+1)}\right)^{\frac{1}{2}} = \frac{\exp -\lambda_0 d_i - \exp -\lambda_0 d_{i+4}}{\exp -\lambda_0 d_i - \exp -\lambda_0 d_{i+1}}. \tag{38}$$

Solving this equation gives λ_0 and thus $G(d_i), G(d_{i+1}), G(d_{i+2}), G(d_{i+3}), G(d_{i+4})$. Then $H(f_{i+k}),\ k = 1 \dots 4$ is computed through :

$$H(f_{i+k}) = \frac{S(i+k,i+k)}{\left(G(d_{i+k}) - G(d_{i+k-1})\right)^2}, \quad k = 1,2,3,4. \tag{39}$$

Every term on the right hand side of the equation is already known. Since we know $G(d_i), G(d_{i+1}), G(d_{i+2}), G(d_{i+3}), G(d_{i+4})$ and $H(f_{i+1}), H(f_{i+2}), H(f_{i+3}), H(f_{i+4})$, we can compute the $S(k,l)$ for all $k, l \in [i+1, i+4]$.

We can notice that, since λ_0 is known, $G(d_l)$ is also known, $\forall l \in [0, i+4]$.

If we assume that we have caplet prices from time d_0, then volatility functions $S(k,k)$ are known for all $k \in [1, i+4]$. This allows the computation of $H(f_k),\ \forall k \in [1, i+4]$. In this case, we know the basic volatility functions $S(k,l),\ \forall (k,l) \in [1, i+4]^2$. This means that we are able to price swaptions with shorter exercise dates than the ones whose prices are observed.

Another interesting consequence of the previous property is that it is possible to completely derive the volatility function with only one swaption price (whatever the exercise date) and the whole set of caplet prices, assuming that the mean reverting parameter is constant. We have just seen this property for dates shorter than the exercise date of the swaption. For longer dates, $G(d_l)$ is known assuming $\lambda(t) = \lambda_0$ is constant through previous G expression. Then $H(f_l)$ is computed through its previous expression. This case is in fact very similar to Cherbonnier and Laurent (1993) where $\lambda(t)$ is assumed to be constant except that no spline interpolation is required to get the important volatility functions present in the pricing formulas.

6.2.2 Calibration of further periods

Let us assume that the calibration over time intervals up to $[d_{i+4(n-1)}, d_{i+4n}[$ has allowed the computation of $G(d_l),\ \forall l \in [0, i+4n]$, of $\lambda_l,\ \forall l \in [0, n-1]$ and of $H(f_n),\ \forall n \in [1, i+4n]$. By computing G', we get :

$$G'(t) = G'(d_{i+4n}) \exp -\lambda_n (t - d_{i+4n}),\ \forall t \in [d_{i+4n}, d_{i+4n+4}]. \tag{40}$$

By integrating G' between d_{i+4n} and d_{i+4n+4}, we get :

$$G(d_{i+4n+k}) - G(d_{i+4n}) = G'(d_{i+4n}) \frac{1 - \exp - \{\lambda_n (d_{i+4n+k} - d_{i+4n})\}}{\lambda_n},\ \forall k \in [1,4]. \tag{41}$$

We can notice that :

$$S(i+1, i+4n+4)^{\frac{1}{2}} = H(f_{i+1})^{\frac{1}{2}}(G(d_{i+4n+4}) - G(d_i)), \quad (42)$$

gives $G(d_{i+4n+4})$, since $S(i+1, i+4n+4)$ is known from swaption prices. Then equation (42) ($k=4$) gives λ_n, since $G(d_{i+4n})$ and $G'(d_{i+4n})$ are known. Once λ_n is known, the computation of $G(d_{i+4n+k}),\ k=1,\ldots,4$ is direct through equation (42). The computations of $H(f_{i+4n+k}),\ k=1,\ldots,4$ are done through the following equation :

$$H(f_{i+4n+k}) = \frac{S(i+4n+k, i+4n+k)}{(G(d_{i+4n+k}) - G(d_{i+4n+k-1}))^2},\ k=1,\ldots,4. \quad (43)$$

This completes the calibration.

7 A NEW NUMERAIRE.

In this section, we introduce a new numeraire that is convenient to price path dependent interest options such as index amortizing swaps.

We have already seen that investing at Libor is a forward investment : the funds are transfered to the borrower the open day after the negociation. We must then distinguish between a set of fixing (or settlement) dates, $f_1, \ldots, f_n, \ldots$ and a set of investment and payment dates, $d_0, d_1, \ldots, d_n, \ldots$. f_1 is the time at which the cap or the amortizing cap will be negociated. In the case of a cap d_0 will be the exercise date, d_1 will be usually the first payment date (if there is a first fixing). The two sets are ordered :

$$f_1 < d_0 < f_2 < d_1 < \ldots < f_n < d_{n-1} < f_{n+1} < \ldots < f_N < d_{N-1} < d_N,$$

d_N is the last payment date. There is no fixing just prior to the last payment date.

Investment at Libor implies negociation of the (forward) rate at time f_n, between times d_{n-1} and d_n. Libor at time f_n is defined by :

$$1 + \text{Libor}(f_n)\left[\frac{J(d_{n-1}, d_n)}{360}\right] = \exp[y(f_n, d_{n-1}, d_n)(d_n - d_{n-1})]. \quad (44)$$

Let us consider now the following iterated (or roll-over) Libor investment strategy :

- At time t, $f_{n-1} \le t < f_n$, the portfolio is invested in discount bonds that mature at time d_{n-1}. Thus, the value of the portfolio should be known at time d_{n-1}.

- At time f_n ($f_n < d_{n-1}$), this known amount is invested forward from time d_{n-1} to time d_n at the prevailing forward rate $y(f_n, d_{n-1}, d_n)$. Thus the portfolio is invested is discount bonds maturing at time d_n.

The initial value of the portfolio at time f_1 is : $\hat{\beta}(f_1) = 1$. This amount is invested in discount bonds maturing at time d_1. This can be duplicated as an investment between time f_1 and time d_0 and a forward investment between time d_0 and time d_1.

To end this iterated strategy, the last forward investement is made at time f_N and thus, the value of the portfolio is defined until time d_N.

The value of the numeraire is then given by :

$$\hat{\beta}(t) = P(t, d_n)\frac{1}{P(f_1, d_0)}\prod_{i=1}^{n}\frac{1}{P(f_i, d_{i-1}, d_i)},\ \forall t,\ f_n \le t \le f_{n+1}, \quad (45)$$

with the conventional notation $f_{N+1} = d_N$. In particular the value of $\hat{\beta}$ at time d_N is compounded at forward rates and is given by :

$$\hat{\beta}(d_N) = \frac{1}{P(f_1, d_0)} \prod_{i=1}^{N} \frac{1}{P(f_i, d_{i-1}, d_i)}, \tag{46}$$

$$\hat{\beta}(d_N) = \exp\left[(d_0 - f_1)y(f_1, d_0) + \sum_{i=1}^{N}(d_i - d_{i-1})y(f_i, d_{i-1}, d_i)\right], \tag{47}$$

$$\hat{\beta}(d_N) = \exp\left[(d_0 - f_1)y(f_1, d_0)\right] \prod_{i=1}^{N} \left(1 + \text{Libor}(f_i)\left[\frac{d_i - d_{i-1}}{360}\right]\right). \tag{48}$$

For $t \in [0, d_N]$, let's define the function $d(t)$ by : $d(t) = d_n$, such that : $f_n \leq t < f_{n+1} < d_n$. $d(t)$ is some kind of "strict" next payment date after time t.

$$d\hat{\beta}(t) = r(t)\hat{\beta}(t)dt + \sigma(t, d(t))\hat{\beta}(t)dW_t. \tag{49}$$

We denote $\hat{Q}$, the probability measure under which $\dfrac{P(t,T)}{\hat{\beta}(t)}$ is a martingale. By Girsanov's theorem, we know that :

$$\boxed{W_t = \hat{W}_t - \int_0^t \sigma(s, d(s))ds.} \tag{50}$$

where $\hat{W}_t$ is a standard brownian motion under $\hat{Q}$.

8 DYNAMICS OF PIBOR RATES UNDER THE RISK NEUTRAL PROBABILITY ASSOCIATED TO THE NEW NUMERAIRE.

Libor rates are directly linked to the associated forward rates $y(f_n, d_{n-1}, d_n)$. The dynamics of these rates are given by :

$$\begin{aligned} y(f_n, d_{n-1}, d_n) &= y(0, d_{n-1}, d_n) + \frac{G(d_n) - G(d_{n-1})}{d_n - d_{n-1}} \int_0^{f_n} \frac{\sigma(s)}{G'(s)} dW_s \\ &+ \frac{1}{2}\frac{G(d_n) - G(d_{n-1})}{d_n - d_{n-1}} \int_0^{f_n} \frac{\sigma^2(s)}{G'^2(s)} \left\{G(d_n) + G(d_{n-1}) - 2G(s)\right\} ds. \end{aligned} \tag{51}$$

The dynamics of the forward rate associated to the Libor savings account risk-neutral probability is given by :

$$\begin{aligned} y(f_n, d_{n-1}, d_n) &= y(0, d_{n-1}, d_n) + \frac{G(d_n) - G(d_{n-1})}{d_n - d_{n-1}} \int_0^{f_n} \frac{\sigma(s)}{G'(s)} d\hat{W}_s \\ &+ \frac{1}{2}\frac{G(d_n) - G(d_{n-1})}{d_n - d_{n-1}} \int_0^{f_n} \frac{\sigma^2(s)}{G'^2(s)} \left\{G(d_n) + G(d_{n-1}) - 2G(d(s))\right\} ds. \end{aligned} \tag{52}$$

This also gives the expression of the forward rate at the previous period :

$$\begin{aligned} y(f_{n-1}, d_{n-2}, d_{n-1}) &= y(0, d_{n-2}, d_{n-1}) + \frac{G(d_{n-1}) - G(d_{n-2})}{d_{n-1} - d_{n-2}} \int_0^{f_{n-1}} \frac{\sigma(s)}{G'(s)} d\hat{W}_s \\ &+ \frac{1}{2}\frac{G(d_{n-1}) - G(d_{n-2})}{d_{n-1} - d_{n-2}} \int_0^{f_{n-1}} \frac{\sigma^2(s)}{G'^2(s)} \left\{G(d_{n-1}) + G(d_{n-2}) - 2G(d(s))\right\} ds. \end{aligned} \tag{53}$$

From (52) and (53), it can be seen that :

$$\hat{E}[y(f_n, d_{n-1}, d_n) \mid y(f_{n-1}, d_{n-2}, d_{n-1})] = L(d_{n-2}, d_{n-1}, d_n) y(f_{n-1}, d_{n-2}, d_{n-1}) + M(f_{n-1}, f_n, d_{n-2}, d_{n-1}, d_n), \quad (54)$$

$$\hat{V}[y(f_n, d_{n-1}, d_n) \mid y(f_{n-1}, d_{n-2}, d_{n-1})] = N(f_{n-1}, f_n, d_{n-2}, d_{n-1}, d_n), \quad (55)$$

$$\boxed{L(d_{n-2}, d_{n-1}, d_n) = \frac{G(d_n) - G(d_{n-1})}{G(d_{n-1}) - G(d_{n-2})} \, \frac{d_{n-1} - d_{n-2}}{d_n - d_{n-1}}.} \quad (56)$$

$$M(f_{n-1}, f_n, d_{n-2}, d_{n-1}, d_n) = y(0, d_{n-1}, d_n) - \frac{G(d_n) - G(d_{n-1})}{G(d_{n-1}) - G(d_{n-2})} \, \frac{d_{n-1} - d_{n-2}}{d_n - d_{n-1}} y(0, d_{n-2}, d_{n-1}) + \frac{1}{2} \frac{G(d_n) - G(d_{n-1})}{d_n - d_{n-1}} \{[G(d_n) - G(d_{n-2})] H_1(f_{n-1}) + [G(d_n) - G(d_{n-1})][H_1(f_n) - H_1(f_{n-1})]\},$$

$$M(f_{n-1}, f_n, d_{n-2}, d_{n-1}, d_n) = y(0, d_{n-1}, d_n) - \frac{G(d_n) - G(d_{n-1})}{G(d_{n-1}) - G(d_{n-2})} \, \frac{d_{n-1} - d_{n-2}}{d_n - d_{n-1}} y(0, d_{n-2}, d_{n-1}) + \frac{1}{2} \frac{G(d_n) - G(d_{n-1})}{d_n - d_{n-1}} \{H_1(f_{n-1})[G(d_{n-1}) - G(d_{n-2})] + H_1(f_n)[G(d_n) - G(d_{n-1})]\}, \quad (57)$$

$$\boxed{M(f_{n-1}, f_n, d_{n-2}, d_{n-1}, d_n) = y(0, d_{n-1}, d_n) - L(d_{n-2}, d_{n-1}, d_n) y(0, d_{n-2}, d_{n-1}) + \frac{1}{2} \frac{G(d_n) - G(d_{n-1})}{d_n - d_{n-1}} \{H_1(f_{n-1})[G(d_{n-1}) - G(d_{n-2})] + H_1(f_n)[G(d_n) - G(d_{n-1})]\},} \quad (58)$$

$$\boxed{N(f_{n-1}, f_n, d_{n-2}, d_{n-1}, d_n) = \left[\frac{G(d_n) - G(d_{n-1})}{d_n - d_{n-1}}\right]^2 (H_1(f_n) - H_1(f_{n-1})).} \quad (59)$$

Once we get these conditional distributions, it is easy to simulate the price path of the numeraire, $\beta(\hat{f_i})$, $i = 1, \ldots, N+1$ and to value by Monte Carlo techniques any path dependent interest rate option, whose payoff is based on Pibor rates taken at fixing dates f_n and paid-off at some payment date d_i. We can notice that the prices of these particular payoffs depend only on the volatility matrix $S(i, j)$, that is determined by cap and swaption prices. Thus, while there may be a large set of probability measures consistent with observed prices, all these measures will give the same price for these particular payoffs. As a practical illustration, some of these index amortizing swap payoffs are embedded in retail mortgages supplied in the French market, or can be used in the hedging of prepayment options by mortgage banks.

9 CONCLUSION

We must emphasize that the LGM risk neutral probability that can be extracted from observed prices may be very different from the actual probability and cannot be used for prediction. Even the yield curve model is well specified, time varying risk premia will make it difficult to infer anything from the risk neutral probability. Moreover, if markets are incomplete, there will be hedging residual risks for most options and multiplicity of equivalent risk neutral probabilities. We do not even claim that the exhibited LGM risk neutral probability is equivalent to the the real one, since it precludes some twists in the yield curve, which we know may happen. The use of a simple complete market model in a maybe incomplete market world can merely be seen as a way to consistently extrapolate derivative prices, knowing the prices of basic derivatives (i.e avoiding simple stactic arbitrage opportunities like a payoff strictly dominating another payoff with a smaller price).

The approximated prices will be accurate provided that the calibrated LGM measure is not too far from the true pricing probability. In our framework, we can reasonably guess that this is true for near-the-money options with payoffs depending primarily on the level of rates, rather than on the shape of the yield curve. This includes a wide range of options such as index amortizing swaps, amortizing swaptions, captions or american style options.

One way to check out our guess would be to look at how stable through time are the implied conditional distributions of the "level of rates" (some average of rates of different maturities with positive weights) under the one factor LGM model. We must stress that the functional parameters may vary quickly over time, while the implied conditional distributions remain much more stable.

Even if these implied conditional distributions of interest rates vary dramatically over time, the quality of interpolation may remain quite high ; since the purpose is not to get parameters with economic meaning, but to provide well approximated and consistent derivative prices, this instability is not relevant[4]. However, in such a case, the hedging performance based on the proxy model would probably be poor.

We must also stress that when the calibration process performs well, we get a quick way to price a wide range of path dependent interest rate options, while still able to reproduce current observed option prices.

The standard approach for calibrating yield curve models is to choose a model (whatever it is) with unknown parameters (maybe functional) to go through some calibration process with observed prices (by forward induction or "hit and try") and then cross fingers with the hope that there is a unique set of implied parameters.

We show in a particular and explicit example (the one factor LGM model) that more attention should be given to whether a solution to the calibration problem exists without simply relying on the fact that "there are as many equations as there are variables". We can reasonably guess that this existence problem will occur with most models specified on a priori grounds.

Some authors (Dupire (1993, Rubinstein (1994)) have addressed the problem of the identification of the risk neutral process for an underlying risky asset. In this approach, the set of probability measures is not a priori constrained, as in the generalized Vasicek model ; relaxing the constraints on the set of probability measures would widen the set of attainable prices. Another related field of future research would be to look when a given model will provide "good approximations" of a set of observed prices.

[4]For instance, in yield curve modelling, it is well known that the parameters in spline functions are unstable even if the fit remains quite good.

A Appendix 1 : swaption and caplet price with date treatments.

The forward price $P(f_i, d_j, d_k)$ has the following representation :

$$P(f_i, d_j, d_k) = P(0, d_j, d_k) \exp - \left\{ R(i,j,k)^{\frac{1}{2}} V + R(i,j,k)/2 \right\}, \qquad (60)$$

where V is Gaussian with zero mean and unit variance under the probability measure Q^{d_j}. The proof is a straightforward consequence of El Karoui et al. (1992).

Let us note A_n the exercise region for the caplet :

$$A_n = \{\omega \in I_{f_n} \; s.t \quad P(f_n, d_{n-1}, d_n) K_n < 1\}.$$

The price at time zero of the caplet is then :

$$P(0, d_{n-1}) E^{Q_{d_{n-1}}} \left[\mathbf{1}_{A_n} - K_n P(f_n, d_{n-1}, d_n) \mathbf{1}_{A_n} \right], \qquad (61)$$

where the expectation is computed under the forward risk neutral probability (associated to the numeraire discount bond $P(t, d_{n-1})$). By (59), we can notice that under this probability :

$$P(f_n, d_{n-1}, d_n) = P(0, d_{n-1}, d_n) \exp - \left[S(n,n)^{\frac{1}{2}} V + \frac{S(n,n)}{2} \right]. \qquad (62)$$

Let us denote V_n^* such that :

$$V_n^* = \left(\log[K_n P(0, d_{n-1}, d_n)] - \frac{S(n,n)}{2} \right) S(n,n)^{-\frac{1}{2}}. \qquad (63)$$

The price of the caplet can then be expressed as :

$$P(0, d_{n-1}) E^{Q_{d_{n-1}}} \left[\mathbf{1}_{V > V_n^*} \right] - K_n P(0, d_n) E^{Q_{d_{n-1}}} \left[\exp - \left\{ S(n,n)^{\frac{1}{2}} V + \frac{S(n,n)}{2} \right\} \mathbf{1}_{V > V_n^*} \right]. \qquad (64)$$

The first expectation is simply equal to $\phi(-V_n^*)$. A simple change of variable gives the second expectation, $\phi(-V_n^* - S(n,n)^{\frac{1}{2}})$. This gives the price of the caplet.

The exercise region of the call swap is given by :

$$\begin{aligned} A_{i,N} &= \{\omega \in I_{f_{i+1}}, V(\omega) < V_N^*\} \text{ with } V_N^* \text{ such that :} \\ 1 &= r \sum_{n=1}^{N} P(0, d_i, d_{i+4n}) \exp - \left\{ S(i+1, i+4n)^{\frac{1}{2}} V_N^* + \frac{1}{2} S(i+1, i+4n) \right\} \\ &\quad + P(0, d_i, d_{i+4N}) \exp - \left\{ S(i+1, i+4N)^{\frac{1}{2}} V_N^* + \tfrac{1}{2} S(i+1, i+4N) \right\}. \end{aligned} \qquad (65)$$

The right hand side is a decreasing function of V_N^*, its limit is 0 when $V_N^* \rightarrow +\infty$ and $+\infty$ when $V_N^* \rightarrow -\infty$. Thus, V_N^* exists and is unique for any positive $\{P(0, d_i, d_{i+4n}), \; , n = 1, \dots, N\}, \{S(i+1, i+4n), \; , n = 1, \dots, N\}$. The right hand side is also an increasing function of the forward discount bond prices $P(0, d_i, d_{i+4n})$. We then can deduce that :

$$\frac{\partial V_N^*}{\partial P(0, d_i, d_{i+4n})} > 0.$$

The pricing of swaptions requires the computation of the following expectations (under the risk neutral probability associated to the zero coupon bond maturing at time d_i).

$$P(0, d_i) E \left[P(f_{i+1}, d_i, d_{i+4n}) \mathbf{1}_{V < V_N^*} \right],$$

$$P(0, d_{i+4n}) E \left[\exp - \left\{ S(i+1, i+4n)^{\frac{1}{2}} V + \frac{1}{2} S(i+1, i+4n) \right\} \mathbf{1}_{V < V_N^*} \right].$$

This is done by a simple change of variable and gives the price of the call swap.

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