Basket default swaps, CDO's and Factor Copulas

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Paper « basket defaults swaps, CDO's and Factor Copulas » available on AFFI web site
« I will survive », technical paper, RISK magazine, june 2003

What are we looking for ?

- A framework where:
  - *One can easily deal with a large number of names,*
  - *Tackle with different time horizons,*
  - *Compute quickly and accurately:*
    - Basket credit derivatives premiums
    - CDO margins on different tranches
    - Deltas with respect to shifts in credit curves
- Main technical assumption:
  - *Default times are independent conditionnally on a low dimensional factor*

Probabilistic Tools: Survival Functions

- $i = 1, \ldots, n$ names
- $\tau_1, \ldots, \tau_n$ default times
- Marginal distribution function $F_i(t) = Q(\tau_i \le t)$
- Marginal survival function $S_i(t) = Q(\tau_i > t)$
  - *Given from CDS quotes*
- Joint survival function:

$$S(t_1, \ldots, t_n) = Q(\tau_1 > t_1, \ldots, \tau_n > t_n)$$

- (Survival) Copula of default times:

$$C(S_1(t_1), \ldots, S_n(t_n)) = S(t_1, \ldots, t_n)$$

  - *C characterizes the dependence between default times.*

Probabilistic Tools: Factor Copulas

- Factor approaches to joint distributions:
  - *V low dimensional factor, not observed « latent factor »*
  - *Conditionally on V default times are independent*
  - *Conditional default probabilities*

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right), \quad q_t^{i|V} = Q\left(\tau_i > t \mid V\right).$$

  - *Conditional joint distribution:*

$$Q(\tau_1 \leq t_1, \dots, \tau_n \leq t_n \mid V) = \prod_{1 \leq i \leq n} p_{t_i}^{i|V}$$

  - *Joint survival function (implies integration wrt V):*

$$Q(\tau_1 > t_1, \dots, \tau_n > t_n) = E\left[\prod_{i=1}^{n} q_{t_i}^{i|V}\right]$$

Probabilistic Tools: Gaussian Copulas

- One factor Gaussian copula *(Basel 2):*
  - $V, \bar{V}_i, \; i = 1, \ldots, n$ *independent Gaussian*

    $V_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$

- Default times:

$$\tau_i = F_i^{-1}(\Phi(V_i))$$

- Conditional default probabilities:

$$p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$$

Probabilistic Tools : Clayton copula

- *Davis & Lo ; Jarrow & Yu ; Schönbucher & Schubert*

- Conditional default probabilities

$$p_t^{i|V} = \exp\left(V\left(1 - F_i(t)^{-\theta}\right)\right)$$

- *V: Gamma distribution with parameter* θ

Probabilistic Tools: Simultaneous Defaults

- *Duffie & Singleton, Wong*

- Modelling of defaut dates: $\tau_i = \min(\bar{\tau}_i, \tau)$

- $Q(\tau_i = \tau_j) \geq Q\left(\tau \leq \min(\bar{\tau}_i, \bar{\tau}_j)\right) > 0$ simultaneous defaults.

- Conditional default probabilities:

$$p_t^{i|\tau} = 1_{\tau > t} Q(\bar{\tau}_i \leq t) + 1_{\tau \leq t}$$

Probabilistic Tools: Affine Jump Diffusion

- *Duffie, Pan & Singleton ;Duffie & Garleanu.*
- $n+1$ *independent affine jump diffusion processes:*

$$X_1, \dots, X_n, X_c$$

- Conditional default probabilities:

$$Q(\tau_i > t \mid V) = q_t^{i|V} = V\alpha_i(t)$$

$$V = \exp\left(-\int_0^t X_c(s)ds\right), \quad \alpha_i(t) = E\left[\exp\left(-\int_0^t X_i(s)ds\right)\right].$$

Risk Management of Basket Credit Derivatives

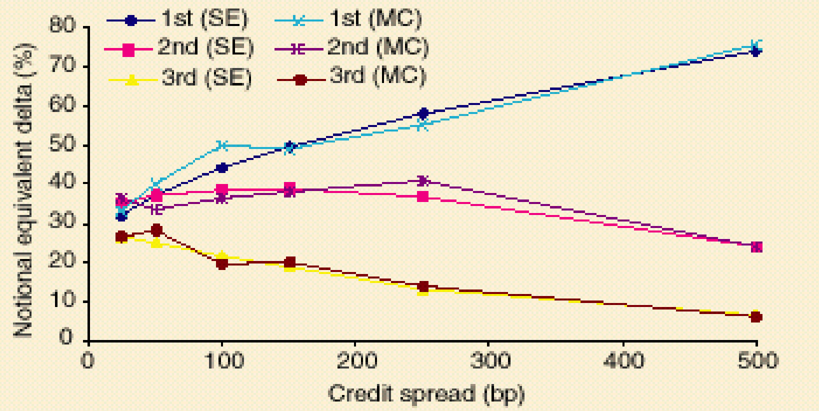
- Example: six names portfolio
- Changes in credit curves of individual names
- Amount of individual CDS to hedge the basket
- Semi-analytical more accurate than 10^5 Monte Carlo simulations.
- Much quicker: about 25 Monte Carlo simulations.

A. Comparison of the semi-explicit formulas with Monte Carlo simulations

	First to default		Second to default		Third to default	
	SE	MC	SE	MC	SE	MC
0%	1,075.1	1,075.9	214.8	214.7	28.2	27.7
20%	927.0	925.9	247.2	247.5	61.4	61.8
30%	859.9	857.9	256.8	257.6	77.6	78.0
40%	796.6	795.2	263.3	264.2	92.7	93.0
60%	679.6	678.0	268.8	268.9	119.5	119.8
80%	573.1	571.7	266.2	266.1	141.0	140.9
100%	500.0	500.0	250.0	250.0	150.0	150.0

Premiums in basis points per annum as a function of correlation for a five-year maturity basket with credit spreads of 25, 50, 100, 150, 250 and 500bp and equal recovery rates of 40%

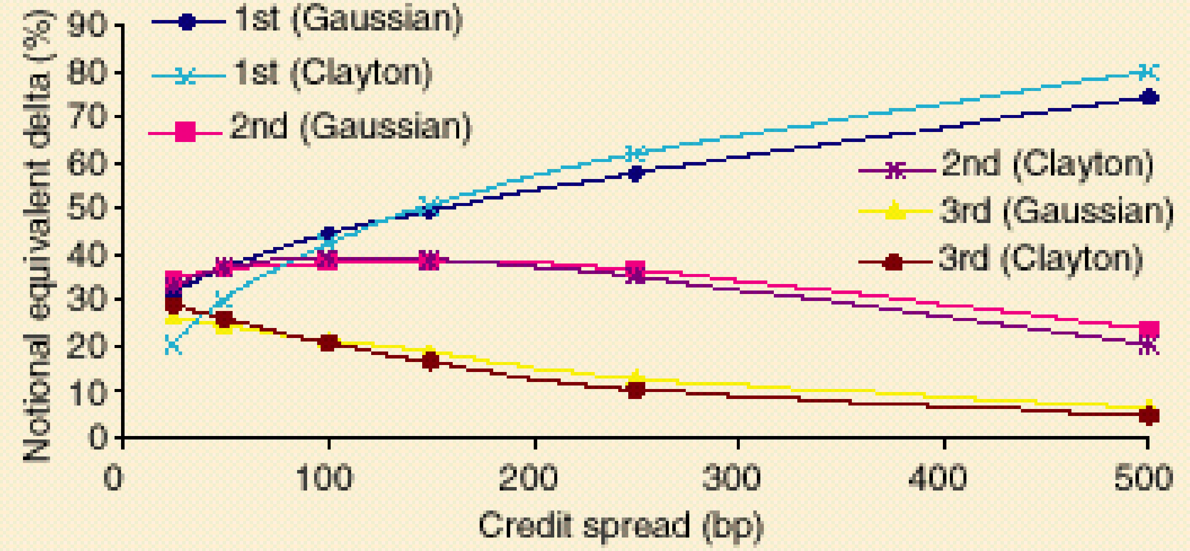
1. Deltas calculated using semi-explicit formulas and Monte Carlo approaches



Comparison of deltas calculated using the analytical formulas and 105 Monte Carlo simulations for the example given in table A. The Monte Carlo deltas are calculated by applying a 10bp parallel shift to each curve

Risk Management of Basket Credit Derivatives

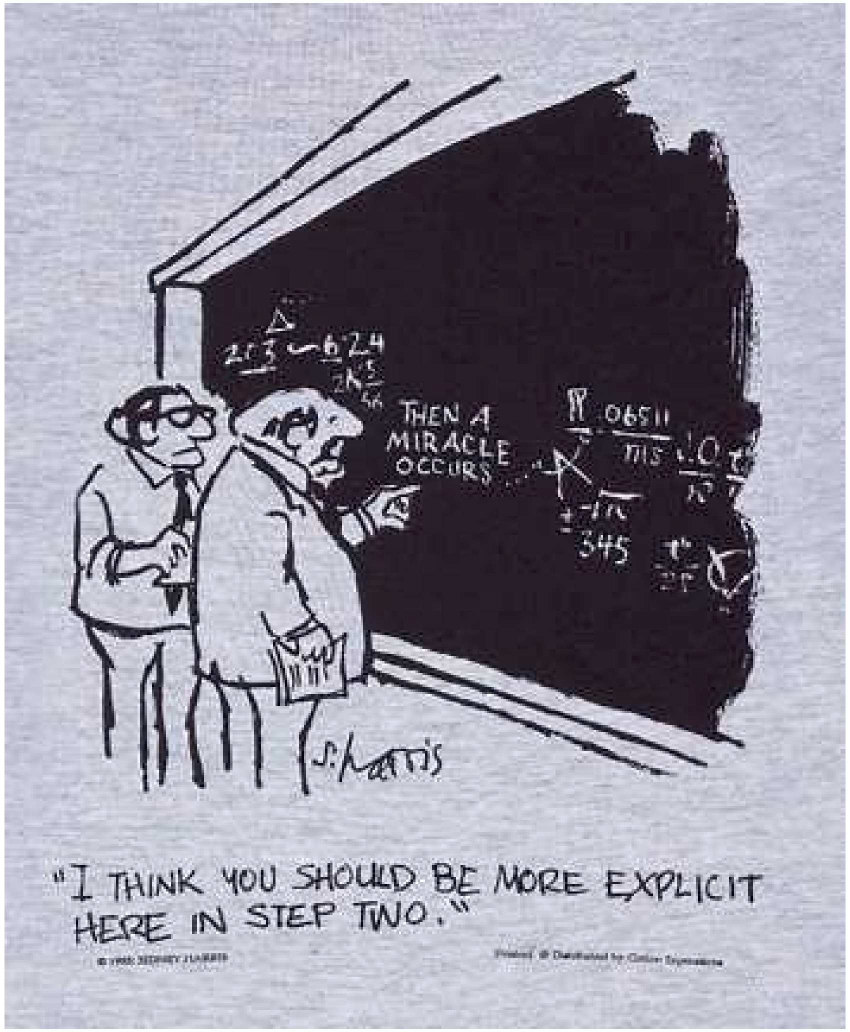
- Changes in credit curves of individual names
  - *Dependence upon the choice of copula for defaults*



Comparison of deltas calculated using Gaussian (30% correlation) and Clayton copulas ($\theta = 0.27$)

CDO Tranches

«Everything should be made as simple as possible, not simpler»



- *Explicit premium computations for tranches*
- *Use of loss distributions over different time horizons*
- *Computation of loss distributions from FFT*
- *Involves integration par parts and Stieltjes integrals*

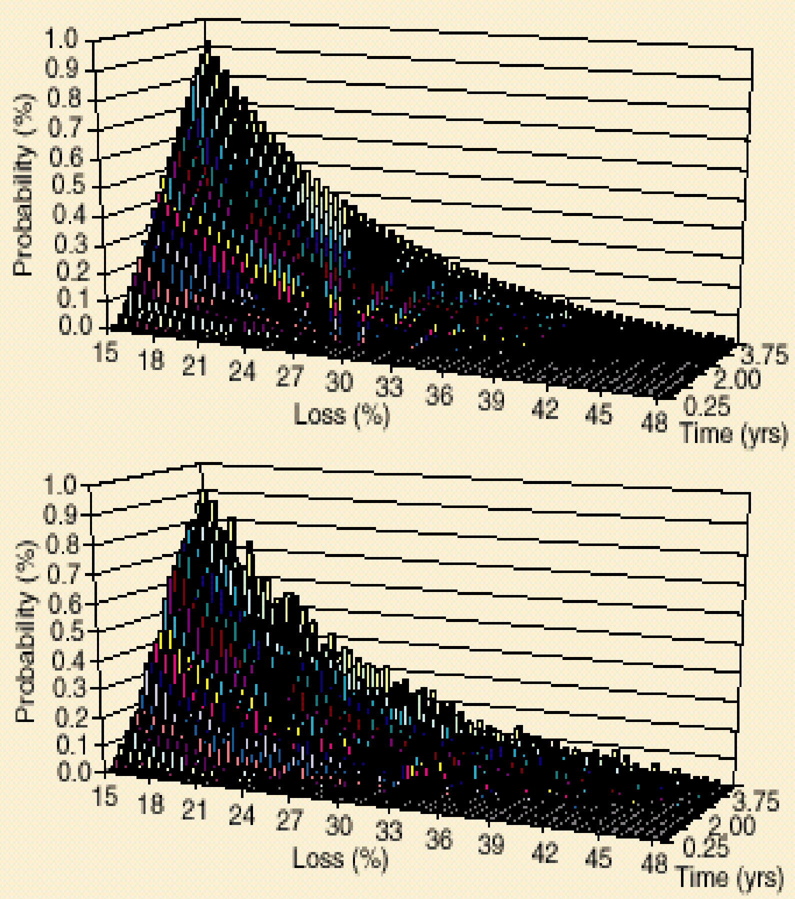
Credit Loss Distributions

- Accumulated loss at t: $L(t) = \sum_{1 \leq i \leq n} N_i(1-\delta_i)N_i(t)$
  - Where $N_i(t) = 1_{\tau_i \leq t}$, $N_i(1-\delta_i)$ loss given default
- Characteristic function $\varphi_{L(t)}(u) = E\left[e^{iuL(t)}\right]$
- By conditioning $\varphi_{L(t)}(u) = E\left[\prod_{1 \leq j \leq n}\left(1 - p_t^{j|V} + p_t^{j|V}\varphi_{1-\delta_j}(uN_j)\right)\right]$
- Distribution of $L(t)$ is obtained by FFT

Credit Loss distributions

- One hundred names, same nominal.
- Recovery rates: 40%
- Credit spreads uniformly distributed between 60 and 250 bp.
- Gaussian copula, correlation: 50%
- 10^5 Monte Carlo simulations

3. Loss distribution



Loss distribution over time for the table B example with 50% correlation for the semi-explicit approach (top) and Monte Carlo simulation (bottom)

Valuation of CDO's

- *Tranches with thresholds* $0 \leq A \leq B \leq \sum N_j$
- *Mezzanine: pays whenever losses are between A and B*
- *Cumulated payments at time t: M(t)*

$$M(t) = (L(t) - A))\, 1_{[A,B]}(L(t)) + (B - A) 1_{]B,\infty[}(L(t))$$

- *Upfront premium:* $E\left[\int_0^T B(t)\,dM(t)\right]$
  - *B(t) discount factor, T maturity of CDO*
- *Stieltjes integration by parts* $B(T)E[M(T)] + \int_0^T E[M(t)]\,dB(t)$
- *where* $E[M(t)] = (B - A)Q(L(t) > B) + \int_A^B (x - A)\,dF_{L(t)}(x)$

Valuation of CDO's

B. Pricing of five-year maturity CDO tranches

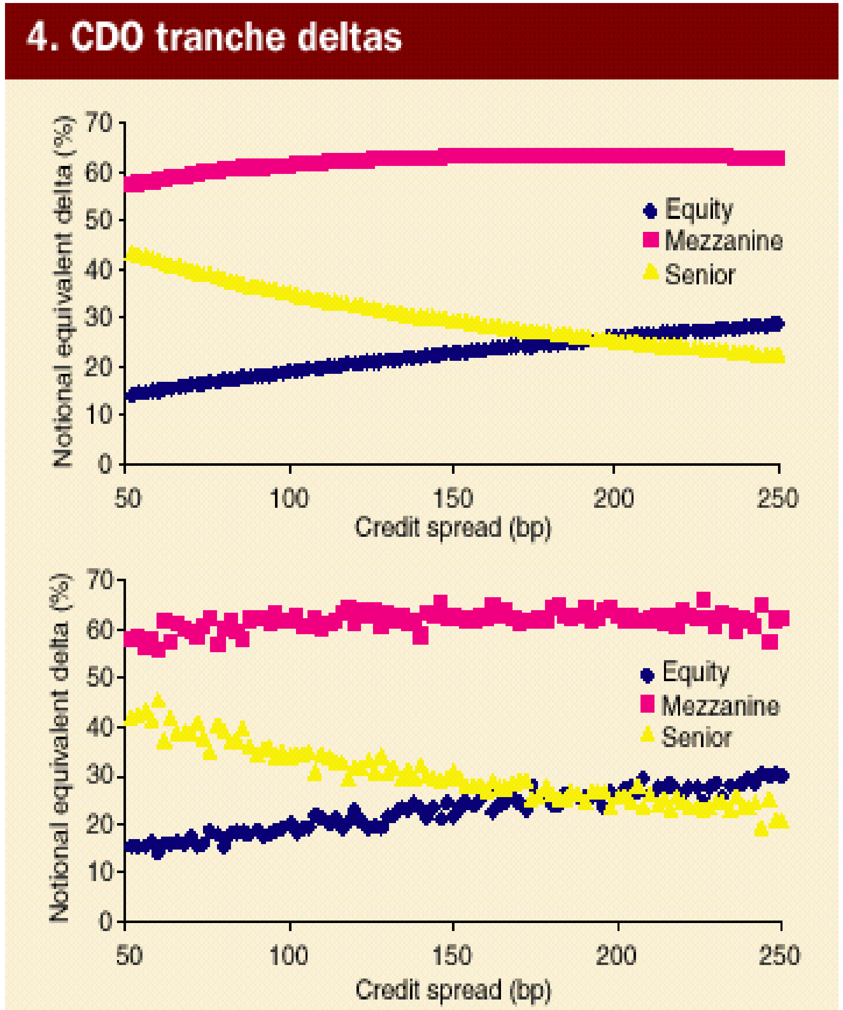
	Equity (0–3%)		Mezzanine (3–14%)		Senior (14–100%)	
	SE	MC	SE	MC	SE	MC
0%	8,219.4	8,228.5	816.2	814.3	0.0	0.0
20%	4,321.1	4,325.3	809.4	806.9	13.7	13.7
40%	2,698.8	2,696.7	734.3	731.4	33.4	33.2
60%	1,750.6	1,738.5	641.0	637.8	54.1	53.7
80%	1,077.5	1,067.9	529.5	526.9	77.0	76.6
100%	410.3	406.6	371.2	367.0	110.4	109.6

Premiums in basis points per annum as a function of correlation for 5-year maturity CDO tranches on a portfolio with credit spreads uniformly distributed between 60 and 250bp. The recovery rates are 40%

- *One factor Gaussian copula*
- *CDO tranches margins with respect to correlation parameter*

Risk Management of CDO's

- Hedging of CDO tranches with respect to credit curves of individual names
- Amount of individual CDS to hedge the CDO tranche
- Semi-analytic : some seconds
- Monte Carlo more than one hour and still shaky



CDO tranche deltas using the analytical method (top) and Monte Carlo (bottom) for a correlation of 50%

Conclusion

- Factor models of default times:
  - *simple computation of basket credit derivatives and CDO's*
  - *deal easily with a large range of names and dependence structures*