What are we looking for?

- A framework where:
  - One can easily deal with a large number of names,
  - Tackle with different time horizons,
  - Compute quickly and accurately:
    - Basket credit derivatives premiums
    - CDO margins on different tranches
    - Deltas with respect to shifts in credit curves

- Main technical assumption:
  - Default times are independent conditionnally on a low dimensional factor
Probabilistic Tools: Survival Functions

- $i = 1, \ldots , n$ names
- $\tau_1, \ldots , \tau_n$ default times
- Marginal distribution function $F_i(t) = Q(\tau_i \leq t)$
- Marginal survival function $S_i(t) = Q(\tau_i > t)$
  - *Given from CDS quotes*
- Joint survival function:
  $$S(t_1, \ldots , t_n) = Q(\tau_1 > t_1, \ldots , \tau_n > t_n)$$
- (Survival) Copula of default times:
  $$C(S_1(t_1), \ldots , S_n(t_n)) = S(t_1, \ldots , t_n)$$
  - *C characterizes the dependence between default times.*
Probabilistic Tools: Factor Copulas

Factor approaches to joint distributions:

- \( V \) low dimensional factor, not observed « latent factor »
- Conditionally on \( V \) default times are independent
- Conditional default probabilities
  \[
  p_t^{i|V} = Q(\tau_i \leq t \mid V), \quad q_t^{i|V} = Q(\tau_i > t \mid V).
  \]
- Conditional joint distribution:
  \[
  Q(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n \mid V) = \prod_{1 \leq i \leq n} p_t^{i|V}
  \]
- Joint survival function (implies integration wrt \( V \)):
  \[
  Q(\tau_1 > t_1, \ldots, \tau_n > t_n) = E \left[ \prod_{i=1}^{n} q_t^{i|V} \right]
  \]
Probabilistic Tools: Gaussian Copulas

- One factor Gaussian copula (*Basel 2*):
  - $V, \tilde{V}_i, \ i = 1, \ldots, n$ independent Gaussian
  - $V_i = \rho_i V + \sqrt{1 - \rho_i^2} \tilde{V}_i$

- Default times:
  - $\tau_i = F_i^{-1}(\Phi(V_i))$

- Conditional default probabilities:
  - $p_{i|V} = \Phi \left( \frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \rho_i^2}} \right)$
Probabilistic Tools: Clayton copula

- *Davis & Lo; Jarrow & Yu; Schönbucher & Schubert*

- Conditional default probabilities

\[ p_t^i | V = \exp \left( V \left( 1 - F_i(t)^{-\theta} \right) \right) \]

- *V: Gamma distribution with parameter \( \theta \)*
Probabilistic Tools: Simultaneous Defaults

- Duffie & Singleton, Wong

- Modelling of default dates: \( \tau_i = \min(\bar{\tau}_i, \tau) \)

- \( Q(\tau_i = \tau_j) \geq Q(\tau \leq \min(\bar{\tau}_i, \bar{\tau}_j)) > 0 \) simultaneous defaults.

- Conditional default probabilities:

\[
\hat{p}_t^i|\tau = 1_{\tau > t}Q(\bar{\tau}_i \leq t) + 1_{\tau \leq t}
\]
Probabilistic Tools: Affine Jump Diffusion

- Duffie, Pan & Singleton; Duffie & Garleanu.
- \( n + 1 \) independent affine jump diffusion processes:
  \[ X_1, \ldots, X_n, X_c \]
- Conditional default probabilities:
  \[ Q(\tau_i > t \mid V) = q_t^i | V = V \alpha_i(t) \]

\[ V = \exp \left( - \int_0^t X_c(s) ds \right), \quad \alpha_i(t) = E \left[ \exp \left( - \int_0^t X_i(s) ds \right) \right]. \]
Risk Management of Basket Credit Derivatives

- Example: six names portfolio
- Changes in credit curves of individual names
- Amount of individual CDS to hedge the basket
- Semi-analytical more accurate than $10^5$ Monte Carlo simulations.
- Much quicker: about 25 Monte Carlo simulations.
Risk Management of Basket Credit Derivatives

- Changes in credit curves of individual names
  - Dependence upon the choice of copula for defaults

![Graph: Deltas using Gaussian and Clayton copula](image-url)
CDO Tranches

«Everything should be made as simple as possible, not simpler»

- Explicit premium computations for tranches
- Use of loss distributions over different time horizons
- Computation of loss distributions from FFT
- Involves integration parts and Stieltjes integrals
Credit Loss Distributions

Accumulated loss at t: \[ L(t) = \sum_{1 \leq i \leq n} N_i(1 - \delta_i)N_i(t) \]

- Where \( N_i(t) = 1_{\tau_i \leq t} \), \( N_i(1 - \delta_i) \) loss given default

- Characteristic function \( \varphi_{L(t)}(u) = E \left[ e^{iuL(t)} \right] \)

- By conditioning \( \varphi_{L(t)}(u) = E \left[ \prod_{1 \leq j \leq n} \left( 1 - p_i^j \varphi_{1-\delta_j}(uN_j) + p_i^j \varphi_{1-\delta_j}(uN_j) \right) \right] \)

- Distribution of \( L(t) \) is obtained by FFT
Credit Loss distributions

- One hundred names, same nominal.
- Recovery rates: 40%
- Credit spreads uniformly distributed between 60 and 250 bp.
- Gaussian copula, correlation: 50%
- $10^5$ Monte Carlo simulations
Valuation of CDO’s

- **Tranches with thresholds** \( 0 \leq A \leq B \leq \sum N_j \)
- **Mezzanine**: pays whenever losses are between \( A \) and \( B \)
- **Cumulated payments at time \( t \)**: \( M(t) \)

\[
M(t) = (L(t) - A) 1_{[A,B]}(L(t)) + (B - A) 1_{[B,\infty]}(L(t))
\]

- **Upfront premium**: 
  \[
  B(t) \text{ discount factor, } T \text{ maturity of CDO}
  \]

\[
E \left[ \int_0^T B(t) dM(t) \right]
\]

- **Stieltjes integration by parts** 
  \[
  B(T) E[M(T)] + \int_0^T E[M(t)] dB(t)
  \]

- **where** 
  \[
  E[M(t)] = (B - A) Q(L(t) > B) + \int_A^B (x - A) dF_{L(t)}(x)
  \]
Valuation of CDO’s

- One factor Gaussian copula
- CDO tranches margins with respect to correlation parameter
Risk Management of CDO’s

- Hedging of CDO tranches with respect to credit curves of individual names
- Amount of individual CDS to hedge the CDO tranche
- Semi-analytic: some seconds
- Monte Carlo more than one hour and still shaky
Conclusion

- Factor models of default times:
  - *simple computation of basket credit derivatives and CDO’s*
  - *deal easily with a large range of names and dependence structures*