Trading book and credit risk: how fundamental is the Basel review?

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January 16, 2015
Preliminary version – Do not quote

Abstract

In its October 2013’s consultative paper for a revised market risk framework (FRTB), the Basel Committee suggests that non-securitization credit positions in the trading book be subject to a separate Incremental Default Risk (IDR) charge, in an attempt to overcome practical challenges raised by the joint modeling of the discrete (default risk) and continuous (spread risk) components of credit risk, enforced in the current Basel 2.5 Incremental Risk Charge (IRC). Banks would no longer have the choice of using either a single-factor or multifactor default risk model but instead, market risk rules would require the use of a two-factor simulation model and a 99.9-VaR capital charge. Proposals are also made as to how to account for diversification effects with regard to calibration of correlation parameters. The Committee finally advocates disclosure of a desk-level calculation, including a breakdown by individual risk factor. In this article, we analyze theoretical foundations of these proposals, particularly the link with one-factor model used for the banking book and with a general J-factor setting. We thoroughly investigate the practical implications of the two-factor and correlation calibration constraints through numerical applications. We introduce the Hoeffding decomposition of the aggregate unconditional loss for a systematic-idiosyncratic representation. Impacts of J-factor correlation structures on risk measures and risk contributions are studied for long/long and long/short credit-sensitive portfolios.

Key words: Portfolio Credit Risk Modeling, Risk Contribution, Fundamental Review of the Trading Book.

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Michael Sestier and Stéphane Thomas acknowledge support from PHAST Solutions Group. Jean-Paul Laurent acknowledges support from the BNP Paribas Cardif chair “Management de la Modélisation”. The usual disclaimer applies. This paper has been presented at the “1\textsuperscript{ère} journée de collaboration ULB-Sorbonne” held in Bruxelles on March 2014, at the colloque “IFRS - Bâle - Solvency” held in Poitiers in October 2014, and in the Finance Seminar of the PRISM Laboratory held in Paris on October 2014. We thank participants of these events for their questions and remarks. All remaining errors are ours.

\textit{Ce travail a été réalisé dans le cadre du laboratoire d’excellence ReFi porté par le Pres heSam, portant la référence ANR-10-LABX-0095. Ce travail a bénéficié d’une aide de l’Etat gérée par l’Agence Nationale de la recherche au titre du projet Investissements d’Avenir Paris Nouveaux Mondes portant la référence n° ANR-11-IDEX-0006-02.}
1 Basel recommendations on credit risk

Created in 1974 by ten leading industrial countries and now including supervisors from twenty-seven countries, the Basel Committee on Banking Supervision (BCBS, henceforth "the Committee") is responsible for strengthening the resilience of the global financial system, ensuring the effectiveness of prudential supervision and improving the cooperation among banking regulators. To accomplish its mandate, the Committee formulates broad supervisory standards and guidelines and it recommends statement of best practices in banking supervision that member authorities and other nations' authorities are expected to implement stepwise within their own national systems. Essential propositions concern standardized regulatory capital requirements, determining how much capital has to be held by financial institutions to be protected against potential losses coming from credit risk realizations (defaults, rating migrations), market risk realizations (losses attributable to adverse market movements), operational risk, etc.

The Committee’s recommendations of 1988 (BCBS 1988)[1] established a minimum required capital amount through the definition of the so-called Cooke ratio and the categorization of credit risk levels into homogeneous buckets based on issuers’ default probability. Nevertheless, this approach ignored the inhomogeneity of banks loans in terms of risk and led the Committee to develop new sets of recommendations in 2004. Also known as Basel II agreements (BCBS 2005)[2], the text defines a regulatory capital through the concept of Risk Weighted Assets (RWAs) and through the McDonough ratio, including operational and market risks in addition to the credit risk. In particular, to make the regulatory credit capital more risk sensitive, the text sets out a more relevant measure for credit risk by considering the borrower’s quality through internal rating system for approved institutions: the Internal Rating Based (IRB) approach. In this framework, the RWA related to credit risk in the banking book measures the exposition of a bank issuing credits by applying a weight according to the intrinsic riskiness of each asset (a function of the issuer’s default probability and effective loss at default time). The Committee went one step further in considering also portfolio risk addressed with a prescribed model based on the Asymptotic-Single-Risk-Factor model (ASRF, described hereafter) along with a set of constrained calibration methods (borrower’s asset value correlation matrix in particular). Despite significant improvements, the Basel II capital requirement calculation for the credit risk remains confined to the banking book. A major gap thus revealed by the 2008 financial crisis was the inability to adequately identify the credit risk of the trading book positions (any component of the trading book: instruments, sub-portfolios, portfolios, desks...), enclosed in credit-quality linked assets. Considering this deficiency, the Committee revised market risk capital requirements in the 2009's reforms, also known as Basel 2.5 agreements (BCBS 2009)[3], that add a new capital requirement, the Incremental Risk Charge (IRC), designed to deal with long term changes in credit spreads, and a specific capital charge for correlation products, the Comprehensive Risk Measures (CRM). More specifically, the IRC is a capital charge that captures default and migration risks through a VaR-type calculation at 99.9% on a 1-year horizon. As opposed to the credit risk treatment in the banking book, the trading book model specification results from a complete internal model validation process whereby financial institutions are led to build their own framework.
In parallel to new rules elaboration, the Committee has recently investigated the RWAs comparability among institutions and jurisdictions, for both the banking book (BCBS 2013)[4] and the trading book (BCBS 2013)[5], through a Regulatory Consistency Assessment Program (RCAP) following previous studies led by the IMF in 2012 (see Le Leslé and Avramova, (2012)[6]). Reports show large differences in risk measure levels, and consequently on RWAs, amongst participating financial institutions. The related causes of such a variability are numerous. Among the foremost is the inhomogeneity of risk profiles, consecutive to institutions' diverse activities, and divergences in local regulation regimes. In conjunction with these structural causes, the Committee also raises important discrepancies among internal methodologies of risk calculation, and in particular, those of the trading book’s RWAs. A main contributor to this variability appears to be the modeling choices made by each institution within their IRC model (for instance, whether it uses spread-based or transition matrix-based models, calibration of the transition matrix or that of the initial credit rating, correlations’ assumptions across obligors, etc.).

In response to these shortcomings, the Committee has been working ever since 2012 towards a new post-crisis review of the market risk global regulatory framework, known as Fundamental Review of the Trading Book (FRTB) (BCBS 2012-2013)[7, 8]. Notwithstanding long-lasting impact studies and ongoing consultative working groups, no consensus seems to be fully reached so far. Main discussions arise from the proposal abandoning the IRC in favor of a default-only risk capital charge, named Incremental Default Risk (IDR) charge. With a one-year 99.9-VaR calculation, IDR would be close to the banking book capital charge but with the noticeable difference that it would be grounded on a two-factor model: "One of the key observations from the Committee’s review of the variability of market risk weighted assets is that the more complex migration and default models were a relatively large source of variation. The Committee has decided to develop a more prescriptive IDR charge in the models-based framework. Banks using the internal model approach to calculate a default risk charge must use a two-factor default simulation model, which the Committee believes will reduce variation in market risk-weighted assets but be sufficiently risk sensitive as compared to multifactor models”. This objective of constraining the modeling choices of the IDR by "limiting discretion on the choice of risk factors” has been mentioned in a recent report to the G20, BCBS (2014)[9]. The Committee would also monitor model calibration through correlation calibration constraints. Especially, Annex 2 of the last consultative document stipulates: “default correlations must be based on listed equity prices and must be estimated over a one year time horizon (based on a period of stress) using a 250 days liquidity horizon. [...] These correlations should be based on objective data and not chosen in an opportunistic way where a higher correlation is used for portfolios with a mix of long and short positions and a low correlation used for portfolio with long only exposures. [...] A bank must validate that its modeling approach for these correlations is appropriate for its portfolio, including the choice and weights of its systematic risk factors. A bank must document its modeling approach and the period of time used to calibrate the model. [...] Firms need to reflect all significant basis risk in recognizing these correlations, including, for example, maturity mismatches, internal or external ratings, vintage, etc.”
Our paper investigates the practical implications of these recommendations, and in particular, studies the impact of factor models and their induced correlation structures on the trading book credit risk measurement. The goal here is to provide a comparative analysis of risk factors contributions within a consistent theoretical framework.

The paper is organized as follows. In Section 2, we describe a two-factor IDR model within the usual Gaussian framework, and analyze the link with the one-factor model used in the current banking book framework on the one hand, and with a general J-factor setting deployed in IRC implementations, on the other. Following the Committee’s recommendations, we look into the effects of correlation calibration constraints on each setting, using the so-called “nearest correlation matrix with J-factor structure” framework. In Section 3, we use the Hoeffding decomposition of the aggregate loss to explicitly derive contributions of systematic and idiosyncratic risks, of particular interest in the trading book. Section 4 is devoted to numerical applications whereby impacts of J-factor correlation structures on risk measures and risk contributions are considered. Representative long/long and long/short credit-sensitive portfolios are tested. The last section gathers some concluding remarks.

2 Two-factor Incremental Default Risk Charge model

2.1 Model specification

The portfolio loss at a one-period horizon is modeled by a random variable \( L \), defined as the sum of the individual losses on issuers’ default over that period. We consider a portfolio with \( K \) positions: \( L = \sum_{k=1}^{K} L_k \) with \( L_k \) the loss on the position \( k \). The individual loss is decomposed as \( L_k = w_k \times \mathbb{1}_k \) where \( w_k \) is the positive or negative effective exposure at the time of default and \( \mathbb{1}_k \) is a random variable referred to as the obligor \( k \)’s creditworthiness index, taking value 1 when default occurs, and 0 otherwise. For conciseness, we assume constant effective exposures at default, hence the sole remaining source of randomness comes from \( \mathbb{1}_k \).

In order to model the portfolio credit risk we need to define the probability distribution of the \( L_k \)’s as well as their dependence structure. We rely on a usual structural factor approach, that is, \( \mathbb{1}_k \) takes its values 1 or 0 depending on a set of – latent or observable – factors \( \mathcal{F} = \{F_m \mid m = 1, \ldots, M\} \). The latter can be expressed through any factor model \( h: \mathbb{R}^M \to \mathbb{R} \) such that creditworthiness is defined as \( \mathbb{1}_k = 1_{\{X_k \leq x_k\}} \), where \( X_k = h(F_1, \ldots, F_M) \) and \( x_k \) is a predetermined threshold. Modeling \( \mathbb{1}_k \) thus boils down to modeling \( X_k \). This model introduced by Vasicek (1987, 2001)[10, 11], and based on seminal work of Merton (1974)[12], is largely used by financial institutions to model default risk either for economic capital calculation or for regulatory purposes. Following these authors, we may interpret \( X_k \) as a latent variable representing obligor \( k \)’s asset value. \( X_k \) evolves according to a \( J \)-factor Gaussian model:

\[ w_k = EAD_k \times LGD_k \]

The effective exposure of the position \( k \) is defined as the product of the Exposure-At-Default (EAD\(_k\)) and the Loss-Given-Default (LGD\(_k\)). Formally: \( w_k = EAD_k \times LGD_k \). While we could think of stochastic LGDs, there is no consensus as regard to proper modelling choices, either regarding marginal LGDs or the joint distribution of LGDs and default indicators. The Basel Committee is not prescriptive at this stage and it is more than likely that most banks will retain constant LGDs.
$X_k = \beta_k Z + \sqrt{1-\beta_k \beta_k^\top} \epsilon_k$ where $Z \sim \mathcal{N}(0, 1)$ is a $J$-dimensional vector of systematic factors, $\epsilon_k \sim \mathcal{N}(0, 1)$ is an idiosyncratic risk – all factors are i.i.d – and $\beta \in \mathbb{R}^{K \times J}$ is the factors loadings matrix. We denote $Z = \{Z_j | j = 1, \ldots, J\}$ the set of all systematic factors and $\mathcal{E} = \{\epsilon_k | k = 1, \ldots, K\}$ the set of all idiosyncratic risks such that $\mathcal{F} = Z \cup \mathcal{E}$. This setting ensures that asset values $(X_k)_{k=1,\ldots,K}$ are standard normal random variables with a correlation matrix
$C(\beta) = \text{Correl}(X_k, X_l)_{k,l=1,\ldots,K} = \beta \beta^\top + \text{diag}(I_d - \beta \beta^\top)$. Threshold $x_k$ is chosen such that $\mathbb{P}(\mathbb{I}_k = 1) = p_k$, where $p_k$ is the observed obligor $k$’s marginal default probability. From standard normality of $X_k$ it comes straightforwardly $x_k = \Phi^{-1}(p_k)$, with $\Phi(.)$ the standard normal cumulative function. The portfolio loss is then written:

$$L = \sum_{k=1}^K w_k \{ \beta_k Z + \sqrt{1-\beta_k \beta_k^\top} \epsilon_k \Phi^{-1}(p_k) \}$$

Since $\mathbb{I}_k$ is discontinuous, $L$ can take only a finite number of realization in the set $\mathbb{L} = \{ \sum_{a \in A} w_a | \forall A \subseteq \{1, \ldots, K\} \}$.

The single factor variant of the model is at the foundation of the Basel II credit risk capital charge. To benefit from asymptotic properties the Committee capital requirement formula is based on the assumption that the portfolio is infinitely fine grained, i.e. it consists of a very large number of credits with small exposures, so that only one systematic risk factor influences portfolio default risk. The aggregate loss can be approximated by the systematic factor projection: $L = L_Z = \mathbb{E}[L \mathbb{I}_k]$, subsequently called "Large Pool Approximation", where $L_Z$ is a continuous random variable. This model is known as Asymptotic Single Risk Factor model (ASRF). Thin granularity implies no name concentrations within the portfolio (idiosyncratic risk being fully diversified) whereas the one-factor assumption implies no sector concentrations such as industry or country-specific risk concentration. Name and sector concentrations are largely looked into in the literature, particularly around the concept of the so-called granularity adjustment. We refer to Fermanian (2014)[13] and Gagliardini and Gourieroux (2014)[14] for recent treatments of this concept. Furthermore, a detailed presentation of the IRB modeling is provided by Gordy (2003)[15]. Note also that under these assumptions, Wilde (2001)[16] expresses a portfolio invariance property that states that the required capital for any given loan does not depend on the portfolio it is added to.

These properties of the IRB modeling for credit risk in the banking book are appealing. Nevertheless, important disparities exist between trading book and banking book positions that make them untenable for the trading book in general. Indeed, banking book positions are supposed to be held until maturity, often long credit-risk $(w_k > 0)$, and exist in a number large enough to make the Large Pool Approximation acceptable (up to a granularity.

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5 The theoretical derivation of this adjustment that accounts for name concentrations was first done by Wilde (2001)[16] and improved then by Gordy (2003)[15]. Their name concentration approach refers to the finite number of credits in the portfolio. In contrast, the semi-asymptotic approach in Emmer and Tasche (2005)[17] refers to position concentrations attributable to a single name while the rest of the portfolio remains infinitely granular. Analytic and semi-analytic approaches that account for sector concentration exist as well. One rigorous analytical approach is Pykhtin (2004)[18]. An alternative is the semi-analytic model of Cespedes and Herrero (2006)[19] that derives an approximation formula through a numerical mapping procedure. Tasche (2005)[20] suggests an ASRF-extension in an asymptotic multifactor setting.
adjustment). Conversely, trading book positions are actively traded, long or short credit-risk, inhomogeneous and may be few, making the Large Pool Approximation or any granularity assumption too restrictive. Consequently, there is a need for taking into account both systematic risk and idiosyncratic risk, furthermore in presence of a discrete loss distribution.

Apart from the previous theoretical issues, an operational question concerns the meaning of underlying factors. In the banking book ASRF model, the systematic factor is usually interpreted as the state of the economy, i.e. a generic macroeconomic variable affecting all firms. Within multi-factor models \((J > 2)\), factors may be industrial sectors, geographical regions, ratings and so on. A fine segmentation allows modelers to postulate a detailed correlation structure among portfolio assets values. While for these two cases \((J = 1\) and \(J > 2\)) a factor interpretation seems straightforward, it is not clear for the two-factor model, for which neither multiple sector and/or region segmentations nor generic macroeconomic variables seem appropriate.

### 2.2 Nearest correlation matrix with two-factor structure

The modeling assumptions on the general framework being made, the Committee provides indications on the calibration of the assets values correlation matrix: "Default correlations must be based on listed equity prices and must be estimated over a one-year time horizon (based on a period of stress) using a \([250]\) day liquidity horizon. [...]. Nevertheless, at this stage the Committee does not provide any guidance on the calibration of factors loadings \(\beta\) needed to pass from a \((J > 2)\)-factor structure to a \((J = 2)\)-factor one. The goal here is then to present a generic method to calibrate a \(J\)-factor model from an initial \((K \times K)\)-correlation matrix.

In the sequel, we assume an initial correlation matrix \(C_0\), estimated from historical stock prices, following the Committee’s proposal\(^6\). At that stage, we let aside theoretical concerns – not addressed by the Committee – as to which estimator of the correlation matrix is to be used, and rely on the empirical estimator. However, to study the impact of the correlation structure on the level of risk and factors contributions (cf. Section 4), we shall consider other candidates as the initial matrix such as the correlation matrix of CDS spreads, the matrix associated with the IRB ASRF model and the one associated with a \(J\)-factor model (like the Moody’s KMV model for instance). The underlying issue of the Committee’s proposition is to build a two-factor model generating a correlation structure as close as possible to the pre-determined correlation structure \(C_0\). Formally, we look for a 2-factor \((J = 2)\) modeled \(X_k\) of which the correlation matrix \(C(\beta), \beta \in \mathbb{R}^{K \times J}\) is as close as possible to \(C_0\) in the sense of a chosen norm. We then define the following optimization problem\(^7\):
\[
\arg\min_{\beta} f(\beta) = \|C(\beta) - C_0\|_F \\
\text{subject to: } \beta \in \Omega = \{\beta \in \mathbb{R}^{K \times J} | \beta_k \beta_k^t \leq 1; k = 1, \ldots, K\}
\]

where \(\|\|_F\) is the Frobenius norm defined as \(\forall A \in \mathbb{R}^{K \times K}; \|A\|_F = \text{trace}(A^t A)^{1/2}\). The above constraint ensures that \(\beta \beta^t\) has diagonal elements bounded by 1, implying that \(C(\beta)\) is positive semi-definite. An important consequence is that some pairwise correlations might increase, whereas others might decrease in comparison with the initial correlation matrix, \(C_0\). Hence the resulting effect may vary significantly depending on the signs of \(w_k\)’s.

The general problem of computing a correlation matrix of \(J\)-factor structure nearest to a given matrix has been tackled in the literature. Andersen, Sidenius and Basu (2003)[25] apply the unconstrained optimization problem to credit basket securities, wherein an \textit{ad hoc} iterative method based on principal component analysis is described. Borsdorff, Higham and Raydan (2010)[26] give a theoretical analysis of the full problem and show how standard optimization methods can be used to solve it. They compare different optimization methods, in particular the principal components-based method and the spectral projected gradient method. \textit{Inter alia}, their experiments show that the principal components-based method, which is not supported by any convergence theory, often performs surprisingly well on the one hand, partly because the constraints are often not active at the solution, but may fail to solve the constrained problem on the other. The spectral projected gradient method solves the full constrained problem and generates a sequence of matrices guaranteed to converge to a stationary point of the convex set \(\Omega\). The authors acknowledge the spectral projected gradient method as being the most efficient. The method allows minimizing \(f(\beta)\) over the convex set \(\Omega\) by iterating over \(\beta\) in the following way: \(\beta_{i+1} = \beta_i + \alpha_id_i\) where \(d_i = \text{Proj}_{\Omega}(\beta_i - \lambda_i \nabla f(\beta_i)) - \beta_i\) is the descent direction, with \(\lambda_i > 0\) a pre-computed scalar, and \(\alpha_i \in [-1; 1]\) is chosen through a non-monotone line search strategy. \(\text{Proj}_{\Omega}\) being cheap to compute, the algorithm is fast enough to enable the calibration of portfolios having a large number of positions. A detailed presentation and algorithms are available in Birgin, Martinez and Raydan (2001)[27].

3 Impact on the risk: the toolbox

Recall our objective is to study the impact of factor model and its induced correlation structures on the trading book credit risk measurement. The particularities of the trading book positions (actively traded positions, the presence of long-short credit risk exposures, inhomogeneous and potentially small number of positions) compel to analyze the risk contribution of both the systematic factors and the idiosyncratic risks. In a first sub-section, we represent the portfolio loss via the Hoeffding decomposition to exhibit the impact of both the systematic factors and the idiosyncratic risks. In this framework, the second sub-section presents analytics of factors contributions to the risk measure. The third subsection discusses methods for reaching an intuitive interpretation of factors.

3.1 Hoeffding decomposition of the loss

The portfolio loss has been defined in the previous section via the sum of the individual losses (cf. \textit{equation [1]}). We consider here a representation of the loss as a sum of terms involving sets of factors. In particular we use a statistical tool, the Hoeffding decomposition, previously
introduced by Rosen and Saunders (2010)[28] within a risk contribution framework. Formally, if $F_1, \ldots, F_M$ and $L \equiv L[F_1, \ldots, F_M]$ are square-integrable random variables, then the Hoeffding decomposition writes the aggregate portfolio loss, $L$, as a sum of terms involving given factor sets:

$$ L = \sum_{S \subseteq \{1, \ldots, M\}} \phi_S(L; F_m, m \in S) = \sum_{S \subseteq \{1, \ldots, M\}} \sum_{\tilde{S} \subseteq S} (-1)^{|S| - |\tilde{S}|} \mathbb{E}[L|F_m; m \in \tilde{S}] $$

Although the Hoeffding decomposition suffers from a practical issue when the number of factors is large, computation for a two-factor model does not present any challenge, especially in the Gaussian framework where an explicit analytical form of each term exists (cf. equations 7.1, 7.2, 7.3). Moreover, even in presence of a large number of factors, the Hoeffding theorem allows decomposing $L$ on any subset of factors. Rosen and Saunders (2010)[28] focus on the heterogeneous Large Pool Approximation, $L_Z$, by considering the set of systematic factors, $\mathcal{Z}$. We define the systematic decomposition with two factors, $Z_1$ and $Z_2$, of the conditional loss by:

$$ L_Z = \phi_0(L_Z) + \phi_1(L_Z; Z_1) + \phi_2(L_Z; Z_2) $$

where $\phi_0(L_Z) = \mathbb{E}[L_Z]$ is the expected loss, $\phi_1(L_Z; Z_1)$ and $\phi_2(L_Z; Z_2)$ are the losses induced by the systematic factors $Z_1$ and $Z_2$ respectively, and $\phi_{1,2}(L_Z; Z_1, Z_2)$ is the remaining loss induced by systematic factors interactions. Each term of the decomposition, $\phi_S(L_Z; Z_{j}, j \in S), S \subseteq \{1, 2\}$, gives the best hedge in the quadratic sense of the residual risk driven by co-movements of the systematic factors $Z_j$ that cannot be hedged by considering any smaller subset of the factors.

In this paper, we propose to extent the decomposition in equation [4] by taking into account idiosyncratic risks. Indeed, the flexibility of Hoeffding decomposition brings the possibility to break the portfolio loss down in terms of aggregated systematic and idiosyncratic parts, yielding an exhaustive representation of the aggregate loss, where we consider all factors $\mathcal{F} = \mathcal{Z} \cup \mathcal{E}$. We define the macro decomposition of the unconditional loss by:

$$ L = \phi_0(L) + \phi_1(L; \mathcal{Z}) + \phi_2(L; \mathcal{E}) + \phi_{1,2}(L; \mathcal{Z}, \mathcal{E}) $$

where $\phi_0(L) = \mathbb{E}[L]$ is the expected loss, $\phi_1(L; \mathcal{Z}) = \mathbb{E}[L|\mathcal{Z}] - \mathbb{E}[L]$ is the loss induced by systematic factors $Z_1$ and $Z_2$ (corresponding, up to the expected loss term, to the heterogeneous Large Pool Approximation), $\phi_2(L; \mathcal{E}) = \mathbb{E}[L|\mathcal{E}] - \mathbb{E}[L]$ is the loss induced by the $K$ idiosyncratic terms $\epsilon_k$, and $\phi_{1,2}(L; \mathcal{Z}, \mathcal{E}) = (L - \mathbb{E}[L|\mathcal{Z}] - \mathbb{E}[L|\mathcal{E}] + \mathbb{E}[L])$ is the remaining risk induced by interactions between idiosyncratic and systematic risk factors. Moreover, since $L_Z = \phi_1(L; Z) + \phi_0(L)$, it is feasible to combine equations [4, 5] to get:

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8 The Hoeffding decomposition is usually applied to independent factors. If this assumption is fulfilled, then all terms of the decomposition are uncorrelated. The decomposition formula is still valid for dependent factors but, in this case, each term depends on the joint distribution of the factors.

9 Consult Van Der Waart (1999)[29] for a detailed presentation of the Hoeffding decomposition.

10 The Hoeffding decomposition may be used also to decompose individual loss $L_k, k = 1, \ldots, K$, to provide finer analysis on the loss origin in the portfolio.

11 The Hoeffding decomposition requires the calculation of $2^M$ terms, where $M$ is the number of factors.
This last expression helps understand the impact of each systematic factor (alone or in interaction) as well as the impact of the idiosyncratic terms, alone or in interaction. Furthermore, equation [6.1] thoroughly expresses the relation between conditional and unconditional portfolio losses.

From a practical point of view, as we consider a Gaussian factor model, we may easily compute each term in the decompositions:

\[
E[L|Z] = \sum_{k=1}^{K} w_k \Phi \left( \frac{\Phi^{-1}(p_k) - \beta_k Z_k}{\sqrt{1 - \beta_k^2}} \right) \quad [7.1]
\]

\[
E[L|E] = \sum_{k=1}^{K} w_k \Phi \left( \frac{\Phi^{-1}(p_k) - \phi_k \varepsilon_k}{\sqrt{\phi_k^2}} \right) \quad [7.2]
\]

\[
E[L|Z_j, j \in \mathcal{S}] = \sum_{k=1}^{K} w_k \Phi \left( \frac{\Phi^{-1}(p_k) - \sum_{j \in \mathcal{S}} \beta_{k,j} Z_j}{\sqrt{1 - \beta_{k,j}^2}} \right) \quad [7.3]
\]

Remark that the Hoeffding decomposition is not an approximation. It is an equivalent representation of the same random variable. In particular, when decomposed, the unconditional loss \( L \) and the conditional loss \( L_Z \) remain discrete and continuous random variables, respectively.

Importantly, considering our model specification of the vector \( X \), we know from standard statistical results on exploratory factor analysis that factor rotations\(^{12}\) (such as the Varimax rotation for instance, cf. section 3.3) of the systematic factors leave the law of the vector \( X \) unchanged. Nevertheless, we may prove\(^{13}\) that a simple rotation of risk factors, modifying the matrix of factor loadings, directly affects the law of Hoeffding terms that depend on subsets included in \( Z \). This feature will have an important consequence on the interpretation of contributions (cf. Section 3.3).

### 3.2 Systematic and idiosyncratic contributions to the risk measure

The portfolio risk is determined by means of a risk measure \( \varphi \), which is a mapping of the loss to a real number: \( \varphi: L \rightarrow \varphi[L] \in \mathbb{R} \). Usual quantile-based risk measures are the Value-at-Risk

\(^{12}\) For further details on this statistic procedure, we refer to common statistical book such as the one of Kline (2014)\[^{30}\].

\(^{13}\) Consider for instance the Large Pool Approximation of the portfolio loss: \( L_Z = \mathbb{E}[L|Z_1, Z_2] \). Case 1: we consider that the betas are equivalent. For instance, \( \forall k, \beta_{k,1} = \beta_{k,2} = b \) (with \( b \in [-1,1] \)) which implies that \( \phi_1(L_Z; Z_1) = \phi_2(L_Z; Z_2) \) in distribution, and the contribution of each element is the same. Case 2: we now consider the special one factor model. For instance \( \beta_{k,1} = \sqrt{2} \times b^2 \) and \( \beta_{k,2} = 0 \) implying that \( \phi_1(L_Z; Z_1) \neq \phi_2(L_Z; Z_2) \) in distribution. Case 3: we finally consider a permutation of the case 2: \( \beta_{k,1} = 0 \) and \( \beta_{k,2} = \sqrt{2} \times b^2 \) leading to the same conclusion. In those 3 cases, the risk is identical but factor contributions differ.
(VaR) and the Conditional-Tail-Expectation\(^{14}\) (CTE). Let \(\alpha \in [0,1]\) be some given confidence level, VaR is the \(\alpha\)-quantile of the loss distribution of \(L\): \(\text{VaR}_\alpha[L] = \inf \{ l \in \mathbb{R} \mid \mathbb{P}(L \leq l) \geq \alpha \}\). On the other hand, CTE is the expectation of the loss conditional to loss occurrences higher than the \(\text{VaR}_\alpha[L]\): \(\text{CTE}_\alpha[L] = \mathbb{E}[L \mid L \geq \text{VaR}_\alpha[L]]\).

By definition, the portfolio loss equals the sum of individual losses: \(L = \sum_{k=1}^{K} L_k\). As we showed earlier it can also be defined as the sum of the Hoeffding decomposition terms: \(L = \sum_{S \subseteq \{1, \ldots, M\}} \phi_S(L; F_m, m \in S)\). To understand risk origin in the portfolio, it is common to refer to a contribution measure \(C_{L_k}^0\) (\(C_{\phi_S}^0\) respectively) of the position \(k\) (of the Hoeffding decomposition term \(\phi_S\)) to the total portfolio risk \(\varrho[L]\). The position risk contribution is of course of great importance for hedging, capital allocation, performance measurement and portfolio optimization and we refer to Tasche (2008)[31] for a detailed presentation. Just as fundamental as the position risk contribution, the factor risk contribution helps unravel alternative sources of portfolio risk. Papers dealing with this topic include the following. Cherny and Madan (2007)[32] consider the conditional expectation of the loss with respect to the systematic factor and name it factor risk brought by that factor. Martin and Tasche (2007)[33] also consider the same conditional expectation, but then apply the Euler’s principle taking the derivative of the portfolio risk in the direction of this conditional expectation and call it risk impact. Rosen and Saunders (2007)[28] apply the Hoeffding decomposition of the loss with respect to sets of systematic factors, the first several terms of this decomposition coinciding with conditional expectations mentioned above.

Theoretical and practical aspects of various allocation schemes have been analyzed in several papers (see Dhaene et al. (2012)[34] for a review). Among them, the marginal contribution method, based on Euler’s allocation rule, is quite a standard one (see Tasche (2007))[35]). To be applied here, an adaptation is required because we face discrete distributions (see Laurent (2003)[36] for an in-depth analysis of the technical issues at hand). Yet, under differentiability conditions, and taking VaR as the risk measure\(^{15}\), it can be shown (see Gouriéroux, Laurent and Scaillet (2001)[37]) that the marginal contribution of the individual loss \(L_k\) to the risk associated with the aggregate loss \(L = \sum_{k=1}^{K} L_k\) is given by \(C_{L_k}^{\text{VaR}} = \mathbb{E}[L_k \mid L = \text{VaR}_\alpha[L]]\). Besides, computing this expectation does not involve any other assumption than integrability and defining risk contributions along these lines fulfills the usual full allocation rule \(L = \sum_{k=1}^{K} C_{L_k}^{\text{VaR}}\) (see Tasche (2008)[31] for details on this rule).

Similarly, we can compute the contributions of the different terms involved in the Hoeffding decomposition of the aggregate loss. For instance, the contribution of the systematic term \(\phi_1(L; Z)\) is readily derived as \(C_{L_k}^{\text{VaR}} = \mathbb{E}[\phi_1(L; Z) \mid L = \text{VaR}_\alpha[L]]\). Likewise, contributions of specific risk and of cross-effect terms can easily be written and added afterwards to the systematic term so as to retrieve the risk measure of the aggregate loss. As shown in subsection 3.1, when we deal with a (systematic) two-factor model, we can further break

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\(^{14}\) Note that Conditional-Tail-Expectation is also known as Expected-Shortfall which is equivalent as long as \(L\) is continuous.

\(^{15}\) By defining \(\text{VaR}_\alpha[L] = \mathbb{E}[L \mid L = \text{VaR}_\alpha[L]]\), similar results hold for CTE (up to a sign change from \(L = \text{VaR}_\alpha[L]\) to \(L \geq \text{VaR}_\alpha[L]\)).
down the Large Pool Approximation loss, $L_Z$, to cope with the marginal effects of the two factors $(Z_1, Z_2)$ and with the cross effects. The additivity property of risk contributions prevails when subdividing the vector of risk factors into multiple blocks. Therefore, our risk contribution approach encompasses that of Rosen and Saunders (2010)[28] since it also includes the important (especially in our trading book context) contribution of specific risk to the portfolio risk.

It is also noteworthy that although our approach could be deemed as belonging to the granularity adjustment corpus, relying as well on large pool approximations, the techniques and the mathematical properties involved here (such as differentiability, associated with smooth distributions) are deeply different.

### 3.3 Interpretation of factor contributions

Note that the risk measures inherit the rotation-invariance property of the underlying asset value vector $X = [X_1, ..., X_k, ..., X_K]^T$. Contrarily, since the Hoeffding terms depending on subsets included in $\mathcal{Z}$ are not rotation-invariant, the computation of the risk contributions at the level of individual risk factors $(Z_1, Z_2)$ cannot be done unambiguously. Therefore, an infinite number of factors loadings combinations might lead to the same risk measure but to very different contributions. To overcome this feature, we consider several ways to cope with this identification problem and expand on the technical details in the appendix. These methods have advantages and drawbacks but, to date, we have not run any computational experiments to test the superiority of one of them$^{16}$.

The rotation of a factor matrix is a problem that dates back to the beginning of multiple factor analysis. Browne (2001)[38] provides an overview of available analytic rotation methods. Among them, the Varimax method, which was developed by Kaiser (1958)[39] is the most popular. Formally, Varimax searches for a rotation (i.e. a linear combination) of the original factors such that the variance of the loadings is maximized. This simplifies the factor interpretation because, after a Varimax rotation, each factor tends to have either large or small loadings for any particular $X_k$. Nevertheless, because the larger weight may be on either the first or the second factor, the contribution interpretation remains delicate.

An alternative iterative method may also be explored, easy to implement and generalizable to any $J$-factor model, that searches for increasing the relative importance of the $Z_1$ in $X_k$. It consists in calibrating a 1-factor model and a 2-factor model from an initial correlation matrix $C_0$ by using a similar optimization problem to equation [2] with $\beta \in \mathbb{R}^{K \times 1}$ and $\beta \in \mathbb{R}^{K \times 2}$ respectively. It follows that the obligor asset values are provided by $X_k^{(1)}$ and $X_k^{(2)}$ respectively.

By stating $\beta_{k,1} = \beta_{k,1}^{(1)}$ and $\beta_{k,2} = \sqrt{\left(\beta_{k,2}^{(2)}\right)^2 + \left(\beta_{k,1}^{(2)}\right)^2 - \left(\beta_{k,1}^{(1)}\right)^2}$, the 1-factor model is completed thanks to the 2-factor model. This procedure ensures that the first systemic factor provides the most important part of the model dynamic whereas the second factor is an

---

$^{16}$ Note that in our numerical applications (see Section 4) we only rely on the macro decomposition of the unconditional loss (see equation 4). Therefore, this interpretation issue is not relevant and we do not use any of these proposed methods.
adjustment. Unfortunately, this procedure produces a correlation structure among the asset values different from a direct computation of a 2-factor model.

Finally, an intuitive and suitable criterion would be the maximization of the first systematic factor contribution by means of a factor rotation; the second factor contribution being interpreted as a systematic adjustment (see Appendix). Unfortunately, this optimization problem suffers from a major drawback: the optimal rotation angle \( \theta^* \) depends on the portfolio composition that can eventually lead to factor loading inconsistencies between two different portfolios containing the same position.

4 Numerical applications

This section is devoted to numerical applications whereby the impacts of \( J \)-factor correlation structures on risk measures and risk contributions are considered.

We based our numerical analysis on representative long/long and long/short credit-sensitive portfolios. Since we reckon to focus on widely traded issuers, who represent a large portion of the banks exposure, we opt for a portfolio with large Investment Grades corporates. This choice is also consistent with hypothetical portfolios enforced by the RCAP showing a higher level of IRC variability for bespoke portfolios than for diversified ones. Specifically, we consider the composition of the iTraxx Europe index, taken on the 31st October 2014. The portfolio\(^{17} \) is composed of \( K = 121 \) European Investment Grade companies, 27 of which are Financials, the remaining being tagged Non-Financials.

We successively look at two types of portfolio: (i) a diversification portfolio, comprised of positive-only exposures (long-only credit risk), (ii) a hedge portfolio, built up of positive and negative exposures (long-short credit risk). The distinction parallels the one between the banking book, containing notably long-credit loans, and the trading book, usually resulting from a mix of long and short positions (like for instance bonds or CDS) – within the latter, an issuer’s default on a negative exposure yields a gain. Regarding the diversification portfolio, we consider a constant and equally weighted effective exposure for each name, and for conciseness, the LGD rate is set to 100% for each position so that \( \forall k, w_k = 1/K \) and \( \sum_{k=1}^{K} w_k = 1 \). Concerning the hedge portfolio, we assume long exposure to Financials and short exposure to Non-Financials. By considering \( w_{k \in \text{Financials}} = 1/K \) and \( w_{k \in \text{Non-Financials}} = -(27/K)/(K-27) \), the hedge portfolio is credit-neutral such that \( \sum_{k=1}^{K} w_k = 0 \).

The remaining terms in equation [1] are the default probabilities, \( p_k \), and the factor-loading matrix, \( \beta \in \mathbb{R}^{K \times J} \). The latter is the concern of the next subsection devoted to calibration of correlation matrix in the so-called "nearest correlation matrix with \( J \)-factor structure" framework. For the sake of numerical application, we use default probabilities\(^{18} \) provided by

\(^{17} \) The index is normally composed of 125 names. Nevertheless, due to lack of data for initial correlation computation, it was not possible to estimate a \((125 \times 125)\)-matrix.

\(^{18} \) Other data sources could be used in further versions of the paper.
the Bloomberg Issuer Default Risk Methodology\textsuperscript{19}. Figure [1] illustrates default probabilities’ frequencies of the portfolio’s companies grouped by Financials and Non-Financials. Financials clearly show higher and more dispersed default probabilities, with a mean and a standard deviation equal to 0.16\% and 0.16\% respectively, compared to 0.08\% and 0.07\% for Non-Financials.

![Figure 1: Histogram of the default probabilities distribution](image)

In the next subsections, we discuss first results on the correlation matrix calibration through the numerical optimization program previously exposed (cf. equation [2]). Then, we consider the impact of such correlation matrix on the risk by using Hoeffding-based representation of the aggregate loss. The analysis is performed for both the diversification and the hedge portfolios. Finally, in the last part, we compute the contribution of the systematic part of the loss to the risk for the two considered portfolios.

### 4.1 Correlation calibration

Following the Committee’s proposal, especially indications provided in Annex 2 of the last consultative document on the FRTB (BCBS 2013)[7], we use listed equity prices of the 121 issuers spanning a one-year period to calibrate the initial default correlation matrix through the empirical estimator.

In order to illustrate the sensitivity to the calibration window, we use two sets of equity time-series. Period 1 is chosen during a time of high market volatility from 07/01/2008 to 07/01/2009, whereas Period 2 spans a comparatively lower market volatility window, from 07/01/2013 to 07/01/2014. For both periods, the computed (121 × 121)-matrix consists in a matrix of pairwise correlations that we retreat\textsuperscript{20} to ensure semi-definite positivity.

\textsuperscript{19} Bloomberg DRSK methodology is based on the model of Merton (1974)[12]. The model does not use credit market variables, rather it is an equity markets-based view of default risk. In addition to the market data and company balance sheet fundamental, the model also includes companies’ income statements.

\textsuperscript{20} Through spectral projection. Note that this treatment is not necessary if we only consider the simulation of the calibrated J-factor model.
Furthermore, Period 2 is used to define other initial correlation matrices with a view to analyze the effects on the $J$-factor model of changes in the correlation structure as computed from different types of financial data. We then consider three alternative sources: (i) the prescribed IRBA correlation formula$^{21}$, grounded on the issuer’s default probability; (ii) the GCorr methodology of Moody’s KMV; (iii) the issuers’ CDS spreads relative changes.

For each of these five initial correlation matrices, $C_0$, the optimization problem in equation [2] is solved, for $J = 1, 2, 5$ factors. Figure [2] to Figure [6] exhibit histograms of the pairwise correlations distribution for each of the five initial correlation matrices along with those of the three $J$-factor models, $J = 1, 2, 5$, and the $(121 \times 121)$-model. Table [1] exposes the calibration results. The column “Terminal value” corresponds to the optimal value of the objective function given by the Frobenius norm whereas the three right hand side columns state the average pairwise correlations from, respectively, the overall portfolio matrix, the Financials sub-matrix and the Non-Financials sub-matrix.

In going through each configuration, the following general comments can be made upon the correlation distributions. We first remark important disparities among correlation distributions depending on the configuration of the initial correlation matrix. As expected, the “Equity – Period 1” configuration (see Figure [2]) shows distributions with large dispersions due to high market volatility, and modes around high levels, whereas the “Equity – Period 2” configuration (see Figure [3]) shows an even more asymetric distribution with a peak around 10%. The “IRBA – Period 2” configuration (see Figure [4]) is prescriptive for all models, yielding similar correlation levels around 25%. The “Moody’s KMV – Period 2” configuration (see Figure [5]) shows asymmetric but centered distributions around somehow low correlation levels (30%). Finally, the “CDS – Period 2” configuration (see Figure [6]) shows normal-like distributions with medium-level modes around 60% to 70%.

Overall, it seems that the factor-model approximation increases some correlations and decreases others compared to the initial correlation matrix, $C_0$. This fact will have important consequences on the risk measure, depending on whether we consider a diversification portfolio or a hedge portfolio where expositions may be positive or negative.

Looking now at a sole configuration, distributions among the $J$-factor models seem close together to that of the initial matrix, meaning that the nearest matrix approach performs correctly. Levels of the Frobenius norm in Table [1] confirm however the intuition that increasing the number of factors $J$ tends to produce a better fit of the $J$-factor model to the $(121 \times 121)$-model. This is also true for less dispersed correlation matrices as can be seen in the case of the “IRBA formula” intial matrix showing the smallest levels of terminal value of the objective function while presenting the most centered correlation distributions. The reason is that more regular correlation structures require fewer factors to be faithfully reproduced.

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$^{21}$ IRB approach is based on a 1-factor model: $X_k = \sqrt{\rho_k Z_1} + \sqrt{1 - \rho_k} \epsilon_k$. Thus, $\text{Correl}(X_k, X_j) = \sqrt{\rho_k \times \rho_j}$, where $\rho_k$ is provided by a prescribed formula: $\rho_k = 0.12 \times \frac{1-e^{-50 \rho_k}}{1-e^{-50}} + 0.24 \times \left(1 - \frac{1-e^{-50 \rho_k}}{1-e^{-50}}\right)$. 

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Table [1]: Factor-model calibration

<table>
<thead>
<tr>
<th>Initial matrix</th>
<th>Correlation matrix</th>
<th>Terminal value of the objective function (Frobenius norm)</th>
<th>Average pairwise correlations</th>
<th>Average pairwise correlation - Financials</th>
<th>Average pairwise correlation - Non-Financials</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity - Period 1</td>
<td>$C_0$</td>
<td>0</td>
<td>0.64</td>
<td>0.30</td>
<td>0.61</td>
</tr>
<tr>
<td>1-factor model</td>
<td>243.49</td>
<td>0.64</td>
<td>0.78</td>
<td>0.61</td>
<td></td>
</tr>
<tr>
<td>2-factor model</td>
<td>51.45</td>
<td>0.64</td>
<td>0.78</td>
<td>0.61</td>
<td></td>
</tr>
<tr>
<td>5-factor model</td>
<td>7.23</td>
<td>0.65</td>
<td>0.80</td>
<td>0.61</td>
<td></td>
</tr>
<tr>
<td>Equity - Period 2</td>
<td>$C_0$</td>
<td>0</td>
<td>0.18</td>
<td>0.23</td>
<td>0.19</td>
</tr>
<tr>
<td>1-factor model</td>
<td>705.61</td>
<td>0.14</td>
<td>0.12</td>
<td>0.16</td>
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</tr>
<tr>
<td>2-factor model</td>
<td>285.12</td>
<td>0.14</td>
<td>0.19</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>5-factor model</td>
<td>24.98</td>
<td>0.18</td>
<td>0.23</td>
<td>0.19</td>
<td></td>
</tr>
<tr>
<td>IRBA - Period 2</td>
<td>$C_0$</td>
<td>0</td>
<td>0.24</td>
<td>0.26</td>
<td>0.24</td>
</tr>
<tr>
<td>1-factor model</td>
<td>0.05</td>
<td>0.24</td>
<td>0.26</td>
<td>0.24</td>
<td></td>
</tr>
<tr>
<td>2-factor model</td>
<td>0.07</td>
<td>0.24</td>
<td>0.26</td>
<td>0.24</td>
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</tr>
<tr>
<td>5-factor model</td>
<td>0.96</td>
<td>0.25</td>
<td>0.27</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>Moody’s KM - Period 2</td>
<td>$C_0$</td>
<td>0</td>
<td>0.28</td>
<td>0.45</td>
<td>0.26</td>
</tr>
<tr>
<td>1-factor model</td>
<td>17.15</td>
<td>0.28</td>
<td>0.41</td>
<td>0.26</td>
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<tr>
<td>2-factor model</td>
<td>5.29</td>
<td>0.28</td>
<td>0.46</td>
<td>0.26</td>
<td></td>
</tr>
<tr>
<td>5-factor model</td>
<td>5.23</td>
<td>0.28</td>
<td>0.46</td>
<td>0.26</td>
<td></td>
</tr>
<tr>
<td>CDS - Period 2</td>
<td>$C_0$</td>
<td>0</td>
<td>0.58</td>
<td>0.80</td>
<td>0.57</td>
</tr>
<tr>
<td>1-factor model</td>
<td>59.06</td>
<td>0.58</td>
<td>0.68</td>
<td>0.56</td>
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<tr>
<td>2-factor model</td>
<td>30.38</td>
<td>0.58</td>
<td>0.80</td>
<td>0.57</td>
<td></td>
</tr>
<tr>
<td>5-factor model</td>
<td>14.41</td>
<td>0.58</td>
<td>0.79</td>
<td>0.57</td>
<td></td>
</tr>
</tbody>
</table>

Figure [2]: Equity - Period 1

Figure [3]: Equity - Period 2

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4.2 Impact on the risk
In this subsection, we analyze the impacts of initial correlation matrices on the portfolio risk. Using Hoeffding-based representation, we also study the contribution of both systematic and idiosyncratic parts of the loss. Numerical applications are based on Monte Carlo simulations of the portfolio loss. We consider $MC \in \mathbb{N}$ i.i.d replications\footnote{Numerical applications are based on simulations using ten millions scenarios.} of the loss random variable $L$ and note $L^{(n)}$ the realization of the loss on scenario $n \in \{1, \ldots, MC\}$. 

\footnote{Numerical applications are based on simulations using ten millions scenarios.}
Recall that the unconditional loss is a discrete random variable that can only take a finite number of realization values\(^{23}\): \(\forall n, L^{(n)} \in \mathbb{L}\). In consequence, the VaR estimator is the value of the \((\alpha \times MC)\)-ordered loss realization. Discreteness\(^{24}\) of \(L\) implies that the mapping \(\alpha \mapsto Var_{\alpha}[L]\) is piecewise constant. Subsequently, jumps in the risk measure are possible for small changes in the default probability. Figures [7 to 8] illustrate this feature for the diversification and the hedge portfolios in the “IRBA – Period 2” configuration. Nevertheless, the “length of each piece” depends on the underlying correlation structure. Thus, with a more complex correlation structure, the mapping \(\alpha \mapsto Var_{\alpha}[L]\) in the “Equity – Period 1” configuration is smoother than in the “IRBA – Period 2” configuration, and is far above (see Figures [9 to 10]).

Recall that the diversification portfolio is composed of long-only positions, i.e. with positive effective exposure. Hence, any issuer’s default is a loss. Conversely, the hedge portfolio is composed of long and short positions so that an issuer’s default may be a gain or a loss. The sum of effective exposures is equal to one for the diversification portfolio but to zero for the hedge portfolio.

For both the diversification portfolio and the hedge portfolio and for each of the five initial correlation matrices, we consider the \(VAR_{0.99}[L]\) computed from losses simulation via the three \(J\)-factor models and the \((121 \times 121)\)-model. Main results are reported in Table [2].

In almost all cases, the 5-factor model over-performed 1-factor and 2-factor models in computing a risk measure as close as the \((121 \times 121)\)-model’s one. For dispersed correlation structures, a higher number of systematic factors allows a better replication of the risk measure obtained on the \((121 \times 121)\)-model’s risk measure, that is, the risk measure of the loss determined through simulations with the non-approximated initial correlation structure. Conversely, for the less complex “IRBA formula” correlation structure, the 1-factor model fully explains VaR levels of both diversification and hedge portfolios.

Comparing the diversification portfolio with the hedge portfolio, we observe a hedge benefit linked to the latter’s long-short configuration inducing a smaller risk level. However, intuitive reading of the loss distribution is made more complex and it appears that a greater number of factors is needed to fully replicate the risk. Hence, the 1-factor model often provides weak results as can be judged by high levels of relative difference with respect to the \((121 \times 121)\)-model’s risk measure. As before, this phenomenon is even more pronounced when considering dispersed correlation matrices, leading to a clustering of defaults over long exposures, not being mitigated by defaults over short ones.

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\(^{23}\) Depending to the default probabilities vector and the correlation structure, the vector of loss realization may contain a large number of zeros. For instance, in our numerical simulation, near 96% of the loss realizations takes zeros as value.

\(^{24}\) Since we deal with discrete distributions, we cannot rely on standard asymptotic properties of sample quantiles. At discontinuity points of VaR, sample quantiles do not converge. This can be solved thanks to the asymptotic framework introduced by Ma, Genton and Parzen (2011)[40] and the use of the mid-distribution function.
Figures [7, 8]: VaR as a function of α in the diversification (left) and hedge (right) portfolios

Figures [7, 8] plot the loss quantiles for the J-factor models calibrated in case of the “IRBA – Period 2” configuration. (*\(J = 1, 2\) and 5)*

Figures [9, 10]: VaR as a function of α in the diversification (left) and hedge (right) portfolios

Figures [9, 10] plot the loss quantiles for the J-factor models calibrated in case of the “Equity – Period 1” configuration. (*\(J = 1, 2\) and 5)*

<table>
<thead>
<tr>
<th>Initial matrix</th>
<th>Risk</th>
<th>(\text{VaR}_{\alpha}[\text{L}])</th>
<th>(\text{VaR}_{\alpha}[\text{C}_0])</th>
<th>(\text{Relative diff. wrt } \text{C}_0)</th>
<th>(\text{VaR}_{\alpha}[\text{L}])</th>
<th>(\text{VaR}_{\alpha}[\text{C}_0])</th>
<th>(\text{Relative diff. wrt } \text{C}_0)</th>
<th>(\text{VaR}_{\alpha}[\text{L}])</th>
<th>(\text{VaR}_{\alpha}[\text{C}_0])</th>
<th>(\text{Relative diff. wrt } \text{C}_0)</th>
<th>(\text{VaR}_{\alpha}[\text{L}])</th>
<th>(\text{VaR}_{\alpha}[\text{C}_0])</th>
<th>(\text{Relative diff. wrt } \text{C}_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity - Period 1</td>
<td>VaR</td>
<td>0.198</td>
<td>0.190</td>
<td>0.190</td>
<td>0.054</td>
<td>0.042</td>
<td>0.042</td>
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</tr>
<tr>
<td>Equity - Period 2</td>
<td>VaR</td>
<td>0.099</td>
<td>0.083</td>
<td>0.091</td>
<td>0.099</td>
<td>0.028</td>
<td>0.017</td>
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<tr>
<td></td>
<td>Risk</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>IRBA - Period 2</td>
<td>VaR</td>
<td>0.041</td>
<td>0.041</td>
<td>0.041</td>
<td>0.041</td>
<td>0.017</td>
<td>0.017</td>
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<tr>
<td>Moody’s KMV -</td>
<td>VaR</td>
<td>0.058</td>
<td>0.050</td>
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<td>0.058</td>
<td>0.031</td>
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<tr>
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<td>Risk</td>
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<td>CDS - Period 2</td>
<td>VaR</td>
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<td>0.132</td>
<td>0.132</td>
<td>0.132</td>
<td>0.058</td>
<td>0.032</td>
<td>0.058</td>
<td>0.058</td>
<td>0.058</td>
<td>0.058</td>
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<tr>
<td></td>
<td>Risk</td>
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</table>

**Table [2]: Number of factors and risk measures**

NB: \(\mu\) stands for the average pairwise correlation and \(\sigma\) for its standard deviation. 1F stands for 1-factor model.
4.3 Systematic and idiosyncratic contributions to risk measure

Turning now to the Hoeffding-based representation (equations [4, 5, 6.1, 6.2]), we note $\Phi_S^{(n)}$ the realization of the projected loss (onto the subset of factors $S$) on the scenario $n$. With these notations, the contribution estimator is:

\[
C_{\Phi_S}^{\text{VaR}}[L, \alpha] = \mathbb{E}[\Phi_S | L = \text{VaR}_\alpha[L]] \Rightarrow \hat{C}_{\Phi_S}^{\text{VaR}}[L, \alpha] = \frac{\sum_{n=1}^{\text{MC}} \Phi_S^{(n)} 1_{\{L^{(n)} = \text{VaR}_\alpha[L]\}}}{\sum_{n=1}^{\text{MC}} 1_{\{L^{(n)} = \text{VaR}_\alpha[L]\}}} \quad [8]
\]

Since the conditional expectation defining the risk contribution is conditioned on rare events, this estimator requires intensive simulations to reach an acceptable confidence interval. In a similar setting, Glasserman (2005)[41] shows that $\hat{C}_{\Phi_S}^{\text{VaR}} \to C_{\Phi_S}^{\text{VaR}}$ as $\text{MC} \to \infty$, almost surely, and develops importance-sampling techniques to address this numerical difficulty. Although appealing, application of its results to the contribution of $\Phi_S$ to the risk is not straightforward.

Consequently, in the remainder of the section, we perform ordinary Monte Carlo simulations and compute the estimator in equation [8]. In order to reduce estimator variance, we perform ten million simulations. For a high level of $\alpha$, we observe good convergence for the diversification portfolios but larger variance for hedge portfolios, especially with equity correlations, less regular than IRBA’s ones.

Figures [11 to 14] illustrate the influence of $\alpha$ on the systematic contribution to the risk by considering the mapping $\alpha \mapsto \hat{C}_{\Phi_1}^{\text{VaR}}[L, \alpha]/\text{VaR}_\alpha[L]$ for both diversification and hedge portfolios in the “Equity – Period 1” configuration and “IRBA – Period 2” configuration.

Considering Figures [11 to 14], many remarks are formulated. First, it is clearly observable that complex correlation structure (such as in the “Equity – Period 1” configuration) induces noisy systematic contribution estimation. This phenomenon is more pronounced in the presence of long and short exposures (hedge portfolio) that structurally increase the estimator variance for a fixed number of scenarios.

Furthermore, we observe an increase in systematic contribution when the number of factors grows. This phenomenon is expected because the relative weight of systematic factors, in comparison with the idiosyncratic risk, increases when adding factors. In relation, we remark than when going further in the tail (up to $\alpha = 0.999$), the one factor approximation becomes increasingly valid for the diversification portfolio, confirming the Committee’s intuitions for the banking book credit modeling. In the “IRBA – Period 2” configuration (Figures [11, 12]), since the factor models are calibrated on a 1-factor correlation structure, it is obvious that all models are equivalent and therefore, the systematic contributions are the same in any $J$-factor model (with $J = 1,2,5$). Conversely, for the hedge portfolios, Figure [12] confirms that the 2-factor approximation may be inoperable when considering complex correlation matrices (as for the “Equity – Period 1” configuration).

\[\text{(Note that negative risk contributions may arise within the hedge portfolio.}\]
Finally, we observe that given the discrete nature of considered distributions, and similarly of the risk measure, the mapping \( \alpha \mapsto \hat{C}_\alpha^{VaR}[L, \alpha]/\overline{VaR}_\alpha[L] \) is piecewise constant.

One step further, still considering the “Equity – Period 1” and “IRBA – Period 2” configurations, we analyze the repartition of risk between the systematic factors, the idiosyncratic risks and their interactions. To avoid noisy estimations, the risk (see Table [3]) is measured via the Value-at-Risk with \( \alpha = 0.99 \).

First, let us point out that an expected feature of the hedge portfolios is their smaller average losses in comparison with the diversification portfolios. Thus, by considering the “IRBA – Period 2” configuration, the hedge benefit is important since the average loss is near zero.

Another important feature reported in Table [3] is the large interaction contributions for these configurations. Recall that interaction contribution, which can be interpreted as a residual, refers to the cross effect between the systematic and idiosyncratic risks that are not taken into account by previous individual contributions. Its high level stems logically from low levels of both the systematic and the idiosyncratic contributions.

Regarding the “Equity – Period 1” configuration, we observe that idiosyncratic risk contribution is small, even for the diversification portfolio. Indeed, a high level of correlation in this setting (see Figure [2]) implies a high level of the coefficient affected to the systematic factors.

**Figures [11, 12]: Systematic contribution to the VaR as a function of \( \alpha \) in the diversification (left) and hedge (right) portfolios**

Figures [11, 12] plot the systematic contribution to the VaR for \( \alpha = [0.99, 1] \) in case of the “Equity – Period 1” configuration. (\( J = 1, 2 \) and 5)
Figures [13, 14]: Systematic contribution to the VaR as a function of $\alpha$ in the diversification (left) and hedge (right) portfolios

Figures [13, 14] plot the systematic contribution to the $f$-factor models’ VaR for $\alpha = [0.99, 1]$ in case of the “IRBA – Period 2” configuration. ($f = 1, 2$ and 5)

<table>
<thead>
<tr>
<th></th>
<th>Average loss: $\phi_0(L)$</th>
<th>Systematic contribution: $\phi_1(L; E)$</th>
<th>Idiosyncratic contribution: $\phi_2(L; E)$</th>
<th>Interaction contribution $\phi_{1,2}(L; Z, E)$</th>
<th>Total contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Equity - Period 1</strong></td>
<td></td>
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<td></td>
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<tr>
<td>Diversification portfolio</td>
<td>6.9%</td>
<td>68.2%</td>
<td>0.7%</td>
<td>24.2%</td>
<td>100.0%</td>
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<tr>
<td>Hedge portfolio</td>
<td>4.4%</td>
<td>18.5%</td>
<td>3.9%</td>
<td>73.1%</td>
<td>100.0%</td>
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<tr>
<td><strong>IRBA - Period 2</strong></td>
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<tr>
<td>Diversification portfolio</td>
<td>3.8%</td>
<td>39.4%</td>
<td>13.0%</td>
<td>43.9%</td>
<td>100.0%</td>
</tr>
<tr>
<td>Hedge portfolio</td>
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<td>5.3%</td>
<td>35.1%</td>
<td>58.9%</td>
<td>100.0%</td>
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</table>

Table [3]: Systematic and idiosyncratic contribution to the $\text{VaR}_{0.99}$ in the 2-factor model.

5 Conclusion

Assessment of default risk in the trading book (IDR, Incremental Default Risk charge) is a key point in the Fundamental Review of the Trading Book. At the core of the approach are some constraints on correlation matrices and factor structure that underpin the dependence structure between defaults. This paper has considered the practical implication of such modeling constraints, based on portfolios inspired by those considered in the RCAP (Regulatory Consistency Assessment Programme) Hypothetical Portfolio Exercises. We have been using Hoeffding decomposition of portfolio exposures to factor and specific risks to monitor risk contributions to the regulatory 99.9% one-year VaR. The key insights are the following:

- For diversification (long only) portfolio of corporate exposures, the well-known Basel II one-factor approximation performs quite well.
- When it comes to long/short hedge credit portfolios, typically involved in the trading book, the two-factor constraint is of little help. Unsurprisingly, the key ingredients are the choice of calibration data. Jointly using widely dispersed pairwise equity correlations and considering tails risks leads to a number of cliff effects and numerical
instabilities, i.e. small changes in exposures or other parameters (default probabilities) may lead to significant changes in risk measures and contributions.

References


Appendix: Maximization of the first systematic factor contribution

The Hoeffding-based representation may lead to ambiguous contributions. Since an infinite number of factors loadings combinations might lead to the same risk measure but to very different contributions, we choose an intuitive and suitable criterion consisting in the maximization of the first systematic factor contribution by means of a factor rotation. The second factor contribution being interpreted as a systematic adjustment.

Let us consider a rotation matrix $R(\theta)$ ($RR^t = I_d$):

$$
R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}
$$

we write $X_k = (\beta_k R)(R^t Z) + \sigma_k \epsilon_k = \tilde{\beta}_k Z + \sqrt{1 - \beta_k^2} \epsilon_k$. Under this notation, the objective is to maximize $C_{\phi_1}^\theta$ by modifying the rotation angle $\theta$. For instance, considering $CTE_\alpha[L]$, the optimization program is as follows:

$$
\arg \max_{\theta} C_{\phi_1}^{CTE} = E \left[ \sum_{k=1}^K w_k \Phi \left( \frac{\phi^{-1}(p_k) - \tilde{\beta}_{k,1} Z_1}{\sqrt{1 - \tilde{\beta}_{k,1}^2}} \right) |L \geq Var_\alpha[L] \right] - E[L]
$$

$$
\tilde{\beta}_{k,1} = \cos(\theta) \beta_{k,1} + \sin(\theta) \beta_{k,2}
$$

This optimization problem suffers from a major drawback though: the optimal angle $\theta^*$ depends on the portfolio composition that can eventually lead to factor loading inconsistencies between two different portfolios containing the same position.