Trading book and credit risk: how fundamental is the Basel review?


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Abstract

In its October 2013's consultative paper for a revised market risk framework (FRTB), the Basel Committee suggests that non-securitization credit positions in the trading book be subject to a separate Incremental Default Risk (IDR) charge, in an attempt to overcome practical challenges raised by the joint modeling of the discrete (default risk) and continuous (spread risk) components of credit risk, enforced in the current Basel 2.5 Incremental Risk Charge (IRC). Banks would no longer have the choice of using either a single-factor or multi-factor default risk model but instead, market risk rules would require the use of a two-factor simulation model and a 99.9-VaR capital charge. Proposals are also made as to how to account for diversification effects with regard to calibration of correlation parameters. In this article, we analyze theoretical foundations of these proposals, particularly the link with one-factor model used for the banking book and with a general J-factor setting. We thoroughly investigate the practical implications of the two-factor and correlation calibration constraints through numerical applications. We introduce the Hoeffding decomposition of the aggregate unconditional loss for a systematic-idiosyncratic representation. Impacts of J-factor correlation structures on risk measures and risk contributions are studied for long-only and long-short credit-sensitive portfolios.

Keywords: Portfolio Credit Risk Modeling, Factor Models, Risk Contribution, Fundamental Review of the Trading Book.
1. Basel recommendations on credit risk

Created in 1974 by ten leading industrial countries and now including supervisors from twenty-seven countries, the Basel Committee on Banking Supervision (BCBS, henceforth “the Committee”) is responsible for strengthening the resilience of the global financial system, ensuring the effectiveness of prudential supervision and improving the cooperation among banking regulators. To accomplish its mandate, the Committee formulates broad supervisory standards and guidelines and it recommends statement of best practices in banking supervision that member authorities and other nations’ authorities are expected to implement step-wise within their own national systems. Essential propositions concern standardized regulatory capital requirements, determining how much capital has to be held by financial institutions to be protected against potential losses coming from credit risk realizations (defaults, rating migrations), market risk realizations (losses attributable to adverse market movements), operational risk, etc.

1.1. Credit risk in the Basel I, II, 2.5 and III agreements

The Committee’s recommendations of 1988 (BCBS, 1988) established a minimum required capital amount through the definition of the so-called Cooke ratio and the categorization of credit risk levels into homogeneous buckets based on issuers’ default probability. Nevertheless, this approach ignored the heterogeneity of banks loans in terms of risk and led the Committee to develop new sets of recommendations.

Updating the earlier recommendations of 1988, Basel II agreements (BCBS, 2005) define a regulatory capital through the concept of Risk Weighted Assets (RWAs) and through the McDonough ratio, including operational and market risks in addition to credit risk. In particular, to make regulatory capital more risk sensitive, the text sets out a more relevant measure for credit risk by considering the borrower’s quality through internal rating system for approved institutions: the Internal Rating Based (IRB) approach. In this framework, the RWA related to credit risk in the banking book measures the exposition of a bank granting loans by applying a weight according to the intrinsic riskiness of each asset (a function of the issuer’s default probability and effective loss at default time). The Committee went one step further in considering also portfolio risk addressed with a prescribed model based on the Asymptotic-Single-Risk-Factor model (ASRF, described hereafter) along with a set of constrained calibration methods (borrower’s asset value correlation matrix in particular). Despite significant improvements, the Basel II capital requirement calculation for the credit risk remains confined to the banking book.

A major gap thus revealed by the 2008 financial crisis was the inability to adequately identify the credit risk of the trading book positions (any component of the trading book: instruments, sub-portfolios, portfolios, desks...), enclosed in credit-quality linked assets. Considering this deficiency, the Committee revised market risk capital requirements in the 2009’s reforms, also known as Basel 2.5 agreements (BCBS, 2009), that add a new capital requirement, the Incremental Risk Charge (IRC), designed to deal with long term changes in credit spreads, and a specific capital charge for correlation products, the Comprehensive Risk Measures (CRM). More exactly, the IRC is a capital charge that captures default and migration risks through a VaR-type calculation at 99.9% on a one-year horizon. As opposed to the credit risk treatment in the banking book, the trading book model specification results from a complete internal model validation process whereby financial institutions are led to build their own
In parallel to new rules elaboration, the Committee has recently investigated the RWAs comparability among institutions and jurisdictions, for both the banking book (BCBS, 2013)[9] and the trading book (BCBS, 2013)[10, 11], through a Regulatory Consistency Assessment Program (RCAP) following previous studies led by the IMF in 2012 (see Le Leslé and Avramova (2012)[43]). Based on a set of hypothetical benchmark portfolios, reports show large discrepancies in risk measure levels, and consequently in RWAs, amongst participating financial institutions. The related causes of such a variability are numerous. Among the foremost is the heterogeneity of risk profiles, consecutive to institutions’ diverse activities, and divergences in local regulation regimes. In conjunction with these structural causes, the Committee also raises important discrepancies among internal methodologies of risk calculation, and in particular, those of the trading book’s RWAs. A main contributor to this variability appears to be the modeling choices made by each institution within their IRC model (for instance, whether it uses spread-based or transition matrix-based models, calibration of the transition matrix or that of the initial credit rating, correlations’ assumptions across obligors, etc.).

1.2. Credit risk in the Fundamental Review of the Trading Book

In response to these shortcomings, the Committee has been working ever since 2012 towards a new post-crisis update of the market risk global regulatory framework, known as Fundamental Review of the Trading Book (FRTB) (BCBS 2012, 2013, 2015)[7, 8, 14]. Notwithstanding long-lasting impact studies and ongoing consultative working groups, no consensus seems to be fully reached so far. Main discussions arise from the proposal transforming the IRC in favor of a default-only risk capital charge (i.e. without migration feature), named Incremental Default Risk (IDR) charge. With a one-year 99.9-VaR calculation, IDR capital charge for the trading book would be grounded on a two-factor model:

“One of the key observations from the Committee’s review of the variability of market risk weighted assets is that the more complex migration and default models were a relatively large source of variation. The Committee has decided to develop a more prescriptive IDR charge in the models-based framework. Banks using the internal model approach to calculate a default risk charge must use a two-factor default simulation model [“with two systemic risk factors” according to (BCBS 2015)[15]], which the Committee believes will reduce variation in market risk-weighted assets but be sufficiently risk sensitive as compared to multi-factor models”. (BCBS 2013)[8].

The objective of constraining the IDR modeling choices by “limiting discretion on the choice of risk factors” has also been mentioned in a report to the G20, BCBS (2014)[13]. Going further, the Committee would especially monitor model risk through correlation calibration constraints. First consultative papers on the FRTB, (BCBS, 2012, 2013)[7, 8], prescribed to use listed equity prices to calibrate the default correlations. From the trading book hypothetical portfolio exercise (BCBS, 2014)[12], the Committee analyses that equity data was prevailing among financial institutions, while some of them chose CDS spreads for the Quantitative Impact Study (QIS). Indeed, equity-based prescribed correlations raise practical problems when data are not available4, as for instance for

4Likewise, no prescription has been yet formulated for the treatment of exposures depending on non-modellable risk-factors, due to a lack of data.
sovereign issuers, leading to consider other data sources. Consequently, the third consultative paper of the Committee (BCBS, 2015)[14], the subsequent ISDA response (ISDA, 2015)[37] and the instructions for the Bale III monitoring (BCBS 2015)[15] recommend the joint use of credit spreads and equity data.

“Default correlations must be based on credit spreads or on listed equity prices. Banks must have clear policies and procedures that describe the correlation calibration process, documenting in particular in which cases credit spreads or equity prices are used. Correlations must be based on a period of stress, estimated over a 10-year time horizon and be based on a 1-year liquidity horizon. These correlations should be based on objective data and not chosen in an opportunistic way where a higher correlation is used for portfolios with a mix of long and short positions and a low correlation used for a portfolio with long only exposures. [...] A bank must validate that its modeling approach for these correlations is appropriate for its portfolio, including the choice and weights of its systematic risk factors. A bank must document its modeling approach and the period of time used to calibrate the model.” (BCBS 2015)[15].

Our paper investigates the practical implications of these recommendations, and in particular, studies the impact of factor models and their induced correlation structures on the trading book credit risk measurement. The goal here is to provide a comparative analysis of risk factors contributions within a consistent theoretical framework. To this end, the scope of the analysis does not include PD and LGD estimations, representing significant challenges in the IDR modeling as well, for which the Committee also provides prescriptions (BCBS 2015)[15].

The paper is organized as follows. In Section 2, we describe a two-factor IDR model within the usual Gaussian latent variables framework, and analyze the link with the one-factor model used in the current banking book framework on the one hand, and with a general $J$-factor ($J > 1$) setting deployed in IRC implementations, on the other. Following the Committee’s recommendations, we look into the effects of correlation calibration constraints on each setting, using the so-called “nearest correlation matrix with $J$-factor structure” framework and we discuss main correlation estimation methods. In Section 3, we use the Hoeffding decomposition of the aggregate loss to explicitly derive contributions of systematic and idiosyncratic risks, of particular interest in the trading book. Section 4 is devoted to numerical applications whereby impacts of $J$-factor correlation structures on risk measures and risk contributions are considered. Representative long-only and long-short credit-sensitive portfolios are tested. The last section gathers concluding remarks.

2. Two-factor Incremental Default Risk Charge model

2.1. Model specification

The portfolio loss at a one-period horizon is modeled by a random variable $L$, defined as the sum of the individual losses on issuers’ default over that period. We consider a portfolio with $K$ positions: $L = \sum_{k=1}^{K} L_k$ with $L_k$ the loss on the position $k$. The individual loss is decomposed as $L_k = w_k \times I_k$ where $w_k$ is the positive or negative effective exposure at the time of default and $I_k$ is a random

\[ w_k = EAD_k \times LGD_k. \]

While we could think of stochastic LGDs, there is no consensus as regard to proper modelling choices, either regarding marginal LGDs or the joint distribution of LGDs and

\[ \text{The effective exposure of the position } k \text{ is defined as the product of the Exposure-At-Default (} EAD_k \text{) and the Loss-Given-Default (} LGD_k \text{). Formally: } w_k = EAD_k \times LGD_k. \]
variable referred to as the obligor $k$’s creditworthiness index, taking value 1 when default occurs, and 0 otherwise. For conciseness, we assume constant effective exposures at default, hence the sole remaining source of randomness comes from $\mathbb{I}_k$.

To define the probability distribution of the $L_k$’s as well as their dependence structure, we rely on a usual structural factor approach, that is, $\mathbb{I}_k$ takes value 1 or 0 depending on a set of - latent or observable - factors $\mathcal{F} = \{F_m|m = 1, \ldots, M\}$. The latter can be expressed through any factor model $h: \mathbb{R}^M \rightarrow \mathbb{R}$ such that creditworthiness is defined as $\mathbb{I}_k = 1(X_k \leq x_k)$, where $X_k = h(F_1, \ldots, F_M)$ and $x_k$ is a predetermined threshold. Modeling $\mathbb{I}_k$ thus boils down to modeling $X_k$. This model, introduced by Vasicek (1987, 2001)[61, 62] and based on seminal work of Merton (1974)[48], is largely used by financial institutions to model default risk either for economic capital calculation or for regulatory purposes.

More precisely, in these approaches, $X_k$ is a latent variable, representing obligor $k$’s asset value, which evolves according to a $J$-factor Gaussian model: $X_k = \beta_k Z + \sqrt{1 - \beta_k^T \beta_k} \varepsilon_k$ where $Z \sim \mathcal{N}(0, 1)$ is a $J$-dimensional random vector of systematic factors, $\varepsilon_k \sim \mathcal{N}(0, 1)$ is an idiosyncratic risk – all factors are i.i.d – and $\beta \in \mathbb{R}^{K \times J}$ is the factor loading matrix. In matrix notation, the random vector $X \in \mathbb{R}^{K \times 1}$ of asset values is written:

$$X = \beta Z + \sigma \varepsilon$$  \hspace{1cm} (1)

where $\sigma \in \mathbb{R}^{K \times K}$ is a diagonal matrix with elements $\sigma_k = \sqrt{1 - \beta_k^T \beta_k}$. This setting ensures that the random vector of asset values, $X$, is standard normal with a correlation matrix depending on $\beta$: $\beta \mapsto C(\beta) = \beta \beta^T + \text{diag}(\text{Id} - \beta \beta^T)$.

Threshold $x_k$ is chosen such that $P(\mathbb{I}_k = 1) = p_k$, where $p_k$ is the observed obligor $k$’s marginal default probability. From standard normality of $X_k$, it comes straightforwardly $x_k = \Phi^{-1}(p_k)$, with $\Phi(.)$ the standard normal cumulative function. The portfolio loss is then written:

$$L = \sum_{k=1}^{K} w_k \mathbb{1}_{\{x_k \leq \Phi^{-1}(p_k)\}}$$  \hspace{1cm} (2)

Since $\mathbb{I}_k$ is discontinuous, $L$ can take only a finite number of values in the set $L = \{\sum_{a \in A} w_a | \forall A \subseteq \{1, \ldots, K\}\}$. In the homogeneous portfolio, where all weights are equals: $\text{Card}(L) = K$. At the opposite, if all weights are different, then $\text{Card}(L) = 2^K$ and the numerical computation of quantile-based risk measures may be more difficult.

In the remainder of the article, we note $Z = \{Z_j|j = 1, \ldots, J\}$ the set of all systematic factors and $\mathcal{E} = \{\varepsilon_k|k = 1, \ldots, K\}$ the set of all idiosyncratic risks such that $\mathcal{F} = Z \cup \mathcal{E}$.

The single factor variant of the model is at the foundation of the Basel II credit risk capital charge. To benefit from asymptotic properties, the Committee capital requirement formula is based on the assumption that the portfolio is infinitely fine grained, i.e. it consists of a very large number of credits with small exposures, so that only one systematic risk factor influences portfolio default risk.

default indicators. The Basel Committee is not prescriptive at this stage and it is more than likely that most banks will retain constant LGDs.
The aggregate loss can be approximated by the systematic factor projection: \( L \approx L_Z = \mathbb{E}[L|Z] \), subsequently called “Large Pool Approximation”, where \( L_Z \) is a continuous random variable. This model is known as Asymptotic Single Risk Factor model (ASRF). Thin granularity implies no name concentrations within the portfolio (idiosyncratic risk being fully diversified) whereas the one-factor assumption implies no sector concentrations such as industry or country-specific risk concentration. Name and sector concentrations are largely looked into in the literature, particularly around the concept of the so-called granularity adjustment\(^6\). We refer to Fermanian (2014)[30] and Gagliardini and Gouriéroux (2014)[31] for recent treatments of this concept. Furthermore, a detailed presentation of the IRB modeling is provided by Gordy (2003)[33]. Note also that under these assumptions, Wilde (2001)[63] expresses a portfolio invariance property stating that the required capital for any given loan does not depend on the portfolio it is added to.

Granularity assumption in the IRB modeling for credit risk in the banking book is appealing since it allows straightforward calculation of risk measures and contributions. Nevertheless, since trading book positions may be few and/or heterogeneous, the Large Pool Approximation or any granularity assumption seems too restrictive in the trading book context. Conversely, there is a need for taking into account both systematic risk and idiosyncratic risk, furthermore in presence of a discrete loss distribution.

Apart from previous theoretical issues, an operational question concerns the meaning of underlying factors. In the banking book ASRF model, the systematic factor is usually interpreted as the state of the economy, i.e. a generic macroeconomic variable affecting all firms. Within multi-factor models\(^7\) \((J > 2)\), factors may be either latent, like in the ASRF model, or observable, thus representing industrial sectors, geographical regions, ratings and so on. A fine segmentation of observable factors allows modelers to define a detailed operational representation of the portfolio correlation structure. In its analysis of the trading book hypothetical portfolio exercise (BCBS, 2014) [12], the Committee reports that most banks currently use an IRC model with three or less factors, and only 3% have more than three factors. For the IDR, a clear meaning of the factors has not been yet provided by the Committee. Consequently, and for conciseness, in the remaining of the article we postulate general latent factor models that could be easily adapted to further Committee’s model specifications.

2.2. Assets values correlation calibration

The modeling assumptions on the general framework being made, we consider here the calibration of the assets values correlation matrix of the structural-type credit model. As previously mentioned, the Committee recommends the joint use of equity and credit spread data (notwithstanding such a

\(^6\)The theoretical derivation of this adjustment accounting for name concentrations was first done by Wilde (2001)[63] and improved then by Gordy (2003)[33]. Their name concentration approach refers to the finite number of credits in the portfolio. In contrast, the semi-asymptotic approach in Emmer and Tasche (2005)[27] refers to position concentrations attributable to a single name while the rest of the portfolio remains infinitely granular. Analytic and semi-analytic approaches that account for sector concentration exist as well. One rigorous analytical approach is Pykhthin (2004)[51]. An alternative is the semi-analytic model of Cespedes and Herrero (2006)[19] that derives an approximation formula through a numerical mapping procedure. Tasche (2005)[56] suggests an ASRF-extension in an asymptotic multi-factor setting.

\(^7\)Multi-factor models for credit-risk portfolio and the comparison to the one-factor model are documented in the literature. For instance, in the context of long-only credit exposure portfolio, Dillmann et al. (2007)[26] compare the correlation and the Value-at-Risk estimates among a one-factor model, a multi-factor model (based on the Moody’s KMV model) and the Basel II IRB model. Their empirical analysis with heterogeneous portfolio shows a complex interaction of credit risk correlations and default probabilities affecting the credit portfolio risk.
combination may raise consistency issues since pairwise correlations computed from credit spreads and equity data are sometimes quite distant). Nevertheless, at that stage, the Committee lets aside theoretical concerns as to which estimator of the correlation matrix is to be used and we here stress that this preliminary recommendation sets aside issues around the sensitivity of the estimation to the underlying calibration period and around the processing of noisy information, although essential to financial risk measurement.

Following the Committee’s prescription, we introduce $\tilde{X}$, the $(K \times T)$-matrix of centered stock or CDS-spread returns (where $T$, the time series length, is equal to 250), and:

$$
\Sigma = T^{-1}\tilde{X}^t \tilde{X} \quad (3)
$$

$$
C = (\text{diag}(\Sigma))^{-\frac{1}{2}} \Sigma (\text{diag}(\Sigma))^{-\frac{1}{2}} \quad (4)
$$

the standard estimators of the sample covariance and correlation matrices. It is well known that those matrices suffer some drawbacks. Indeed, when the number of variables (equities or CDS-spreads), $K$, is close to the number of historical returns, $T$, the total number of parameters is of the same order as the total size of the data set, which is problematic for the estimator stability. Moreover, when $K$ is larger than $T$, the matrices are always singular$^8$.

Within the vast literature dedicated to covariance/correlation matrix estimation from equities, we refer particularly to Michaud (1989)[49] for a proof of the instability of the empirical estimator, to Alexander and Leigh (1997)[1] for a review of covariance matrix estimators in VaR models and to Disatnik and Benninga (2007)[25] for a brief review of covariance matrix estimators in the context of the shrinkage method$^9$. Shrinkage methods are statistical procedures which consist in imposing low-dimensional factor structure to a covariance matrix estimator to deal with the trade-off between bias and estimation error. Indeed, the sample covariance matrix can be interpreted as a $K$-factor model where each variable is a factor (no residuals) so that the estimation bias is low (the estimator is asymptotically unbiased) but the estimation error is large. At the other extreme, we may postulate a one-factor model which may have a large bias from likely misspecified structural assumptions but little estimation error. According to seminal work of Stein (1956)[54], reaching the optimal trade-off may be done by taking a properly weighted average of the biased and the unbiased estimators: this is called shrinking the unbiased estimator. Within the context of default correlation calibration, we here focus on the approach of Ledoit and Wolf (2003)[42] who define a weighted average of the sample covariance matrix with the single-index model estimator of Sharpe (1963)[53]: $\Sigma_{\text{shrink}} = \alpha_{\text{shrink}} \Sigma_J + (1 - \alpha_{\text{shrink}}) \Sigma$, where $\Sigma_J$ is the covariance matrix generated by a $(J = 1)$-factor model and the weight $\alpha_{\text{shrink}}$ controls how much structure to impose. The authors show how to determine the optimal shrinking intensity ($\alpha_{\text{shrink}}$) and, based on historical data, illustrate their approach through numerical experiments where the method out-performs all other standard estimators.

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$^8$Note that this feature is problematic when considering the Principal Component Analysis (PCA) to estimate factor models because the method requires the invertibility of $\Sigma$ or $C$. To overcome this problem, Connor and Korajczyk (1986,1988) [21, 22] introduce the Asymptotic PCA which consist in applying PCA on the $(T \times T)$-matrix, $K^{-1}\tilde{X}^t \tilde{X}$, rather than on $\Sigma$. Authors prove that APCA is asymptotically equivalent to the PCA on $\Sigma$.

$^9$See also Laloux, Cizeau, Bouchaud and Potters (1999) [40] for evidences of ill-conditioning and of the “curse of dimension” within a random matrix theory approach, and Papp, Kaifa, Nowak and Kondor (2005) [50] for an application of random matrix theory to portfolio allocation.
Remark that we here consider the sole static case where the covariance/correlation matrices are supposed to be estimated on a unique and constant period of time, since those methods only are relevant in the current version of the Committee’s proposition.\footnote{A number of academic papers also address the estimation of dynamic correlations. See for instance the paper of Engle [2002] [28] introducing the Dynamic Conditional Correlation (DCC) or the paper of Engle and Kelly [2012] [29] for a brief overview of dynamic correlation estimation and the presentation of the Dynamic Equicorrelation (DECO) approach.}

In the sequel, we assume an initial correlation matrix $C_0$, estimated from historical stock or CDS spread returns, following the Committee’s proposal. However, to study the impact of the correlation structure on the levels of risk and factors contributions (cf. Section 4), we shall consider other candidates as the initial matrix such as the “shrunked” correlation matrix (computed from $\Sigma_{shrink}$), the matrix associated with the IRB ASRF model and the one associated with a standard $J$-factor model (like the Moody’s KMV model for instance).

2.3. Nearest correlation matrix with $J$-factor structure

Factor models have been very popular in Finance as they offer parsimonious explanations of asset returns and correlations. The underlying issue of the Committee’s proposition is to build a factor model (with a specified number of factors) generating a correlation structure as close as possible to the pre-determined correlation structure $C_0$. At this stage the Committee does not provide any guidance on the calibration of factors loadings $\beta$ needed to pass from a $(J > 2)$-factor structure to a $(J = 2)$-factor one. The objective here is then to present generic methods to calibrate a model with a $J$-factor structure from an initial $(K \times K)$-correlation matrix.

Among popular exploratory methods used to calibrate such models, Principal Components Analysis (PCA) aims at specifying a linear factor structure between variables. Indeed, by considering the random vector $X$ of asset values, and using the spectral decomposition theorem on the initial correlation matrix: $C_0 = \Gamma \Lambda \Gamma^t$ (where $\Gamma$ is the diagonal matrix of ordered eigenvalues of $C_0$ and $\Lambda$ is an orthogonal matrix whose columns are the associated eigenvectors), the principal components transform of $X$ is: $\Psi = \Gamma^t X$, where the random vector $\Psi$ contains the ordered principal components. Since this transformation is invertible, we may finally write: $X = \Gamma \Psi$. In this context, an easy way to postulate a $J$-factor model is to partition $\Psi$ according to $(\Psi_1, \Psi_2)^t$ where $\Psi_1 \in \mathbb{R}^{J \times 1}$ and $\Psi_2 \in \mathbb{R}^{(K-J) \times 1}$, and to partition $\Gamma$ according to $(\Gamma_1, \Gamma_2)$ where $\Gamma_1 \in \mathbb{R}^{K \times J}$ and $\Gamma_2 \in \mathbb{R}^{K \times (K-J)}$. This truncation leads to: $X = \Gamma_1 \Psi_1 + \Gamma_2 \Psi_2$. Hence, by considering $\Gamma_1$ as the factors loadings (composed of the $J$ first eigenvectors of $C_0$), $\Psi_1$ as the factors (composed of the $J$ first principal components of $X$) and $\Gamma_2 \Psi_2$ as the residuals, we get a $J$-factor model.

Nevertheless, as mentioned by Andersen, Sidius, Basus (2003) [2], the specified factor structure in Equation (1) cannot be merely calibrated by a truncated eigen-expansion since it requires arbitrary residuals, that depends on $\beta$. In fact, here, we look for a $(J = 2)$-factor modeled $X_k$ of which the correlation matrix $C(\beta) = \beta \beta^t + \text{diag}(\text{Id} - \beta \beta^t)$, $\beta \in \mathbb{R}^{K \times J}$ is as close as possible to $C_0$ in the sense of a chosen norm. Thus, we define the following optimization problem \footnote{\(\Omega\) is a closed and convex set in $\mathbb{R}^{K \times J}$. Moreover, the gradient of the objective function is given by: $\nabla f(\beta) = 4(\beta(\beta^t) - C_0 \beta + \beta + \text{diag}(\beta \beta^t \beta))$}:
\[ \begin{align*}
\text{arg min}_{\beta} f(\beta) &= \|C(\beta) - C_0\|_F \\
\text{subject to: } \beta &\in \Omega = \{ \beta \in \mathbb{R}^{K \times J} | \beta_k \beta_k^T \leq 1; k = 1, \ldots, K \} 
\end{align*} \tag{5} \]

where \( \| \cdot \|_F \) is the Frobenius norm defined as \( \forall A \in \mathbb{R}^{K \times K} : \|A\|_F = \text{tr}(A^T A)^{1/2} \) (with \( \text{tr}(\cdot) \), the trace of a square matrix). The above constraint ensures that \( \beta \beta^T \) has diagonal elements bounded by 1, implying that \( C(\beta) \) is positive semi-definite.

The general problem of computing a correlation matrix of \( J \)-factor structure nearest to a given matrix has been tackled in the literature. In the context of credit basket securities, Andersen, Sidenius and Basu (2003) [2] use the fact that the solution of the unconstrained problem may be found by PCA for a particular \( \sigma \). Specifically, authors show that the solution of the unconstrained problem is of the following forms: \( \beta = \Gamma_\sigma \sqrt{\Lambda_J} \) where \( \Gamma_\sigma \) is the matrix of eigenvectors of \( (C_0 - \sigma) \) and \( \Lambda_J \) is a diagonal matrix containing the \( J \) largest eigenvalues of \( (C_0 - \sigma) \). As the solution found does not generally satisfy the constraint, authors advocate to use an iterative procedure to respect it.

Borsdorff, Higham and Raydan (2010) [17] give a theoretical analysis of the full problem and show how standard optimization methods can be used to solve it. They compare different optimization methods, in particular the PCA-based method and the spectral projected gradient (SPG) method. Inter alia, their experiments show that the principal components-based method, which is not supported by any convergence theory, often performs surprisingly well on the one hand, partly because the constraints are often not active at the solution, but may fail to solve the constrained problem on the other. The SPG method solves the full constrained problem and generates a sequence of matrices guaranteed to converge to a stationary point of the convex set \( \Omega \). The authors acknowledge the SPG method as being the most efficient. The method allows minimizing \( f(\beta) \) over the convex set \( \Omega \) by iterating over \( \beta \) in the following way: \( \beta_{i+1} = \beta_i + \alpha_i d_i \) where \( d_i = \text{Proj}_\Omega (\beta_i - \lambda_i \nabla f(\beta_i)) - \beta_i \) is the descent direction, with \( \lambda_i > 0 \) a pre-computed scalar, and \( \alpha_i \in [-1, 1] \) is chosen through a non-monotone line search strategy. \( \text{Proj}_\Omega \) being cheap to compute, the algorithm is fast enough to enable the calibration of portfolios having a large number of positions. A detailed presentation and algorithms are available in Birgin, Martinez and Raydan (2001) [16].

Finally, an important point for the validity of a factor model is the correct specification of the number of factors. Until now, in accordance with the Committee’s specification, we have assumed arbitrary \( J \)-factor models where \( J \) is specified by the modeler (\( J = 2 \) for the Committee). Based on the data, we may also consider the problem of determining the optimal number of factors in approximate factors models. Some previous academic papers deal with this issue. Among them, we particularly refer to Bai and Ng (2002) [3] who propose panel criteria to consistently estimate the optimal number of factor from historical data\(^{12}\).

\(^{12}\)Authors consider the sum of squared residuals, noted \( V(j, Z_j) \) where the \( j \) factors are estimated by PCA (\( \forall j \in 1, \ldots, J \)), and introduce the Panel Criteria and the Information Criteria to be used in practice for determining the optimal number of factors: \( PC_{IC}(j) = V(j, Z_j) + \text{Penalty}_{PC} \) and \( IC_m = \ln(V(j, Z_j)) + \text{Penalty}_{IC}^m \) \( (m = 1, 2, 3) \) where \( \text{Penalty}_{PC} \) and \( \text{Penalty}_{IC}^m \) are some penalty functions. To validate their method, the authors consider several numerical experiments. In particular, in the strict factor model (where the idiosyncratic errors are uncorrelated as in our framework), the preferred criteria could be the following: \( PC_1, PC_2, IC_1 \) and \( IC_2 \). We refer to Bai and Ng (2002) [3] for a complete description of these criteria and the general methodology.
3. Impact on the risk

The paper objective is to study the impact of factor model and its induced correlation structures on the trading book credit risk measurement. The particularities of the trading book positions (actively traded positions, the presence of long-short credit risk exposures, heterogeneous and potentially small number of positions) make the Large Pool Approximation or any granularity assumption too restrictive so that they compel to analyze the risk contribution of both the systematic factors and the idiosyncratic risks. In a first subsection, we represent the portfolio loss via the Höftdding decomposition to exhibit the impact of both the systematic factors and the idiosyncratic risks. In this framework, the second subsection presents analytics of factors contributions to the risk measure.

3.1. Höffding decomposition of the loss

The portfolio loss has been defined in the previous section via the sum of the individual losses (cf. Equation(2)). We consider here a representation of the loss as a sum of terms involving sets of factors. In particular we use a statistical tool, the Höffding decomposition, previously introduced by Rosen and Saunders (2010) [52] within a risk contribution framework. Formally, if $F_1, \ldots, F_M$ and $L = L[F_1, \ldots, F_M]$ are square-integrable random variables, then the Höffding decomposition writes the aggregate portfolio loss, $L$, as a sum of terms involving conditional expectations given factor sets:

$$L = \sum_{S \subseteq \{1, \ldots, M\}} \phi_S(L; F_m, m \in S) = \sum_{S \subseteq \{1, \ldots, M\}} \sum_{\tilde{S} \subseteq S} (-1)^{|S|-|\tilde{S}|} \mathbb{E} [L|F_m; m \in \tilde{S}]$$

Although the Höffding decomposition suffers from a practical issue when the number of factors is large, computation for a two-factor model does not present any challenge, especially in the Gaussian framework where an explicit analytical form of each term exists (cf. Equations (10), (11), (12)). Moreover, even in presence of a large number of factors, the Höffding theorem allows decomposing $L$ on any subset of factors. Rosen and Saunders (2010) [52] focus on the heterogeneous Large Pool Approximation, $L_Z$, by considering the set of systematic factors, $Z$. We define the systematic decomposition with two factors, $Z_1$ and $Z_2$, of the conditional loss by:

$$L_Z = \phi_0(L_Z) + \phi_1(L_Z; Z_1) + \phi_2(L_Z; Z_2) + \phi_{1,2}(L_Z; Z_1, Z_2)$$

where $\phi_0(L_Z) = \mathbb{E}[L]$ is the expected loss, $\phi_1(L_Z; Z_1)$ and $\phi_2(L_Z; Z_2)$ are the losses induced by the systematic factors $Z_1$ and $Z_2$ respectively, and the last term $\phi_{1,2}(L_Z; Z_1, Z_2)$ is the remaining loss induced by systematic factors interactions. Each term of the decomposition, $\phi_S(L_Z; Z_j, j \in S), S \subseteq \{1, 2\}$, gives the best hedge in the quadratic sense of the residual risk driven by co-movements of the systematic factors $Z_j$ that cannot be hedged by considering any smaller subset of the factors.

---

13 The Höffding decomposition is usually applied to independent factors. If this assumption is fulfilled, then all terms of the decomposition are uncorrelated. The decomposition formula is still valid for dependent factors but, in this case, each term depends on the joint distribution of the factors.

14 Consult Van Der Waart (1999) [60] for a detailed presentation of the Höffding decomposition.

15 The Höffding decomposition may be used also to decompose individual loss $L_k, k \in \{1, \ldots, K\}$, to provide finer analysis on the loss origin in the portfolio.

16 The Höffding decomposition requires the calculation of $2^M$ terms, where $M$ is the number of factors.
In this paper, we propose to extend the decomposition in Equation (7) by taking into account idiosyncratic risks. Indeed, the flexibility of Hoeffding decomposition brings the possibility to break the portfolio loss down in terms of aggregated systematic and idiosyncratic parts, yielding an exhaustive representation of the aggregate loss, where we consider all factors $F = Z \cup E$. We define the \textit{macro} decomposition of the unconditional loss by:

$$L = \phi_0(L) + \phi_1(L; Z) + \phi_2(L; E) + \phi_{1,2}(L; Z, E) \tag{8}$$

where $\phi_0(L) = \mathbb{E}[L]$ is the expected loss, $\phi_1(L; Z) = \mathbb{E}[L|Z] - \mathbb{E}[L]$ is the loss induced by the systematic factors $Z_1$ and $Z_2$ (corresponding, up to the expected loss term, to the heterogeneous Large Pool Approximation), $\phi_2(L; E) = \mathbb{E}[L|E] - \mathbb{E}[L]$ is the loss induced by the $K$ idiosyncratic terms $\epsilon_k$, and $\phi_{1,2}(L; Z, E) = (L - \mathbb{E}[L|Z] - \mathbb{E}[L|E] + \mathbb{E}[L])$ is the remaining risk induced by interactions between idiosyncratic and systematic risk factors. Moreover, since $L_Z = \phi_1(L; Z) + \phi_0(L)$, it is feasible to combine Equations (7) and (8) to get:

$$L = L_Z + \phi_2(L; E) + \phi_{1,2}(L; Z, E) \tag{9}$$

This last expression helps understand the impact of each systematic factor (alone or in interaction) as well as the impact of the idiosyncratic terms, alone or in interaction. Furthermore, Equation (9) thoroughly expresses the relation between conditional and unconditional portfolio losses.

Remark that the Hoeffding decomposition is not an approximation. It is an equivalent representation of the same random variable. In particular, when decomposed, the unconditional loss $L$ and the conditional loss $L_Z$ remain discrete and continuous random variables, respectively. Note also that $\phi_1(L; Z)$ and $\phi_2(L; E)$ are continuous random variables whereas $\phi_{1,2}(L; Z, E)$ is discrete. It follows that the latter, stemming from the interactions of the three former, accounts also for the discreteness of $L$ which is not captured by the others.

From a practical point of view, as we consider a Gaussian factor model, we may easily compute each term of the decomposition:

$$\mathbb{E}[L|Z] = \sum_{k=1}^{K} w_k \Phi \left( \frac{\Phi^{-1}(p_k) - \beta_k Z}{\sqrt{1 - \beta_k^2 \beta_t^2}} \right) \tag{10}$$

$$\mathbb{E}[L|E] = \sum_{k=1}^{K} w_k \Phi \left( \frac{\Phi^{-1}(p_k) - \sqrt{1 - \beta_k^2 \beta_t^2} \epsilon_k}{\sqrt{\beta_k^2 \beta_t^2}} \right) \tag{11}$$

$$\mathbb{E}[L_Z|Z_j, j \in \tilde{S}] = \sum_{k=1}^{K} w_k \Phi \left( \frac{\Phi^{-1}(p_k) - \sum_{j \in \tilde{S}} \beta_{k,j} Z_j}{\sqrt{1 - \beta_k^2 \beta_t^2}} \right) \tag{12}$$

Importantly, considering our model specification of the vector $X$, we know from standard statistical results on exploratory factor analysis that factor rotations\(^{17}\) of the systematic factors leave the law of the vector $X$ unchanged. Moreover, the risk measures inherit the rotation-invariance property of the underlying asset value random vector $X$. Nevertheless, we may prove\(^{18}\) that a simple rotation of
risk factors modifying factor loading matrix directly affects the law of Hoeffding terms that depend on subsets included in $Z$. Therefore, an infinite number of combinations of factors loadings might lead to the same risk measure but to very different contributions. The rotation of a factor matrix is a problem that dates back to the beginning of multiple factor analysis. Browne (2001) [18] provides an overview of available analytical rotation methods. Among them, the Varimax method, which was developed by Kaiser (1958) [38] is the most popular. Formally, Varimax searches for a rotation (i.e. a linear combination) of the original factors such that the variance of the loadings is maximized. This simplifies the factor interpretation because, after a Varimax rotation, each factor tends to have either large or small loading for any particular $X_k$. Nevertheless, because the larger weight may be on either the first or the second factor, the contribution interpretation may remain delicate.

3.2. Systematic and idiosyncratic contributions to the risk measure

The portfolio risk is determined by means of a risk measure $\varphi$, which is a mapping of the loss to a real number: $\varphi : L \mapsto \varphi(L) \in \mathbb{R}$. Usual quantile-based risk measures are the Value-at-Risk (VaR) and the Conditional-Tail-Expectation (CTE). Let $\alpha \in [0, 1]$ be some given confidence level, VaR is the $\alpha$-quantile of the loss distribution: $\text{VaR}_\alpha[L] = \inf \{ l \in \mathbb{R} | P(L \leq l) \geq \alpha \}$. On the other hand, CTE is the expectation of the loss conditional to loss occurrences higher than the $\text{VaR}_\alpha[L]$.

$\text{CTE}_\alpha[L] = E[L | L \geq \text{VaR}_\alpha[L]]$. Since both IRC in Basel 2.5 and IDR in the Fundamental Review of the Trading Book prescribe the use of a one-year 99.9% VaR, we will further restrict to this risk measure even though risk decompositions can readily be extended to the set of spectral risk measures.

By definition, the portfolio loss equals the sum of individual losses: $L = \sum_{k=1}^{K} L_k$. As we showed earlier it can also be defined as the sum of the Hoeffding decomposition terms: $L = \sum_{S \subseteq \{1,\ldots,M\}} \phi_S(L; F_m, m \in S)$. To understand risk origin in the portfolio, it is common to refer to a contribution measure $C_k^\alpha (C_k^\varphi$, respectively) of the position $k$ (of the Hoeffding decomposition term $\phi_S$) to the total portfolio risk $\varphi(L)$. The position risk contribution is of great importance for hedging, capital allocation, performance measurement and portfolio optimization and we refer to Tasche (2007) [57] for a detailed presentation. Just as fundamental as the position risk contribution, the factor risk contribution helps unravel alternative sources of portfolio risk. Papers dealing with this topic include the following. Cherny and Madan (2007) [20] consider the conditional expectation of the loss with respect to the systematic factor and name it factor risk brought by that factor. Martin and Tasche (2007) [45] also consider the same conditional expectation, but then apply the Euler’s principle taking the derivative of the portfolio risk in the direction of this conditional expectation and call it risk impact. Rosen and Saunders (2007) [52] apply the Hoeffding decomposition of the loss with respect to sets of systematic factors, the first several terms of this decomposition coinciding with the conditional expectations mentioned above.

the betas are equivalent. For instance, $\forall k, \beta_{k,1} = \beta_{k,2} = b$ (with $b \in [-1, 1]$) which implies that $\phi_1(L; Z_1) = \phi_2(L; Z_2)$ in distribution, and the contribution of each element is the same. Case 2: we now consider the special one factor model. For instance $\beta_{k,1} = \sqrt{2} \times b^2$ and $\beta_{k,2} = 0$ implying that $\phi_1(L; Z_1) \neq \phi_2(L; Z_2)$ in distribution. Case 3: we finally consider a permutation of the case 2: $\beta_{k,1} = 0$ and $\beta_{k,2} = \sqrt{2} \times b^2$ leading to the same conclusion. In those 3 cases, the risk is identical but factor contributions differ.

Note that in our numerical applications (see Section 4) we only rely on the macro decomposition of the unconditional loss (see Equation [8]) so that this interpretation coindrum is not relevant.

Note that as long as $L$ is continuous Conditional-Tail-Expectation is equivalent to Expected-Shortfall.
Theoretical and practical aspects of various allocation schemes have been analyzed in several papers (see Dhaene et al. (2012) [24] for a review). Among them, the marginal contribution method, based on Euler’s allocation rule, is quite a standard one (see Tasche (2007) [58]). To be applied here, an adaptation is required because we face discrete distributions (see Laurent (2003) [41] for an in-depth analysis of the technical issues at hand). Yet, under differentiability conditions, and taking VaR as the risk measure\(^{21}\), it can be shown (see Gouriéroux, Laurent and Scaillet (2001) [34]) that the marginal contribution of the individual loss \(L_k\) to the risk associated with the aggregate loss \(L = \sum_{k=1}^{K} L_k\) is given by \(C_{VaR}^{VaR}(L_k) = \mathbb{E}[L_k | L = VaR[\alpha][L]]\). Besides, computing this expectation does not involve any other assumption than integrability and defining risk contributions along these lines fulfills the usual full allocation rule \(L = \sum_{k=1}^{K} C_{VaR}^{VaR}(L_k)\) (see Tasche (2008) [57] for details on this rule).

Similarly, we can compute the contributions of the different terms involved in the Hoeffding decomposition of the aggregate loss. For instance, the contribution of the systematic term is readily derived as \(C_{VaR}^{VaR}(\phi_1) = \mathbb{E}[\phi_1(L, Z) | L = VaR[\alpha][L]]\). Likewise, contributions of specific risk and of cross-effect terms can easily be written and added afterwards to the systematic term so as to retrieve the risk measure of the aggregate loss. As shown in subsection 3.1, when we deal with a (systematic) two-factor model, we can further break down the Large Pool Approximation loss, \(L_Z\), to cope with the marginal effects of the two factors \((Z_1, Z_2)\) and with the cross effects. The additivity property of risk contributions prevails when subdividing the vector of risk factors into multiple blocks. Therefore, our risk contribution approach encompasses that of Rosen and Saunders (2010) [52] since it also includes the important (especially in the trading book context) contribution of specific risk to the portfolio risk.

It is also noteworthy that although our approach could be deemed as belonging to the granularity adjustment corpus, relying as well on large pool approximations, the techniques and the mathematical properties involved here (such as differentiability, associated with smooth distributions) are deeply different.

### 4. Empirical implications for diversification and hedge portfolios

This section is devoted to the empirical study of the effects of the correlation structure (sample-based and factor model-based) on risk measure and risk contributions. In particular, it aims at analyzing the impacts of modeling constraints for both the future IDR charge prescription and the current Basel 2.5 IRC built on constrained and unconstrained factor models. We base our numerical analysis on representative long-only and long-short credit-sensitive portfolios. Since we reckon to focus on widely traded issuers, who represent a large portion of banks exposures, we opt for a portfolio with large Investment Grades companies. This choice is also consistent with hypothetical portfolios enforced by the R-CAP showing a higher level of IRC variability for bespoke portfolios than for diversified ones. Specifically, we consider the composition of the iTraxx Europe index, taken on the 31st October 2014. The portfolio is composed of 121 European Investment Grade companies\(^{22}\), 27 are Financials, the remaining being tagged Non-Financials.

\(^{21}\)By defining \(VaR[\alpha][L] = \mathbb{E}[L | L = VaR[\alpha][L]]\), similar results hold for CTE (up to a sign change from \(L = VaR[\alpha][L]\) to \(L \geq VaR[\alpha][L]\).

\(^{22}\)The index is genuinely composed of 125 names. Nevertheless, due to lack of data for initial correlation computation, it was not possible to estimate a (125 x 125)-matrix.
We successively look at two types of portfolio: (i) a *diversification portfolio*, comprised of positive-only exposures (long-only credit risk), (ii) a *hedge portfolio*, built up of positive and negative exposures (long-short credit risk). The distinction parallels the one between the banking book, containing notably long-credit loans, and the trading book, usually resulting from a mix of long and short positions (like for instance bonds or CDS) – within the latter, an issuer’s default on a negative exposure yields a gain. For conciseness, the LGD rate is set to 100% for each position of the two portfolios. Regarding the *diversification portfolio* ($K = 121$), we consider a constant and equally weighted effective exposure for each name so that $\forall k, w_k = 1/K$ and $\sum_{k=1}^{K} w_k = 1$. Concerning the *hedge portfolio* ($K = 54$), we assume long exposure to 27 Financials and short exposure to 27 Non-Financials chosen such that the average default probability between the two groups is nearly the same (around 0.16%). By considering $w_{k\in Financials} = 1/27$ and $w_{k\not\in Financials} = -1/27$, the *hedge portfolio* is thus credit-neutral: $\sum_{k=1}^{K} w_k = 0$.

For the sake of numerical application, we use default probabilities\(^{23}\) provided by the Bloomberg Issuer Default Risk Methodology\(^{24}\). Figure (1) illustrates default probabilities’ frequencies of the portfolio’s companies grouped by Financials and Non-Financials.

![Figure 1: Histogram of the default probabilities distribution](image)

Financials clearly show higher and more dispersed default probabilities, with a mean and a standard deviation equal to 0.16% and 0.16% respectively, compared to 0.08% and 0.07% for Non-Financials. We also note that the floor value (at 0.03%) prescribed by the Committee is restrictive for numerous (34 over 121) Financial and Non-Financial issuers.

In the next subsections, we discuss results on the calibration of both the initial correlation matrix ($C_0$) and the loading matrix ($\beta \in \mathbb{R}^{K \times J}$) of the $J$-factor models. We then consider the impact of these different models on the risk for both portfolios. By using Hoeffding-based representation of the aggregate loss, we finally compute the contributions of the systematic and idiosyncratic parts of the loss to the risk.

\(^{23}\)Other data sources could be used in further versions of the paper.  
\(^{24}\)Bloomberg DRSK methodology is based on the model of Merton (1974) \cite{merton1974}. The model does not use credit market variables, rather it is an equity market-based view of default risk. In addition to market data and companies balance sheet fundamental, the model also includes companies’ income statements.
4.1. Correlation calibration

Following the Committee’s proposal, we use listed equity prices (daily returns) of the 121 issuers spanning a one-year period to calibrate the initial default correlation matrix through the empirical estimator $C$ (cf. Equations (4)). To illustrate the sensitivity to the calibration window, we use two sets of equity time-series. Period 1 is chosen during a time of high market volatility from 07/01/2008 to 07/01/2009, whereas Period 2 spans a comparatively lower market volatility window, from 07/01/2013 to 07/01/2014. For both periods, the computed unconstrained (i.e. with no factor structure) $(121 \times 121)$-matrix consists in a matrix of pairwise correlations that we retreat\(^{25}\) to ensure semi-definite positivity. To limit estimation error, we also apply the shrinkage methodology on the two periods.

Furthermore, Period 2 is used to define other initial correlation matrices with a view to analyze the effects on the $J$-factor model of changes in the correlation structure as computed from different types of financial data. We consider three alternative sources: (i) the prescribed IRBA correlation formula\(^{26}\), grounded on the issuer’s default probability; (ii) the GCorr methodology of Moody’s KMV; (iii) the issuers’ CDS spreads relative changes (also preconized by the Committee).

For each initial correlation matrices, $C_0$, the optimization problem in Equation (5) is solved with the PCA-based and the SPG-based algorithms (for $J = 1, 2$). All characteristics are reported in Table (1).

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Data for estimating $C_0$</th>
<th>Period</th>
<th>Estimation method for $C_0$</th>
<th>Calibration method for the $J$-factor models</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Equity - P1</td>
<td>Equity returns</td>
<td>1</td>
<td>Sample correlation</td>
<td>PCA and SPG algorithms</td>
</tr>
<tr>
<td>(2) Equity - P2</td>
<td>Equity returns</td>
<td>2</td>
<td>Sample correlation</td>
<td>PCA and SPG algorithms</td>
</tr>
<tr>
<td>(3) Equity - P1 Shrinked</td>
<td>Equity returns</td>
<td>1</td>
<td>Shrinkage ($\alpha_{shrink} = 0.32$)</td>
<td>PCA and SPG algorithms</td>
</tr>
<tr>
<td>(4) Equity - P2 Shrinked</td>
<td>Equity returns</td>
<td>2</td>
<td>Shrinkage ($\alpha_{shrink} = 0.43$)</td>
<td>PCA and SPG algorithms</td>
</tr>
<tr>
<td>(5) IRBA</td>
<td>-</td>
<td>-</td>
<td>IRBA formula</td>
<td>PCA and SPG algorithms</td>
</tr>
<tr>
<td>(6) KMV - P2</td>
<td>-</td>
<td>2</td>
<td>GCorr methodology</td>
<td>PCA and SPG algorithms</td>
</tr>
<tr>
<td>(7) CDS - P2</td>
<td>CDS spreads</td>
<td>2</td>
<td>Sample correlation</td>
<td>PCA and SPG algorithms</td>
</tr>
</tbody>
</table>

Table 1: Initial correlation matrix estimation and $J$-factor model calibration.

Period 1: from 07/01/2008 to 07/01/2009. Period 2: from 07/01/2013 to 07/01/2014.

The optimization problem is also considered for the calibration of $J^*$-factor models (where $J^*$ is the data-based “optimal number” of factors) for both the “Equity - P1” and the “Equity - P2” configurations. It is defined here as the integer part of the arithmetic average of the panel and information criteria. Applying this methodology to the historical time series, we get $J^* = 6$ for the

\(^{25}\)Through spectral projection. Note that this treatment is not necessary if we only consider the simulation of the $J$-factor model calibrated via the optimization problem in Equation (5).

\(^{26}\)IRBA approach is based on a 1-factor model: $X_k = \sqrt{\rho_k}Z_1 + \sqrt{1 - \rho_k}\epsilon_k$. Thus, $\text{Correl}(X_k, X_j) = \sqrt{\rho_k \times \rho_j}$ where $\rho_k$ is provided by a prescribed formula: $\rho_k = 0.12 \times \frac{1 - \exp^{-50\rho_k}}{1 - \exp^{-50}} + 0.24 \times \left(1 - \frac{1 - \exp^{-50\rho_k}}{1 - \exp^{-50}} \right)$. 

15
“Equity - P1” configuration \((IC_1 = 6, IC_2 = 5, PC_1 = 8 \text{ and } PC_2 = 7)\) and \(J_\ast = 3\) for the “Equity - P2” configuration \((IC_1 = 2, IC_2 = 2, PC_1 = 4 \text{ and } PC_2 = 4)\). To make the results comparable, these optimal numbers are the same for the diversification portfolio and the hedge portfolio. Remark that these experimental results are consistent with empirical conclusions of Connor and Korajczyk (1993)\cite{23} who find out a number of factors included between 1 to 2 factors for “non-stressed” periods and 3 to 6 factors for “stressed” periods for the monthly stock returns of the NYSE and the AMEX, over the period 1967 to 1991. Moreover, it is also in line with the results of Bai and Ng (2001)\cite{3} who exhibit the presence of two factors when studying the daily returns on the NYSE, the AMEX and the NASDAQ, over the period 1994 to 1998.

Table (2) exposes the calibration results of the \(J\)-factor models for both the PCA-based and the SPG-based algorithms.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Nb factors</th>
<th>Frobenius Norm</th>
<th>Average Correlation</th>
<th>Average Correlation</th>
<th>Average Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SPG</td>
<td>PCA</td>
<td>SPG</td>
<td>PCA</td>
</tr>
<tr>
<td>(1) Equity - P1</td>
<td>C_0</td>
<td>0.00</td>
<td>0.00</td>
<td>0.46</td>
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</tr>
<tr>
<td></td>
<td>1 factor</td>
<td>8.75</td>
<td>8.73</td>
<td>0.47</td>
<td>0.46</td>
</tr>
<tr>
<td></td>
<td>2 factors</td>
<td>6.10</td>
<td>6.01</td>
<td>0.47</td>
<td>0.46</td>
</tr>
<tr>
<td>((J_\ast = 6)) factors</td>
<td>4.26</td>
<td>3.84</td>
<td>0.46</td>
<td>0.46</td>
<td>0.63</td>
</tr>
<tr>
<td>(2) Equity - P2</td>
<td>C_0</td>
<td>0.00</td>
<td>0.00</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>1 factor</td>
<td>8.69</td>
<td>8.66</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>2 factors</td>
<td>6.99</td>
<td>6.94</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td>((J_\ast = 3)) factors</td>
<td>6.36</td>
<td>6.24</td>
<td>0.28</td>
<td>0.28</td>
<td>0.44</td>
</tr>
<tr>
<td>(3) Equity - P1 Shrinked</td>
<td>C_0</td>
<td>0.00</td>
<td>0.00</td>
<td>0.46</td>
<td>0.46</td>
</tr>
<tr>
<td></td>
<td>1 factor</td>
<td>5.92</td>
<td>5.88</td>
<td>0.47</td>
<td>0.46</td>
</tr>
<tr>
<td></td>
<td>2 factors</td>
<td>4.18</td>
<td>4.05</td>
<td>0.47</td>
<td>0.46</td>
</tr>
<tr>
<td>(4) Equity - P2 Shrinked</td>
<td>C_0</td>
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<td>0.00</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>1 factor</td>
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</tr>
<tr>
<td></td>
<td>2 factors</td>
<td>4.07</td>
<td>3.97</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td>(5) IRBA</td>
<td>C_0</td>
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<td>0.00</td>
<td>0.25</td>
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<tr>
<td></td>
<td>1 factor</td>
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<td>2 factors</td>
<td>0.34</td>
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<tr>
<td>(6) KMV - P2</td>
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<td>0.00</td>
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<td>0.29</td>
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<td></td>
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<td>4.09</td>
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<td>(7) CDS - P2</td>
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<td>0.58</td>
<td>0.58</td>
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<td>7.66</td>
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<td>0.58</td>
</tr>
<tr>
<td></td>
<td>2 factors</td>
<td>5.51</td>
<td>5.44</td>
<td>0.59</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Table 2: Factor-model calibration over the 121 iTraxx issuers. The column “Frobenius norm” corresponds to the optimal value of the objective function whereas the three right hand side columns state the average pairwise correlations from, respectively, the overall portfolio matrix, the Financial sub-matrix and the Non-Financial sub-matrix. Note that the Frobenius norm can only be compared within a configuration, but cannot be used to compare configurations between them, as it depends on \(C_0\).

We remark that the fitting results among the \(J\)-factor models seem close together to that of the
initial matrix, meaning that the two nearest matrix approaches perform correctly. Moreover, shrinking the correlation matrix allows a better fit for all of the $J$-factor models, as we can judge by smaller Froebenius norms in comparison with “Equity - P1” and “Equity - P2” configurations. Finally, increasing the number of factors, $J$, tends to produce a better fit of the $J$-factor model to the unconstrained ($121 \times 121$)-model.

Figure (2) exhibits histograms of the pairwise correlation frequencies for each configuration within each $J$-factor models ($J = 1, 2, J^*$ with PCA-based calibration) $^{27}$. 

$^{27}$Note that results with the SPG-based algorithm are very similar.
We observe important disparities on frequencies’ level and dispersion depending on the configuration. The “Equity − P1” configuration shows frequencies with large dispersion due to high market volatility, and modes around high levels, whereas the “Equity − P2” configuration shows frequencies with a peak around 30%. The shrinkage seems to have small effect on the level of the pairwise correlations but slightly decreases disparities among the models. The “IRBA” configuration yields concentrated correlation levels around 25%. The “CDS − P2” configuration somehow presents the most disparate results of which we may say that factor models tend to overestimate central correlations and underestimate tail correlations (note that this is also true for other configurations but to a lesser extent). Overall, the factor models seem to accurately reproduce the underlying pairwise correlation
distribution and, by combining Figure (2) and Table (2), we may conclude that the more regular the correlation structure, the fewer the number of factors needed to be faithfully reproduce it.

4.2. Impact on the risk

In this subsection, we analyze the impacts of initial correlation matrices on portfolio risk. Numerical applications are based on Monte Carlo simulations of portfolio loss. We consider $MC \in \mathbb{N}$ i.i.d replications$^{28}$ of the loss random variable $L$ and note $L^{(n)}$ the realization of the loss on scenario $n \in \{1, \ldots, MC\}$. Since the unconditional loss is a discrete random variable that can only take a finite number of realization values$^{29}(\forall n, L^{(n)} \in \mathbb{L})$, the $VaR_{\alpha}$ estimator is the value of the $(\alpha \times MC)$-ordered loss realization.

In Table (3), we present the $VaR_{\alpha=0.999}[L]$ for all model specifications (unconstrained and constrained $J$-factor ($J = 1, 2, J^*$) models) for both the diversification portfolio and the hedge portfolio and for each of the seven initial correlation matrices.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Diversification portfolio</th>
<th>Hedge portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_0$</td>
<td>1 factor</td>
</tr>
<tr>
<td>(1) Equity - P1</td>
<td>9.92</td>
<td>9.09</td>
</tr>
<tr>
<td>(2) Equity - P2</td>
<td>5.79</td>
<td>5.79</td>
</tr>
<tr>
<td>(3) Equity - P1 Shrinked</td>
<td>9.09</td>
<td>9.09</td>
</tr>
<tr>
<td>(4) Equity - P2 Shrinked</td>
<td>5.79</td>
<td>5.79</td>
</tr>
<tr>
<td>(5) IRBA</td>
<td>4.13</td>
<td>4.13</td>
</tr>
<tr>
<td>(6) KMV - P2</td>
<td>5.79</td>
<td>4.96</td>
</tr>
<tr>
<td>(7) CDS - P2</td>
<td>13.22</td>
<td>12.40</td>
</tr>
</tbody>
</table>

Table 3: Comparison of $VaR_{\alpha=0.999}[L]$ level between constrained and unconstrained models ($\%$).

A first remark is that with these equally weighted portfolios, constrained $J$-factor models tend to produce lower tail risk measures than the unconstrained model. This phenomenon is even more pronounced when considering dispersed initial correlation matrices, particularly in the hedge portfolio where constrained model may lead to substantial risk mitigation (such that on the “CDS - P2” configuration). Comparing the diversification portfolio with the hedge portfolio, we observe a hedge benefit linked to the latter’s long-short configuration inducing a smaller risk level.

In order to study the impact of the correlation structure on the risk, we consider $[0.99, 1][\exists \alpha \mapsto VaR_{\alpha}[L]]$. As shown in Figures (3) and (4), discreteness$^{30}$ of $L$ implies that this mapping is piecewise

$^{28}$Numerical applications are based on simulations using twenty million scenarios.

$^{29}$Depending on the default probabilities vector and the correlation structure, the vector of loss realizations may contain a large number of zeros. In our numerical simulation, this is the case for 96\% of realizations.

$^{30}$Since we deal with discrete distributions, we cannot rely on standard asymptotic properties of sample quantiles. At discontinuity points of VaR, sample quantiles do not converge. This can be solved thanks to the asymptotic framework introduced by Ma, Genton and Parzen (2011) [44] and the use of the mid-distribution function.
constant so that jumps in the risk measure are possible for small changes in the default probability.

Figure 3: Risk measure as a function of $\alpha$ for the diversification portfolio (PCA-based calibration). $J^*$-factor model is only active for “Equity − P1” and “Equity − P2” configurations.
Figure 4: Risk measure as a function of $\alpha$ for the hedge portfolio (PCA-based calibration). $J^*$-factor model is only active for “Equity – P1” and “Equity – P2” configurations.

In Figures (3) and (4), the level of the risk measure and the “length of each piece” depend heavily on the underlying correlation structure. For instance, the “Equity – P1” configuration having a more complex correlation structure than the “IRBA” configuration, its plot of $\alpha \mapsto \text{VaR}_\alpha[L]$ is steeper and above. Moreover (and as expected from calibration results), for dispersed correlation structures, a higher number of systematic factors allows a better replication of the risk measure obtained with the unconstrained $(121 \times 121)$-model. Conversely, for the less complex “IRBA formula” correlation structure, the one-factor model fully explains VaR levels, which is observed for both the diversification portfolio and the hedge portfolio. Indeed, comparing these clearly shows that the role of the number of factors is particularly underlined within the latter presenting a dispersed correlation structure. Thus,
in the *hedge portfolio*, Figure (4), it appears that a greater number of factors is needed to fully replicate the risk of the unconstrained *(121 × 121)-model* and that the two-factor model tends to decrease the VaR for dispersed correlation structures, since the calibrated correlation matrix is smoother than it should be (see for instance the longer first piece around $\alpha = 0.995$ in “Equity - P1- Shrinked” configuration). We finally remark that high levels of $\alpha$ may lead to relative discrepancies among factor models. Take for instance the “Equity - P1” configuration, Figure (4), with $\alpha = 0.999$, the risk measure for the one-factor model is 40% smaller than that of the two-factor model.

### 4.3. Systematic and idiosyncratic contributions to risk measure

Turning now to the Hoeffding-based representation (Equations (8) and (9)), we note $\phi_S^{(n)}$ the realization of the projected loss (onto the subset of factors $S$) on the scenario $n$. With these notations, given the pre-calculated risk measure $v = VaR_\alpha[L]$, the contribution estimator is:

$$ C_{\phi_S}^{VaR}[L, \alpha] = \mathbb{E}[\phi_S[L = v]] \implies C_{\phi_S}^{VaR}[L, \alpha] = \frac{\sum_{n=1}^{MC} \phi_S^{(n)} 1\{L^{(n)} = v\}}{\sum_{n=1}^{MC} 1\{L^{(n)} = v\}} $$

(13)

Since the conditional expectation defining the risk contribution is conditioned on rare events, this estimator requires intensive simulations to reach an acceptable confidence interval. Tasche (2009) [59] and Glasserman 2005 [32] have already addressed the issue of computing credit risk contributions of individual exposures or sub-portfolios from numerical simulations. Our framework is similar to theirs, except that we focus on the contributions of the different terms involved in the Hoeffding decomposition of the aggregate risk. We are thus able to derive contribution of factors, idiosyncratic risks and cross-effects.

Figures (5) and (6) expose the risk allocation between the systematic factors, the idiosyncratic risks and their interactions when considering the $VaR_{0.999}[L]$ for the two portfolios. This analysis provides a detailed understanding of the risk composition on this prescribed level of confidence.

Figures (7) and (8) illustrate the influence of $\alpha$ on the systematic contribution to the risk by considering the mapping $\alpha \mapsto C_{\phi_S}^{VaR}[L, \alpha]/v$ for the two portfolios and for each configuration. Remark that given the discrete nature of considered distributions, and similarly of the risk measure, the mapping is piecewise constant.

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31Note that negative risk contributions may arise within the *hedge portfolio.*

32Numerical experiments demonstrate that complex correlation structures (such as in the “Equity - P1” configuration) may induce noisy contribution estimation. This phenomenon is even more pronounced in the presence of a large number of loss combinations which implies frequent changes in value for the mapping $\alpha \mapsto C_{\phi_S}^{VaR}[L, \alpha]/v$. Indeed, for a given loss level, there may have only a few simulated scenarios such that $L^{(n)} = v$, leading to more volatile estimators.

33While we could think of various optimized Monte Carlo simulations methods, we have not implemented any in the current version of the article. Yet, several papers are worth noticing and could represent a basis for extensions of our work in a future version. Glasserman (2005) [32] develops efficient methods based on importance sampling, though not directly applicable to Hoeffding representation of the discrete loss. Martin, Thompson and Browne (2001) [46] pioneer the saddle point approximation of the unconditional moment generating function (MGF) for the calculation of VaR and VaR contributions. Huang et al. (2007) [36] computes risk measures (VaR and ES) and contributions with the saddle point method applied to the conditional MGF, while Huang et al. (2007) [35] presents a comparative study for the calculation of VaR and VaR contributions with the saddle point method, the importance sampling method and the normal approximation (ASRF) method. Takano and Hashiba (2008) [55] propose to calculate marginal contributions using a numerical Laplace transform inversion of the MGF. Recently, Maslenmont and Ortize-Gracia (2014) [47] applies a fast expansion wavelet approximation to the unconditional MGF for the calculation of VaR, ES and contributions, through numerically optimized techniques.
In the *diversification portfolio*, Figure (5), the most important risk contributor is the systematic part of the loss whereas the specific risk is weak for all configurations except for the “IRBA” for which the lower level of correlation leads to a smaller contribution of the systematic part, necessarily balanced by the specific terms in the Hoeffding decomposition. Importantly, the second most significant contributor for all configurations is the last term of the Hoeffding decomposition, \( \phi_{1,2}(L; Z, E) \), dealing with the interaction of both the systematic and the specific terms. Particularly, we may observe that the more complex the correlation structure, the lesser the interaction term. Here again, this term can also be viewed as expressing the discreteness of the portfolio loss, not captured by the continuous three first Hoeffding terms: \( \phi_0(L) \), \( \phi_1(L|Z) \) and \( \phi_2(L|E) \).
Concerning the *hedge portfolio*, Figure (6), the first contributor is the interaction term for all configurations. Its high level stems logically from low levels of both the systematic and the idiosyncratic contributions due to a credit-neutral configuration: potential losses are balanced by potential gains so that the average loss is near zero. Nevertheless, since the *hedge portfolio* contains a smaller number of elements than the *diversification portfolio*, specific risk plays a greater role in the resulting risk measure. Contrarily to the *diversification portfolio*, the number of factors influences the systematic contributions for all configurations (see the explicit case of “Equity − P1”) except for “IRBA”. Indeed, for the latter, the prescriptive low level of correlation coupled with a credit-neutral composition restrict the systematic contribution to a minimum, independently of the number of factors.
Figure 7: Systematic contribution as a function of $\alpha$ for the diversification portfolio (PCA-based calibration). $J^*$-factor model is only active for “Equity – P1” and “Equity – P2” configurations.

Regarding the systematic contribution in the diversification portfolio, Figure (7), we observe a high level of systematic contribution for correlation matrices with a high level of average pairwise correlation. It appears that the different factor models lead roughly to the same level of systematic risk contribution. Specifically, for the “IRBA” configuration (which presents the smallest level of systematic contribution), all models are equivalent. Observations also confirm that the shrinkage method involves a tightening of models.
In the hedge portfolio case however, Figure (8), we observe smaller levels of systematic contributions. It is also striking that the one-factor and two-factor approximations may be inoperable when considering complex correlation matrices (like for the “Equity − P1” configuration), the majority of the risk being explained by the other terms of the Hoeffding decomposition (cf. Figure (6)). We should nonetheless nuance that adding one factor to the one-factor model suffices to produce a significant tightening towards the $J^*$-factor model. For this hedge portfolio setting, the “IRBA” configuration presents an interesting feature since the systematic contribution is smaller than the average loss ($L_Z < \phi(L)$), so that its systematic contribution is negative and constant.
5. Conclusion

Assessment of default risk in the trading book (IDR, Incremental Default Risk charge) is a key point in the Fundamental Review of the Trading Book. Within the current Committee’s approach, the dependence structure of defaults has to be modeled through a systematic factor model with constraints on (i) the calibration data of the initial correlation matrix (ii) and on the number of factors in the underlying correlation structure. Equity and CDS spreads data are suggested to approximate the pairwise default correlations.

Based on representative long-only and long-short portfolios, this paper has considered the practical implications of such modeling constraints for both the future IDR charge prescription and the current Basel 2.5 IRC built on constrained and unconstrained factor models.

Various correlation structures have been considered. Based on a structural-type credit model, we assessed the impacts on the regulatory 99.9% one-year VaR of the calibration data, by using several types of data, as well as those of the estimation of the constrained correlation structure and the chosen number of factors, by relying on different estimation procedures. Eventually, the Hoeffding decomposition of portfolio exposures to factor and specific risks has been introduced to monitor risk contributions to the risk measure.

The key insights of our empirical analysis, based on latent factor models with $J$ factors ($J = 1, 2, J^*$), are the following:

- The strength of the factor constraint depends on the smoothness of the pairwise correlations frequencies in the initial correlation matrix: the more complex (dispersed) the underlying correlation structure, the greater the number of factors needed to approximate it.

In our case study, we observed two sources of that complexity: the nature of data (Equity returns, CDS spread returns,…) and the period of calibration (stressed or non-stressed). On the contrary, the estimation methods for both the initial correlation (standard or shrunked estimators) and the factor-based correlation matrices (SPG-based or PCA-based algorithms) have smaller effects, at least on the diversification portfolio (long-only exposures).

- The impact of the correlation structure on the risk measure mainly depends on the composition of the portfolio (long only or short only).

For the particular case of a diversification portfolio (long-only exposures) with a smooth initial correlation structure (e.g. estimated on non-stressed equity returns), constrained factor models (mostly when considering at least two factors) and unconstrained model produce almost similar risk measure.

For the specific case of a hedge portfolio (long-short exposures) for which widely dispersed pairwise equity or CDS-spread correlations and far tail risks (99.9-VaR) are jointly considered, a certain number of cliff effects arises from discreteness of loss: small changes in exposures or other parameters (default probabilities) may lead to significant changes in the risk measure and contributions. Risk decomposition into specific risk, systematic risk and their interactions
Interestingly highlights this point, exhibiting an important contribution of the discrete part of the Hoefding decomposition. In this context, constrained factor models and unconstrained model may produce quite different risk measure and risk contributions.

- When dealing with equally-weighted portfolios, the two-factor constraint being more restrictive on dispersed initial correlation matrix (e.g. estimated on stressed equity returns or CDS spreads), the two-factor model tends to lower the VaR since the induced correlation matrix is smoother than for the unconstrained model.

Within the current Committee’s approach, financial institutions could be brought to take arbitrary choices regarding the calibration of the initial default correlation structure, which might then cause an unsought variability in the IDR making the comparison among institutions harder. To address the problem, the ISDA (2015) proposed to use stressed IRBA-type correlations, in the spirit of the banking book approach. While in our case study, the (non-stressed) IRBA correlations are the smoothest, thus providing the lowest VaR variability, further empirical analyses should be led to validate ISDA’s proposal.

Overall, the usefulness of the two-factor constraint can be challenged: in our case study, it drives down the VaR. Moreover, it is unclear that it would enhance model comparability and reduce RWA variability. On the other hand, the Basel Committee prescriptions might prove quite useful when dealing with a large number of assets. In such a framework, reasonably standard for large financial institutions with active credit trading activities, the unconstrained empirical correlation matrix would be associated with zero eigenvalues. This would ease the building of opportunistic portfolios, seemingly with low risk and would jeopardize the reliance on internal models.

References


