

Trading book and credit risk: how fundamental is the Basel review?^{☆,☆☆}

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Abstract

In its October 2013's consultative paper for a revised market risk framework (FRTB), and subsequent versions published thereafter, the Basel Committee suggests that non-securitization credit positions in the trading book be subject to a separate Default Risk Charge (DRC, formally Incremental Default Risk charge or IDR). This evolution is an attempt to overcome practical challenges raised by the current Basel 2.5 Incremental Risk Charge (IRC). Banks using the internal model approach would no longer have the choice of using either a single-factor or a multi-factor default risk model but instead, market risk rules would require the use of a two-factor simulation model and a 99.9%-VaR capital charge. In this article, we analyze the theoretical foundations of these proposals, particularly the link with the one-factor model used for the banking book and with a general J -factor setting. We thoroughly investigate the practical implications of the two-factor and the correlation calibration constraints through numerical applications. We introduce the Hoeffding decomposition of the aggregate unconditional loss to provide a systematic-idiosyncratic representation. Impacts of a J -factor correlation structure on risk measures and risk contributions are studied for long-only and long-short credit-sensitive portfolios.

Keywords: Fundamental Review of the Trading Book, Portfolio Credit Risk Modeling, Factor Models, Risk Contribution

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1. Basel recommendations on credit risk

1.1. Credit risk in the Basel II, 2.5 and III agreements

Created in 1974 by ten leading industrial countries and now including supervisors from twenty-seven countries, the Basel Committee on Banking Supervision (BCBS, henceforth “the Committee”) is responsible for strengthening the resilience of the global financial system, ensuring the effectiveness of prudential supervision and improving the cooperation among banking regulators.

The Basel II agreements (BCBS, 2005)[4] define regulatory capital through the concept of Risk Weighted Assets (RWAs) and through the McDonough ratio. Under the Advanced Internal Rating Based (IRB) approach, the RWAs in the banking book measure the exposition of a bank granting loans by applying a weight according to the intrinsic riskiness of each asset. Issuer’s default probability and effective loss at default time are based on bank’s own internal estimates, though correlation parameters are regulatory prescribed. The Committee also addressed portfolio risk prescribing a model based on the Asymptotic Single Risk Factor model (ASRF).

A major gap revealed by the 2008 financial crisis, was the inability to adequately identify the credit risk of the trading book positions, enclosed in credit-quality linked assets. Considering this deficiency, the Committee revised the market risk capital requirements in the 2009 reforms, also known as Basel 2.5 agreements (BCBS, 2009) [5], that add two new capital requirements, the Incremental Risk Charge (IRC) and the Comprehensive Risk Measures (CRM). The former was designed to deal with long term changes in credit quality, whereas the latter specifically controls correlation products. More exactly, the IRC is a capital charge that captures default and migration risks through a VaR-type calculation at 99.9% on a one-year horizon. As opposed to the credit risk treatment in the banking book, the trading book model specification results from a complete internal model validation process, whereby financial institutions are required to implement their own framework.

In parallel to the elaboration of new rules, the Committee has recently investigated the RWAs comparability among institutions and jurisdictions, for both the banking book (BCBS, 2013)[8] and the trading book (BCBS, 2013)[9, 10], through a Regulatory Consistency Assessment Program (RCAP) following previous

studies led by the International Monetary Fund (IMF) in 2012 (see Le Leslé and Avramova (2012)[40]). Based on a set of hypothetical benchmark portfolios, reports show large discrepancies in risk measure levels, and consequently in RWAs, amongst participating financial institutions. The related causes of such a variability are numerous. Among the foremost is the heterogeneity of risk profiles, consecutive to institutions' diverse activities, and divergences in local regulation regimes. In conjunction with these structural causes, the Committee also raises important discrepancies among internal methodologies of risk calculation, and in particular, those of the trading book's RWAs. A main contributor to this variability appear to be the modeling choices made by each institution within their IRC model (for instance, whether it uses spread-based or transition matrix-based models, calibration of the transition matrix or that of the initial credit rating, correlations' assumptions across obligors, etc.).

1.2. Credit risk in the Fundamental Review of the Trading Book

In response to these shortcomings, the Committee has been working since 2012 towards a new post-crisis update of the market risk global regulatory framework, known as Fundamental Review of the Trading Book (FRTB) (BCBS 2012, 2013, 2015)[6, 7, 14, 15, 16]. Notwithstanding long-lasting impact studies and ongoing consultative working groups, no consensus seems to have been achieved so far. Main discussions arise from the proposal transforming the IRC in favor of a default-only risk capital charge (i.e. without migration feature), named DRC. With a one-year 99.9%-VaR calculation, DRC for the trading book would be based on a two-factor model:

“One of the key observations from the Committee’s review of the variability of market risk weighted assets is that the more complex migration and default models were a relatively large source of variation. The Committee has decided to develop a more prescriptive IDR charge [amended to DRC for Default Risk Charge] in the models-based framework. Banks using the internal model approach to calculate a default risk charge must use a two-factor default simulation model [“with two types of systematic risk factors” according to (BCBS 2015)[16]], which the Committee believes will reduce variation in market risk-weighted assets but be sufficiently risk sensitive as compared to multi-factor models”. (BCBS 2013)[7].

The objective of constraining the default risk modeling choices by “*limiting discretion on the choice of risk factors*” has also been mentioned in a report to the G20, BCBS (2014)[12]. Going further, the Committee would especially monitor model risk through correlation calibration constraints. First consultative papers on the FRTB, (BCBS, 2012, 2013)[6, 7], prescribed the usage of listed equity prices to calibrate the default correlations. Based on the trading book hypothetical portfolio exercise (BCBS, 2014)[11], the Committee observes that the usage of equity data was dominant among financial institutions, while some of them chose CDS spreads for the Quantitative Impact Study (QIS). Indeed, equity-based prescribed correlations raise practical problems when data are not available⁴, as for instance for sovereign issuers, resulting in the consideration of other data sources. Consequently, the third consultative paper of the Committee (BCBS, 2015)[14], the subsequent ISDA response (ISDA, 2015)[35] and the instructions for the Basel III monitoring (BCBS 2015)[15] recommend the joint use of credit spreads and equity data.

“Default correlations must be based on credit spreads or on listed equity prices. Banks must have clear policies and procedures that describe the correlation calibration process, documenting in particular in which cases credit spreads or equity prices are used. Correlations must be based on a period of stress, estimated over a 10-year time horizon and be based on a [one]-year liquidity horizon.[...] These correlations should be based on objective data and not chosen in an opportunistic way where a higher correlation is used for portfolios with a mix of long and short positions and a low correlation used for portfolio with long only exposures. [...] A bank must validate that its modeling approach for these correlations is appropriate for its portfolio, including the choice and weights of its systematic risk factors. A bank must document its modeling approach and the period of time used to calibrate the model.” (BCBS 2015)[15].

Our paper investigates the practical implications of these recommendations, and in particular, studies the impact of factor models and their induced correlation structures on the trading book credit risk measurement. The goal here is to provide a comparative analysis of risk factors modeling to assess the relevance of the Committee’s proposals of prescribing model and calibration procedure to reduce

⁴Likewise, no guidance has been yet formulated for the treatment of exposures depending on non-modellable risk-factors.

the RWAs variability and enhance comparability between financial institutions. For this purpose, the scope of the analysis is only focused on the correlation part of the modeling and therefore does not include Probability of Default (PD) and Loss Given Default (LGD) estimations for which the Committee also provides directions (BCBS 2015)[15]. For a general modeling approach, compliant with the Basel recommendations and covering the correlation structure, the default probabilities and the recovery rates, see Wilkens and Predescu (2015)[59].

The paper is organized as follows. In Section 2, we describe a two-factor default risk model within the usual Gaussian latent variables framework, already used in the current banking book setting (one-factor model) and in the IRC implementations (general J -factor ($J \geq 1$) models). Following the Committee’s recommendations, we discuss main correlation estimation methods. In Section 3, we use the Hoeffding decomposition of the aggregate loss to explicitly derive contributions of systematic and idiosyncratic risks, of particular interest in the trading book. Section 4 is devoted to numerical applications on representative long-only and long-short credit-sensitive portfolios, whereby impacts of J -factor correlation structures on risk measures and risk contributions are considered. The last section gathers concluding remarks providing answers to the question raised in the title.

2. Two-factor Default Risk Charge model

2.1. Model specification

The portfolio loss at a one-period horizon is modeled by a random variable L , defined as the sum of the individual losses on issuers’ default over that period. We consider a portfolio with K positions: $L = \sum_{k=1}^K L_k$ with L_k denoting the loss on the position k . The individual loss is decomposed as $L_k = w_k \times \mathbb{I}_k$, where w_k is the positive or negative effective exposure⁵ at the time of default and \mathbb{I}_k is a random variable referred to as the obligor k ’s creditworthiness index, taking value 1 when default occurs, and 0 otherwise. For conciseness, we assume constant effective exposures at default, hence the sole remaining source of randomness is \mathbb{I}_k .

⁵The effective exposure of the position k is defined as the product of the Exposure At Default (EAD_k) and the Loss Given Default (LGD_k). Formally: $w_k = EAD_k \times LGD_k$. While we could think of stochastic LGDs, there is no consensus as regard to proper choices, either regarding marginal LGDs or the joint distribution of LGDs and default indicators. The Basel Committee is not prescriptive at this stage and it is more than likely that most banks will retain constant LGDs.

To define the probability distribution of the L_k 's as well as their dependence structure, we rely on a usual structural factor approach. Creditworthiness is then defined as $\mathbb{I}_k = \mathbf{1}_{\{X_k \leq x_k\}}$, where x_k is a predetermined threshold. Modeling \mathbb{I}_k thus amounts to modeling X_k . This model, introduced by Vasicek (1987, 2001)[56, 57] and based on seminal work of Merton (1974) [45], is widely used by financial institutions and regulators to model default risk. More precisely, in this approach, $X \in \mathbb{R}^{K \times 1}$ is a vector of latent variables, representing obligor's asset values, which evolves according to a J -factor Gaussian model:

$$X = \beta Z + \sigma(\beta)\varepsilon \quad (1)$$

where $Z \sim \mathcal{N}(\mathbf{0}, \Sigma_Z)$ is a J -dimensional random vector of centered systematic factors, $\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{1})$ is a K -dimensional random vector of centered and independent specific risks, $\beta \in \mathbb{R}^{K \times J}$ is the factor loading matrix and $\sigma(\beta) \in \mathbb{R}^{K \times K}$ is a diagonal matrix with elements $\sigma_k = \sqrt{1 - \beta_k \Sigma_Z \beta_k^t}$, ($\beta_k \in \mathbb{R}^{1 \times J}$). This setting ensures that the random vector of asset values, X , is standard normal with a correlation matrix depending on the factor loadings:

$$\beta \mapsto C(\beta) = \beta \Sigma_Z \beta^t + \sigma^2(\beta) \quad (2)$$

In the remainder of the article, we note $\mathcal{Z} = \{Z_j | j = 1, \dots, J\}$ the set of all systematic factors and $\mathcal{E} = \{\varepsilon_k | k = 1, \dots, K\}$ the set of all idiosyncratic risks such that $\mathcal{F} = \mathcal{Z} \cup \mathcal{E}$.

The threshold x_k is chosen such that $\mathbb{P}(\mathbb{I}_k = 1) = p_k$, where p_k is the obligor k 's marginal default probability. From normality of X_k , we have $x_k = \Phi^{-1}(p_k)$, with $\Phi(\cdot)$ denoting the standard normal cumulative function. The portfolio loss⁶ is then written as follows:

$$L = \sum_{k=1}^K w_k \mathbf{1}_{\{\beta_k Z + \sigma_k \varepsilon_k \leq \Phi^{-1}(p_k)\}} \quad (3)$$

The single factor variant of this model, where correlation coefficients are regulatory prescribed, is at the

⁶Since \mathbb{I}_k is discontinuous, L can take only a finite number of values in the set $\mathbb{L} = \{\sum_{a \in A} w_a | \forall A \subseteq \{1, \dots, K\}\} \cup \{0\}$. In the homogeneous portfolio, where all weights are equal: $Card(\mathbb{L}) = K + 1$. In contrast, if the weights are different, then the number of possible loss values can go up to 2^K and the numerical computation of quantile-based risk measures may be more difficult.

heart of the Basel II credit risk capital charge. In this model, known as Asymptotic Single Risk Factor (ASRF) model (cf. Gordy (2003)[31]), the latent systematic factor is usually interpreted as the state of the economy, i.e. a generic macroeconomic variable affecting all firms. Within multi-factor models⁷ ($J \geq 2$), factors may be either latent or observable. For the later, a fine segmentation (into industrial sectors, geographical regions, ratings and so on) of factors allows modelers to define a detailed operational representation of the portfolio correlation structure.

2.2. Assets values correlation calibration

The modeling assumptions on the general framework being made, we now consider the calibration of the assets values correlation matrix of the structural-type credit model. As previously mentioned, the Committee recommends the joint use of equity and credit spread data (notwithstanding such a combination may raise consistency issues as pairwise correlations computed from credit spreads and equity data are sometimes distant). Nevertheless, at that stage, the Committee sets aside concerns as to which estimator of the correlation matrix is to be used and we here stress that this recommendation neglects issues regarding the sensitivity of the estimation to the underlying calibration period and regarding the processing of noisy information, although essential to financial risk measurement.

In the following Subsection, for ease of expositions we consider the latent factor models with $\Sigma_Z = Id$, where Id is the identity matrix. We introduce \tilde{X} , the $(K \times T)$ -matrix of centered stock or CDS-spread returns (T is the time series length), and the standard estimators of the sample covariance and correlation matrices:

$$\Sigma = T^{-1} \tilde{X} \tilde{X}^t \quad (4)$$

$$C = (diag(\Sigma))^{-\frac{1}{2}} \Sigma (diag(\Sigma))^{-\frac{1}{2}} \quad (5)$$

⁷In its analysis of the trading book hypothetical portfolio exercise (BCBS, 2014) [11], the Committee reports that most banks currently use an IRC model with three or less factors, and only 3% have more than three factors. Multi-factor models for credit-risk portfolio and the comparison to the one-factor model are documented in the literature. For instance, in the context of long-only credit exposure portfolio, Düllmann et al. (2007)[25] compare the correlation and the Value-at-Risk estimates among a one-factor model, a multi-factor model (based on the Moody's KMV model) and the Basel II IRB model. Their empirical analysis with a heterogeneous portfolio shows a complex interaction of credit risk correlations and default probabilities affecting the credit portfolio risk.

It is well known that these matrices suffer some drawbacks. Indeed, when the number of variables (equities or CDS-spreads), K , is close to the number of historical returns, T , the total number of parameters is of the same order as the total size of the data set, which is puzzling for the estimator stability. Moreover, when K is larger than T , the matrices are always singular⁸.

Within the vast literature dedicated to covariance/correlation matrix estimation from equities⁹, we refer particularly to Michaud (1989)[46] for a proof of the instability of the empirical estimator, to Alexander and Leigh (1997)[1] for a review of covariance matrix estimators in VaR models and to Disatnik and Benninga (2007)[24] for a brief review of covariance matrix estimators in the context of the shrinkage method¹⁰.

Shrinkage methods are statistical procedures which consist in imposing low-dimensional factor structure to a covariance matrix estimator to deal with the trade-off between bias and estimation error. Indeed, the sample covariance matrix can be interpreted as a K -factor model where each variable is a factor (no residuals) so that the estimation bias is low (the estimator is asymptotically unbiased) but the estimation error is large. At the opposite, we may postulate a one-factor model which has a large bias from likely misspecified structural assumptions, but a little estimation error. According to seminal work of Stein (1956)[50], reaching the optimal trade-off may be done by taking a properly weighted average of the biased and the unbiased estimators: this is called shrinking the unbiased estimator. Within the context of default correlation calibration, we here focus on the approach of Ledoit and Wolf (2003)[39] who define a weighted average of the sample covariance matrix with the single-index model estimator of Sharpe (1963)[49]: $\Sigma_{shrink} = \alpha_{shrink} \Sigma_J + (1 - \alpha_{shrink}) \Sigma$, where Σ_J is the covariance matrix generated by a

⁸Note that this feature is problematic when considering the Principal Component Analysis (PCA) to estimate factor models because the method requires the invertibility of Σ or C . To overcome this problem, Connor and Korajczyk (1986,1988) [20, 21] introduce the Asymptotic PCA which consists in applying PCA on the $(T \times T)$ -matrix, $K^{-1} \tilde{X}^t \tilde{X}$, rather than on Σ . The authors show that APCA is asymptotically equivalent to the PCA on Σ .

⁹A number of academic papers also address the estimation of dynamic correlations. See for instance the paper of Engle (2002) [26] introducing the Dynamic Conditional Correlation (DCC) or the paper of Engle and Kelly (2012) [27] for a brief overview of dynamic correlation estimation and the presentation of the Dynamic Equicorrelation (DECO) approach. This is out of the scope of this paper. For practical purposes we focus on techniques which can easily account for a credit universe that encompasses several hundreds/thousands of names.

¹⁰See also Laloux, Cizeau, Bouchaud and Potters (1999) [37] for evidences of ill-conditioning and of the ‘‘curse of dimension’’ within a random matrix theory approach, and Papp, Kafka, Nowak and Kondor (2005) [47] for an application of random matrix theory to portfolio allocation.

($J = 1$)-factor model and the weight α_{shrink} controls how much inertia to impose. The authors show how to determine the optimal shrinking intensity (α_{shrink}) and, based on historical data, illustrate their approach through numerical experiments where the method out-performs all other standard estimators.

In the sequel, following the Committee’s proposal, we consider an initial correlation matrix C_0 , estimated from historical stock or CDS spread returns. To study the impact of the correlation structure on the levels of risk and factors contributions (cf. Section 4), we shall also consider other candidates as the initial matrix such as the “*shrunked*” correlation matrix (computed from Σ_{shrink}), the matrix associated with the IRB ASRF model and the one associated with a standard J -factor model (like the Moody’s KMV model for instance).

2.3. Nearest correlation matrix with J -factor structure

Factor models have been very popular in finance as they offer parsimonious explanations of asset returns and correlations. The underlying goal of the Committee’s proposition is to build a factor model (with a specified number of factors) generating a correlation structure as close as possible to the pre-determined correlation structure C_0 . At this stage, the Committee does not provide any guidance on the calibration of factors loadings β needed to pass from a ($J > 2$)-factor structure to a ($J = 2$)-factor structure. The objective here is then to present generic methods to calibrate a model with a J -factor structure from an initial ($K \times K$)-correlation matrix.

Among popular exploratory methods used to calibrate such models, Principal Components Analysis (PCA) aims at specifying a linear factor structure between variables. Indeed, by considering the random vector X of asset values, and using the spectral decomposition theorem on the initial correlation matrix: $C_0 = \Gamma\Lambda\Gamma^t$ (where Λ is the diagonal matrix of ordered eigenvalues of C_0 and Γ is an orthogonal matrix whose columns are the associated eigenvectors), the principal components transform of X is: $Y = \Gamma^t X$, where the random vector Y contains the ordered principal components. Since this transformation is invertible, we may finally write: $X = \Gamma Y$. In this context, an easy way to postulate a J -factor model is to partition Y according to $(Y_J^t, Y_{\bar{J}}^t)^t$ where $Y_J \in \mathbb{R}^{J \times 1}$ and $Y_{\bar{J}} \in \mathbb{R}^{(K-J) \times 1}$, and to partition Γ according to $(\Gamma_J, \Gamma_{\bar{J}})$ where $\Gamma_J \in \mathbb{R}^{K \times J}$ and $\Gamma_{\bar{J}} \in \mathbb{R}^{K \times (K-J)}$. This truncation leads to: $X = \Gamma_J Y_J + \Gamma_{\bar{J}} Y_{\bar{J}}$. Hence,

by considering Γ_J as the factors loadings (composed of the J first eigenvectors of C_0), Y_J as the factors (composed of the J first principal components of X) and $\Gamma_{\bar{J}}Y_{\bar{J}}$ as the residuals, we obtain a J -factor model.

Nevertheless, as mentioned by Andersen, Sidenius and Basu (2003) [2], the specified factor structure in Equation (1) cannot be merely calibrated by a truncated eigen-expansion, due to the specifications of residuals that depend on β . In fact, here, we look for a ($J = 2$)-factor modeled X for which the correlation matrix¹¹ $C(\beta) = \beta\beta^t + \sigma^2(\beta)$, with $\sigma^2(\beta) = Id - diag(\beta\beta^t)$, is as close as possible to C_0 in the sense of a chosen norm. Thus, we define the following optimization problem¹²:

$$\begin{cases} \arg \min_{\beta} f(\beta) & = \|C_0 - C(\beta)\|_F \\ \text{subject to: } \beta \in \Omega & = \{\beta \in \mathbb{R}^{K \times J} \mid \beta_k \beta_k^t \leq 1; k = 1, \dots, K\}, \end{cases} \quad (6)$$

where $\|\cdot\|_F$ is the Froebenius norm: $\forall A \in \mathbb{R}^{K \times K} : \|A\|_F = \text{tr}(A^t A)$ (with $tr(\cdot)$ denoting the trace of a square matrix). The constraint, $\beta \in \Omega$, ensures that $\beta\beta^t$ has diagonal elements bounded by 1, implying that $C(\beta)$ is positive semi-definite.

The general problem of computing a correlation matrix of J -factor structure nearest to a given matrix has been tackled in the literature. In the context of credit basket securities, Andersen, Sidenius and Basu (2003) [2] determine a sequence $\{\beta_i\}_{i>0}$ in the following way. Given the spectral decomposition of $(C_0 - \sigma^2(\beta_i))$, $\Gamma_{\sigma} \Lambda_{\sigma} \Gamma_{\sigma}^t$: $\beta_{i+1} = \Gamma_{\sigma, J} \Lambda_{\sigma, J}^{1/2}$, i.e. they consider the eigenvectors associated with the J largest eigenvalues. The iterations stop when $\sigma^2(\beta_{i+1})$ is sufficiently close to $\sigma^2(\beta_i)$.

Borsdorff, Higham and Raydan (2010) [18] show that this PCA-based method, which is not supported by any convergence theory, often performs surprisingly well on the one hand, partly because the constraints are often not active at the solution, but may fail to solve the constrained problem on the other. They acknowledge the Spectral Projected Gradient (SPG) method, as being the most efficient to solve the

¹¹Recall that, for ease of expositions, we consider the latent factor models with $\Sigma_Z = Id$.

¹² Ω is a closed and convex set in $\mathbb{R}^{K \times J}$. Moreover, the gradient of the objective function is given by: $\nabla f(\beta) = 4(\beta(\beta\beta^t) - C_0\beta + \beta + diag(\beta\beta^t\beta))$

constrained problem. This method allows minimizing $f(\beta)$ over the convex set Ω by iterating over β in the following way: $\beta_{i+1} = \beta_i + \alpha_i d_i$ where $d_i = \text{Proj}_{\Omega}(\beta_i - \lambda_i \nabla f(\beta_i)) - \beta_i$ is the descent direction¹³, with $\lambda_i > 0$ a pre-computed scalar, and α_i is a positive scalar chosen through a non-monotone line search strategy. Proj_{Ω} being uncostly to compute, the algorithm is fast enough to enable the calibration of portfolios having a large number of positions. A detailed presentation and algorithms are available in Birgin, Martinez and Raydan (2001) [17].

An important point for the validity of a factor model is the correct specification of the number of factors. Until now, in accordance with the Committee's specification, we have assumed arbitrary J -factor models where J is specified by the modeler ($J = 2$ for the Committee). Based on the data, we may also consider the problem of determining the optimal number of factors. Some previous academic papers deal with this issue. Among them, we particularly refer to Bai and Ng (2002) [3] who propose Panel Criteria to consistently estimate the optimal number of factor from historical data¹⁴.

Finally, it is noteworthy that the presented methods, approximating the initial correlation matrix (methods based on PCA or SPG to find the nearest correlation matrix with J -factor structure based, or the shrinkage method to make the correlation matrix more robust), may sometimes smooth the pairwise correlations. For instance, the shrinkage method does not specifically treat the case where the pairwise correlations are near or equal to one. It rather tends to reverse these correlations to the mean level even if, statistically, the variance of the correlation estimator with a value close to the unit is often very low or even null.

3. Hoeffding decomposition of losses and risk contributions

In the banking book, the Basel II capital requirement formula (IRB modeling) is based on the assumption that the portfolio is infinitely fine grained, i.e. it consists of a very large number of credits with small

¹³Note that $\beta_{i+1} = (1 - \alpha_i)\beta_i + \alpha_i \text{Proj}_{\Omega}(\beta_i - \lambda_i \nabla f(\beta_i))$. Thus, if $\beta_1 \in \Omega$, then $\beta_{i+1} \in \Omega, \forall i \geq 1$.

¹⁴The authors consider the sum of squared residuals, noted $V(j, Z_j)$ where the j factors are estimated by PCA ($\forall j \in 1, \dots, J$), and introduce the Panel Criteria and the Information Criteria to be used in practice for determining the optimal number of factors: $PC_m(j) = V(j, Z_j) + \text{Penalty}_m^{PC}$ and $IC_m = \ln(V(j, Z_j)) + \text{Penalty}_m^{IC}$ (for $m = 1, 2, 3$), where Penalty_m^{PC} and Penalty_m^{IC} are penalty functions. To validate their method, the authors consider several numerical experiments. In particular, in the strict factor model (where the idiosyncratic errors are uncorrelated as in our framework), the preferred criteria would be the following: PC_1, PC_2, IC_1 and IC_2 . We refer to Bai and Ng (2002) [3] for a complete description of these criteria and the general methodology.

exposures, so that only one systematic risk factor influences portfolio default risk (ASRF model). Thus, the aggregate loss can be approximated by the systematic factor projection: $L \approx L_Z = \mathbb{E}[L|Z]$, subsequently called “Large Pool Approximation”. Under this assumption, Wilde (2001)[58] expresses a portfolio invariance property stating that the required capital for any given loan does not depend on the portfolio it is added to. This makes the IRB modeling appealing as it allows the straightforward calculation of risk measures and contributions.

At the opposite, the particularities of the trading book positions (actively traded positions, the presence of long-short credit risk exposures, heterogeneous and potentially small number of positions) make this assumption too restrictive and compel to analyze the risk contribution of both the systematic factors and the idiosyncratic risks.

In this Section, we first represent the portfolio loss via the Hoeffding decomposition to exhibit the impact of both the systematic factors and the idiosyncratic risks. In this framework, the second Subsection presents analytics of factors contributions to the risk measure.

3.1. Hoeffding decomposition of the aggregate loss

The portfolio loss has been defined in the previous Section via the sum of the individual losses (cf. Equation (3)). We consider here a representation of the loss as a sum of terms involving sets of factors through the Hoeffding decomposition, previously used in a credit context by Rosen and Saunders (2010) [48]. Formally, if F_1, \dots, F_M and $L \equiv L[F_1, \dots, F_M]$ are square-integrable random variables¹⁵, then the Hoeffding decomposition¹⁶ expresses the aggregate portfolio loss, L , as a sum of terms involving conditional expectations given factor sets:

$$L = \sum_{S \subseteq \{1, \dots, M\}} \phi_S(L; F_m, m \in S) = \sum_{S \subseteq \{1, \dots, M\}} \sum_{\tilde{S} \subseteq S} (-1)^{|S| - |\tilde{S}|} \mathbb{E} \left[L | F_m; m \in \tilde{S} \right]. \quad (7)$$

¹⁵The Hoeffding decomposition is usually applied to independent factors. If this assumption is fulfilled, then all terms of the decomposition are uncorrelated, easing the interpretation of each term.

¹⁶Consult Van Der Waart (1999) [55] for a detailed presentation of the Hoeffding decomposition.

Although the Hoeffding decomposition may be unadapted¹⁷ when the number of factors is large, computation for a two-factor model does not present any challenge, especially in the Gaussian framework where an explicit analytical form of each term exists (cf. Equations (10) and (11)).

An appealing feature of the Hoeffding decomposition is that it can be applied to subsets of risk factors. Therefore, we use the decomposition on the set of systematic factors, \mathcal{Z} , and the set of specific risks, \mathcal{E} , to break the portfolio loss down in terms of aggregated systematic and idiosyncratic parts:

$$L = \phi_0(L) + \phi_1(L; \mathcal{Z}) + \phi_2(L; \mathcal{E}) + \phi_{1,2}(L; \mathcal{Z}, \mathcal{E}), \quad (8)$$

where $\phi_0(L) = \mathbb{E}[L]$ is the expected loss, $\phi_1(L; \mathcal{Z}) = \mathbb{E}[L|\mathcal{Z}] - \mathbb{E}[L]$ is the loss induced by the systematic factors Z_1 and Z_2 (corresponding, up to the expected loss term, to the heterogeneous Large Pool Approximation: $L_{\mathcal{Z}}$), $\phi_2(L; \mathcal{E}) = \mathbb{E}[L|\mathcal{E}] - \mathbb{E}[L]$ is the loss induced by the K idiosyncratic terms ϵ_k , and $\phi_{1,2}(L; \mathcal{Z}, \mathcal{E}) = (L - \mathbb{E}[L|\mathcal{Z}] - \mathbb{E}[L|\mathcal{E}] + \mathbb{E}[L])$ is the remaining risk induced by interaction (cross-effect) between idiosyncratic and systematic risk factors.

Since $L_{\mathcal{Z}} = \phi_1(L; \mathcal{Z}) + \phi_0(L)$, the relation between unconditional and conditional portfolio losses is:

$$L = L_{\mathcal{Z}} + \phi_2(L; \mathcal{E}) + \phi_{1,2}(L; \mathcal{Z}, \mathcal{E}). \quad (9)$$

Remark that the Hoeffding decomposition is not an approximation. It is an equivalent representation of the same random variable into uncorrelated terms (thanks to the independence between the sets of idiosyncratic and systematic risk factors).

From a practical point of view, as we consider a Gaussian factor model, we may easily compute each term of the decomposition:

$$\mathbb{E}[L|\mathcal{Z}] = \sum_{k=1}^K w_k \Phi \left(\frac{\Phi^{-1}(p_k) - \beta_k Z}{\sigma_k} \right), \quad (10)$$

¹⁷The Hoeffding decomposition requires the calculation of 2^M terms, where M is the number of factors.

$$\mathbb{E}[L|\mathcal{E}] = \sum_{k=1}^K w_k \Phi \left(\frac{\Phi^{-1}(p_k) - \sigma_k \epsilon_k}{\sqrt{\beta_k \Sigma_Z \beta_k^t}} \right). \quad (11)$$

Rosen and Saunders (2010) [48] focus on the Hoeffding decomposition of heterogeneous Large Pool Approximation, $L_{\mathcal{Z}}$. In this context, they define the decomposition with two factors, Z_1 and Z_2 , of the conditional loss $L_{\mathcal{Z}}$ by:

$$L_{\mathcal{Z}} = \phi_{\emptyset}(L_{\mathcal{Z}}) + \phi_1(L_{\mathcal{Z}}; Z_1) + \phi_2(L_{\mathcal{Z}}; Z_2) + \phi_{1,2}(L_{\mathcal{Z}}; Z_1, Z_2), \quad (12)$$

where $\phi_{\emptyset}(L_{\mathcal{Z}}) = \mathbb{E}[L]$ is the expected loss, $\phi_1(L_{\mathcal{Z}}; Z_1)$ and $\phi_2(L_{\mathcal{Z}}; Z_2)$ are the losses induced by the systematic factors Z_1 and Z_2 respectively, and the last term $\phi_{1,2}(L_{\mathcal{Z}}; Z_1, Z_2)$ is the remaining loss induced by systematic factors interaction (cross-effect of Z_1 and Z_2).

Equation (12) provides a factor-by-factor decomposition of the aggregate loss, but raises questions concerning the significance of each term. Indeed, in the case of models with endogenous factors, Σ_Z is the identity matrix and the terms are uncorrelated. When it comes to exogenous factors, where we account for the factors' correlation structure, the elements in Equation (12) are correlated as well, and the meaning of each term is unclear. More generally, the significance of the terms is unobvious since factor rotations¹⁸ of the systematic factors leave the distribution of X unchanged but affects the computation of Hoeffding terms that depend on subsets included in \mathcal{Z} ¹⁹. Since we further focus on systematic factors on one hand and specific risks on the others, these two conundrums are not relevant in our case.

3.2. Systematic and idiosyncratic contributions to the risk measure

The portfolio risk is determined by means of a risk measure ϱ , which is a mapping of the loss to a real number: $\varrho : L \mapsto \varrho[L] \in \mathbb{R}$. Usual quantile-based risk measures are the VaR and the

¹⁸For further details on this statistical procedure, we refer to common statistical literature such as Kline (2014) [36].

¹⁹Consider for instance the Large Pool Approximation of the portfolio loss: $L_{\mathcal{Z}} = \mathbb{E}[L|Z_1, Z_2]$. Case 1: we consider that the betas are equivalent. For instance, $\forall k, \beta_{k,1} = \beta_{k,2} = b$ (with $b \in [-1, 1]$) which implies that $\phi_1(L_{\mathcal{Z}}; Z_1) = \phi_2(L_{\mathcal{Z}}; Z_2)$ in distribution, and the contribution of each element is the same. Case 2: we now consider the special one factor model. For instance $\beta_{k,1} = \sqrt{2} \times b^2$ and $\beta_{k,2} = 0$ implying that $\phi_1(L_{\mathcal{Z}}; Z_1) \neq \phi_2(L_{\mathcal{Z}}; Z_2)$ in distribution. Case 3: we finally consider a permutation of the case 2: $\beta_{k,1} = 0$ and $\beta_{k,2} = \sqrt{2} \times b^2$ leading to the same conclusion. In those 3 cases, the risk is identical but factor contributions differ.

Conditional-Tail-Expectation²⁰ (CTE). For a given confidence level $\alpha \in [0, 1]$, the VaR is the α -quantile of the loss distribution: $VaR_\alpha[L] = \inf\{l \in \mathbb{R} | \mathbb{P}(L \leq l) \geq \alpha\}$. On the other hand, the CTE is the expectation of the loss conditional to loss occurrences higher than the $VaR_\alpha[L]$: $CTE_\alpha[L] = \mathbb{E}[L | L \geq VaR_\alpha[L]]$. Since both IRC in Basel 2.5 and DRC in the Fundamental Review of the Trading Book prescribe the use of a one-year 99.9%-VaR, we will further restrict to this risk measure even though risk decompositions can readily be extended to the set of spectral risk measures.

By definition, the portfolio loss equals the sum of individual losses: $L = \sum_{k=1}^K L_k$. As we showed earlier, it can also be defined as the sum of the Hoeffding decomposition terms: $L = \sum_{S \subseteq \{1, \dots, M\}} \phi_S(L; F_m, m \in S)$. To understand risk origin in the portfolio, it is common to refer to a contribution measure $C_{L_k}^g$ ($C_{\phi_S}^g$ respectively) of the position k (of the Hoeffding decomposition term ϕ_S) to the total portfolio risk $\varrho[L]$. The position risk contribution is of great importance for hedging, capital allocation, performance measurement and portfolio optimization and we refer to Tasche (2007) [52] for a detailed presentation.

As fundamental as the position risk contribution, the factor risk contribution helps unravel alternative sources of portfolio risk. Papers dealing with this topic include the following. Cherny and Madan (2007) [19] consider the conditional expectation of the loss with respect to the systematic factor and refer it to as *factor risk brought by that factor*. Martin and Tasche (2007) [42] consider the same conditional expectation, but then apply the Euler's principle taking the derivative of the portfolio risk in the direction of this conditional expectation and call it *risk impact*. Rosen and Saunders (2007) [48] apply the Hoeffding decomposition of the loss with respect to sets of systematic factors, the first several terms of this decomposition coinciding with the conditional expectations mentioned above.

Theoretical and practical aspects of various allocation schemes have been analyzed in several papers (see Dhaene et al. (2012) [23] for a review). Among these, the marginal contribution method, based on Euler's allocation rule, is quite a standard one (see Tasche (2007) [53]). To be applied here, an adaptation is required because we face discrete distributions (see Laurent (2003) [38] for an analysis of the technical

²⁰Note that as long as L is continuous, the Conditional-Tail-Expectation is equivalent to the Expected-Shortfall.

issues at hand). Yet, under differentiability conditions, and choosing the VaR as the risk measure²¹, it can be shown (see Gouriéroux, Laurent and Scaillet (2001) [32]) that the marginal contribution of the individual loss L_k to the risk associated with the aggregate loss $L = \sum_{k=1}^K L_k$ is given by $C_{L_k}^{VaR} = \mathbb{E}[L_k | L = VaR_\alpha[L]]$. Besides, computing this expectation does not involve any other assumption than integrability and defining risk contributions along these lines fulfills the usual full allocation rule $L = \sum_{k=1}^K C_{L_k}^{VaR}$ (see Tasche (2008) [52] for details on this rule).

Similarly, we can compute the contributions of the different terms involved in the Hoeffding decomposition of the aggregate loss. For instance, the contribution of the systematic term is readily derived as $C_{\phi_1}^{VaR} = \mathbb{E}[\phi_1(L; \mathcal{Z}) | L = VaR_\alpha[L]]$. Likewise, contributions of specific risk and of interaction terms can easily be written and added afterwards to the systematic term so as to retrieve the risk measure of the aggregate loss. The additivity property of risk contributions prevails when subdividing the vector of risk factors into multiple blocks. This parallels the convenient invariance property well known in the Basel ASRF framework (cf. Wilde (2001)[58]).

It is also noteworthy that although our approach could be deemed as belonging to the granularity adjustment corpus, relying as well on the Large Pool Approximation, the techniques and the mathematical properties involved here (such as differentiability, associated with smooth distributions) are completely different. We refer to Fermanian (2014)[28] and Gagliardini and Gouriéroux (2014)[29] for recent studies involving this concept.

4. Empirical implications for diversification and hedge portfolios

This section is devoted to the empirical study of the effects of the correlation structure on risk measure and risk contributions. In particular, it aims at analyzing the impacts of modeling constraints for both the future DRC prescription and the current Basel 2.5 IRC built on constrained (factor-based) and unconstrained models.

²¹By defining $VaR_\alpha[L] = \mathbb{E}[L | L = VaR_\alpha[L]]$, similar results hold for CTE (up to a sign change from $L = VaR_\alpha[L]$ to $L \geq VaR_\alpha[L]$).

We base our numerical analysis on representative long-only and long-short credit-sensitive portfolios. Since we reckon to focus on widely traded issuers, who represent a large portion of banks' exposures, we opt for portfolios with large Investment Grade companies. Specifically, we consider the names in the iTraxx Europe index, retrieved on the 31st October 2014. The portfolio is composed of 121 European Investment Grade companies²², 27 are Financials, the remaining being tagged Non-Financials.

We successively look at two types of portfolio: (i) a *diversification portfolio*, comprised of positive-only exposures (long-only credit risk), (ii) a *hedge portfolio*, built up of positive and negative exposures (long-short credit risk). The distinction parallels the one between the banking book, containing notably long-credit loans, and the trading book, usually including long and short positions (like for instance bonds or CDS) – within the latter, an issuer's default on a negative exposure yields a gain. For conciseness, the LGD rate is set to 100% for each position of the two portfolio types.

Regarding the *diversification portfolio* ($K = 121$), we consider a constant and equally weighted effective exposure for each name so that $\forall k, w_k = 1/K$ and $\sum_{k=1}^K w_k = 1$.

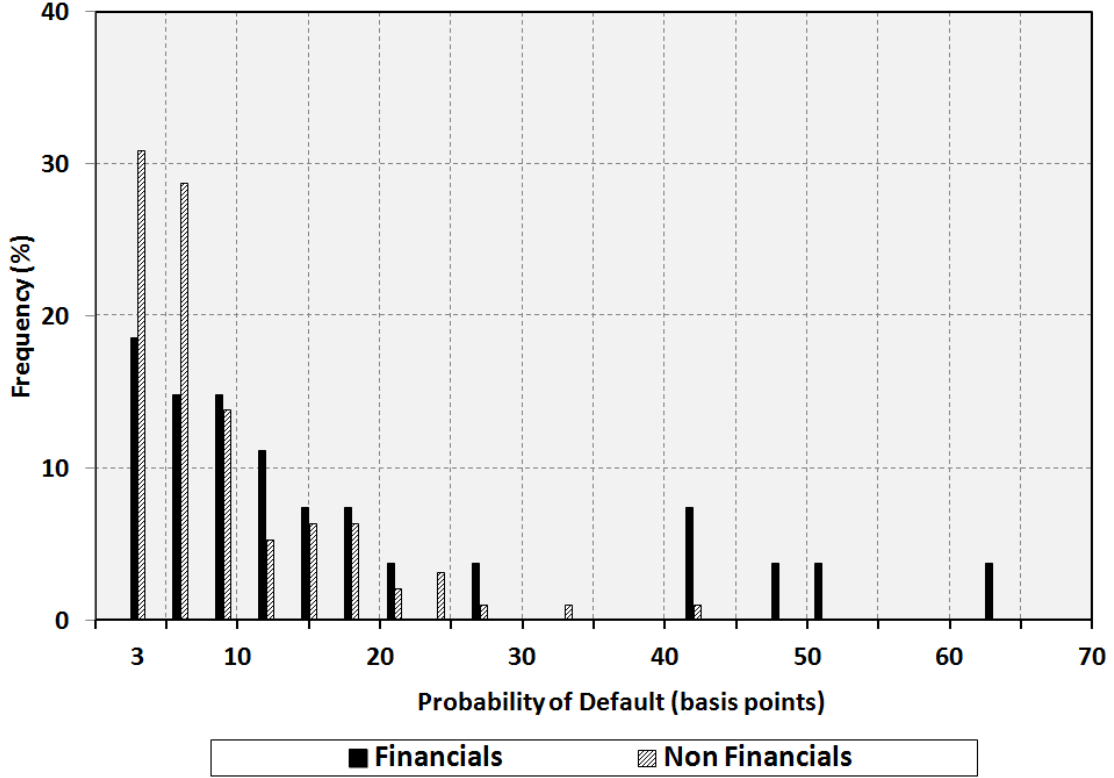
Concerning the *hedge portfolio* ($K = 54$), we assume long exposure to 27 Financials and short exposure to 27 Non-Financials, selected such that the average default probability between the two groups is the same (approximately 0.16%). By considering $w_{k \in \text{Financials}} = 1/27$ and $w_{k \notin \text{Financials}} = -1/27$, the *hedge portfolio* is thus credit-neutral: $\sum_{k=1}^K w_k = 0$.

For the sake of numerical application, we use default probabilities provided by the Bloomberg Issuer Default Risk Methodology²³. Figure (1) illustrates default probabilities' frequencies of the portfolio's companies grouped by Financials and Non-Financials.

²²The index is genuinely composed of 125 names. Nevertheless, due to lack of data for initial correlation computation, it was not possible to estimate a (125×125) -matrix.

²³This methodology, referred to as Bloomberg DRISK methodology, is based on the model of Merton (1974) [45]. The model does not use credit market variables, rather it is an equity markets-based view of default risk. In addition to market data and companies balance sheet fundamental, the model also includes companies' income statements.

Figure 1: Histogram of the default probabilities distribution.



Financials clearly show higher and more dispersed default probabilities, with a mean and a standard deviation equal to 0.16% and 0.16%, respectively, compared to 0.08% and 0.07% respectively, for Non-Financials. We also note that the floor value (at 0.03%) prescribed by the Committee is restrictive for numerous (34 over 121) Financial and Non-Financial issuers.

In the next Subsections, we discuss results on the calibration of both the initial correlation matrix (C_0) and the loading matrix ($\beta \in \mathbb{R}^{K \times J}$) of the J -factor models. We then consider the impact of these different models on the risk for both portfolios. By using a Hoeffding-based representation of the aggregate loss, we finally compute the contributions of the systematic, the idiosyncratic and the interaction parts of the loss to the risk.

4.1. Unconstrained and constrained correlation matrices

Following the Committee’s proposal, we use listed equity prices²⁴ of the 121 issuers, spanning over a one-year period, to calibrate the initial default correlation matrix through the empirical estimator (cf. Equation (5))²⁵. To illustrate the sensitivity to the calibration window, we use two sets of equity time-series. Period 1 covers a time of high market volatility from 07/01/2008 to 07/01/2009, whereas Period 2 spans a comparatively lower market volatility window, from 09/01/2013 to 09/01/2014. For both periods, the computed unconstrained (i.e. with no factor structure) (121×121) -matrix consists of a matrix of pairwise correlations that we adjust²⁶ to ensure semi-definite positivity. To limit the estimation error, we also apply the shrinkage methodology on the two periods.

Furthermore, Period 2 is used to define other initial correlation matrices in order to analyze the effects on the J -factor model of changes in the correlation structure as computed from different types of financial data. We consider three alternative sources: (i) the imposed IRB correlation formula²⁷, based on the issuer’s default probability; (ii) the GCorr methodology of Moody’s KMV; (iii) the issuers’ CDS spreads relative changes (also advocated by the Committee). For each initial correlation matrix, C_0 , the optimization problem in Equation (6) is solved with the PCA-based and the SPG-based algorithms.

In addition to these uncorrelated latent factor models, we also consider models based on two correlated exogenous factors: one region factor (identical across all issuers) and one sector factor. For both periods, the factor loadings are calibrated by projecting (through linear regression) each issuer’s equity returns onto the returns of the MSCI Europe index and the returns of a MSCI sector index corresponding to the issuer’s sector (10 indexes overall).

All characteristics are reported in Table (1).

²⁴Data provided by Bloomberg Professional.

²⁵In the FAQ of August 2015 (BCBS, 2015)[13], the Committee suggests to calibrate correlations over a ten year period, which includes a one-year stress period, using annual co-movements. To reduce estimation errors, our approach differs from this proposal and is based on daily returns on a one-year stress period, selected over a ten year period.

²⁶We use a spectral projection method. Note that this treatment is not needed if we only consider the simulation of the J -factor model calibrated via the optimization problem in Equation (6). As for the unconstrained matrix, we could have used other approaches such as EM algorithm to deal with missing data.

²⁷The IRB approach is based on a 1-factor model: $X_k = \sqrt{\rho_k}Z_1 + \sqrt{1 - \rho_k}\epsilon_k$. Thus, $Correl(X_k, X_j) = \sqrt{\rho_k\rho_j}$ where ρ_k is provided by a regulatory formula: $\rho_k = 0.12 \frac{1 - \exp^{-50\rho_k}}{1 - \exp^{-50}} + 0.24 \left(1 - \frac{1 - \exp^{-50\rho_k}}{1 - \exp^{-50}}\right)$.

Table 1: Initial correlation matrix estimation and J -factor model calibration.

Configuration	Data for estimating C_0	Period	Estimation method for C_0	Calibration method for the J -factor models
(1) Equity - P1	Equity returns	1	Sample correlation	PCA and SPG algorithms
(2) Equity - P1 Shrunked	Equity returns	1	Shrinkage ($\alpha_{shrink} = 0.32$)	PCA and SPG algorithms
(3) Equity - P1 Exogenous Factors	Equity returns	1	Sample correlation	Linear Regression
(4) Equity - P2	Equity returns	2	Sample correlation	PCA and SPG algorithms
(5) Equity - P2 Shrunked	Equity returns	2	Shrinkage ($\alpha_{shrink} = 0.43$)	PCA and SPG algorithms
(6) Equity - P2 Exogenous Factors	Equity returns	2	Sample correlation	Linear Regression
(7) IRBA	-	-	IRBA formula	PCA and SPG algorithms
(8) KMV - P2	-	2	GCorr methodology	PCA and SPG algorithms
(9) CDS - P2	CDS spreads	2	Sample correlation	PCA and SPG algorithms

Period 1: from 07/01/2008 to 07/01/2009. Period 2: from 09/01/2013 to 09/01/2014.

The optimization problem is also considered for the calibration of J^* -factor models (where J^* is the data-based “optimal number” of factors) for both the “(1) Equity - P1” and the “(4) Equity - P2” configurations. It is defined here as the integer part of the arithmetic average of the panel and information criteria (see Bai and Ng (2001)[3]). Applying this methodology to the historical time series, we get $J^* = 6$ for the “(1) Equity - P1” configuration and $J^* = 3$ for the “(4) Equity - P2” configuration²⁸. To make the results comparable, these optimal numbers²⁹ are the same for the *diversification portfolio* and the *hedge portfolio*.

Table (2) shows the calibration results for models with endogenous factors for both the PCA-based and the SPG-based algorithms while Figure (2) displays histograms of the pairwise correlations frequencies

²⁸In particular, we get $IC_1 = 6$, $IC_2 = 5$, $PC_1 = 8$ and $PC_2 = 7$ for the “(1) Equity - P1” configuration and $IC_1 = 2$, $IC_2 = 2$, $PC_1 = 4$ and $PC_2 = 4$ for the “(4) Equity - P2” configuration.

²⁹Remark that these experimental results are consistent with the empirical conclusions of Connor and Korajczyk (1993)[22], who find a number of factors between 1 to 2 factors for “non-stressed” periods and 3 to 6 factors for “stressed” periods for the monthly stock returns of the NYSE and the AMEX, over the period 1967 to 1991. It is also in line with the results of Bai and Ng (2001)[3] who exhibit the presence of two factors when studying the daily returns on the NYSE, the AMEX and the NASDAQ, over the period 1994 to 1998.

for each configuration within each J -factor model ($J = 1, 2, J^*$ with PCA-based calibration) ³⁰.

Table 2: Factor-model calibration over the 121 iTraxx issuers.

Configuration	Number of factors	Froebenius Norm		Average Correlation		Average Correlation Financial		Average Correlation Non-Financial	
		SPG	PCA	SPG	PCA	SPG	PCA	SPG	PCA
(1) Equity - P1	C_0	0,00	0,00	0,46	0,46	0,62	0,62	0,46	0,46
	1 factor	8,75	8,73	0,47	0,46	0,54	0,54	0,45	0,45
	2 factors	6,10	6,01	0,47	0,46	0,60	0,59	0,46	0,46
	($J^* = 6$) factors	4,26	3,84	0,46	0,46	0,63	0,61	0,46	0,46
(2) Equity - P1 Shrunked	C_0	0,00	0,00	0,46	0,46	0,60	0,60	0,45	0,45
	1 factor	5,92	5,88	0,47	0,46	0,55	0,55	0,45	0,45
	2 factors	4,18	4,05	0,47	0,46	0,59	0,58	0,46	0,45
(4) Equity - P2	C_0	0,00	0,00	0,28	0,28	0,44	0,44	0,26	0,26
	1 factor	8,69	8,66	0,28	0,28	0,41	0,41	0,26	0,25
	2 factors	6,99	6,94	0,28	0,28	0,43	0,43	0,26	0,26
	($J^* = 3$) factors	6,36	6,24	0,28	0,28	0,44	0,43	0,26	0,26
(5) Equity - P2 Shrunked	C_0	0,00	0,00	0,28	0,28	0,43	0,43	0,26	0,26
	1 factor	4,98	4,95	0,28	0,28	0,41	0,41	0,26	0,26
	2 factors	4,07	3,97	0,28	0,28	0,42	0,42	0,26	0,26
(7) IRBA	C_0	0,00	0,00	0,25	0,25	0,29	0,29	0,25	0,25
	1 factor	0,22	0,00	0,25	0,25	0,29	0,29	0,25	0,25
	2 factors	0,34	0,00	0,25	0,25	0,29	0,29	0,25	0,25
(8) KMV - P2	C_0	0,00	0,00	0,29	0,29	0,47	0,47	0,27	0,27
	1 factor	4,14	4,09	0,29	0,29	0,43	0,43	0,26	0,26
	2 factors	2,29	2,10	0,29	0,29	0,47	0,47	0,27	0,26
(9) CDS - P2	C_0	0,00	0,00	0,58	0,58	0,81	0,81	0,57	0,57
	1 factor	7,69	7,66	0,59	0,58	0,70	0,69	0,57	0,56
	2 factors	5,51	5,44	0,59	0,58	0,80	0,80	0,58	0,57

The column “Froebenius norm” corresponds to the optimal value of the objective function whereas the three right hand side columns state the average pairwise correlations from, respectively, the overall portfolio matrix, the Financial sub-matrix and the Non-Financial sub-matrix.

Fitting results among the endogenous factor models, in Table (2), suggests that the two considered nearest correlation matrix approaches (SPG-based and PCA-based) perform similarly and correctly. As expected, increasing the number of factors tends to produce a better fit to the unconstrained model. We also note that shrinking the correlation matrix allows a better fit for all models.

³⁰Note that results with the SPG-based algorithm are very similar.

Figure 2: Histogram of the pairwise correlations among the 121 iTraxx issuers (PCA-based calibration).

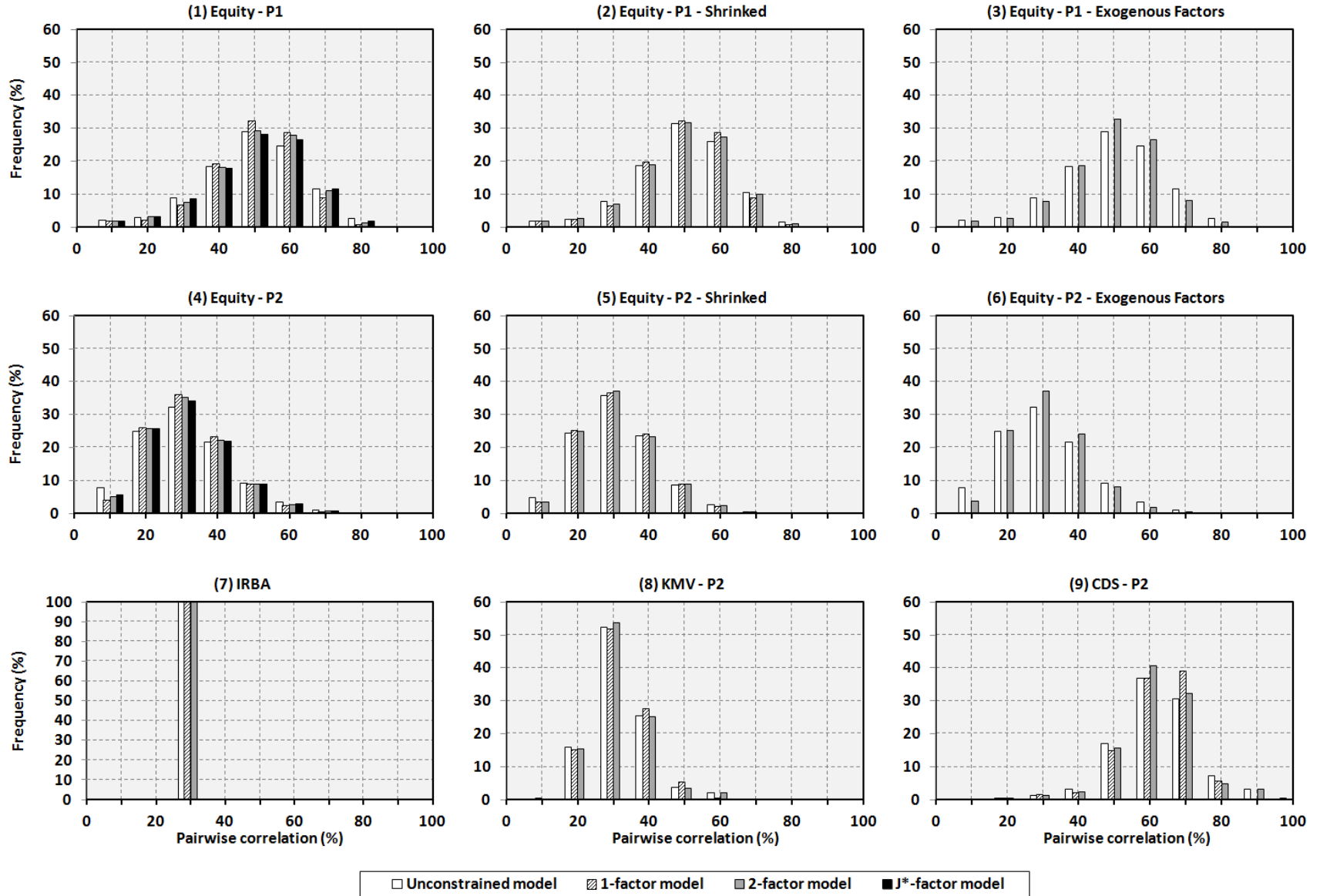


Figure (2) presents important disparities on average level and dispersion in pairwise correlations among the configurations, reflecting discrepancies in the data used to calibrate the correlation matrices. For instance, the “(1) Equity – P1” shows large dispersion and high average levels (approximately 50%) whereas the “(4) Equity – P2” configuration shows more concentrated frequencies with a peak around 30%, much lower than in the stress period 1. The shrinkage seems to have small effect on the level of the pairwise correlations but slightly decreases disparities among the models. The “(7) IRBA” configuration yields concentrated correlation levels quite close to the upper bound of 24%. The “(9) CDS – P2” configuration somehow presents the most dispersed distribution of pairwise correlation. In this case, factor models tend to overestimate central correlations and underestimate tail correlations (this is also true for other configurations but to a lesser extent).

Concerning the factor models, they accurately reproduce the underlying pairwise correlations distribution. In particular, combining Figure (2) and Table (2), the less dispersed the distribution of pairwise correlations, the fewer the number of factors needed to correctly reproduce the correlation structure.

4.2. Correlation impacts on regulatory VaR

In this Subsection, we analyze the impacts of correlation matrices on portfolio risk. The VaR is computed via the Monte-Carlo method with 2×10^7 scenarios. Note that the discreteness³¹ of L implies that the mapping $\alpha \mapsto VaR_\alpha[L]$ is piecewise constant so that jumps in the risk measure are possible for small changes in the default probability.

For both the *diversification portfolio* (cf. Figure (3)) and the *hedge portfolio* (cf. Figure (4)), we simulate the $VaR_\alpha[L]$ for $\alpha \in \{0.99, 0.995, 0.999\}$ for each of the nine configurations and the unconstrained and factor-based models.

³¹Since we deal with discrete distributions, we cannot rely on standard asymptotic properties of sample quantiles. At discontinuity points of VaR, sample quantiles do not converge. This can be solved considering the asymptotic framework introduced by Ma, Genton and Parzen (2011) [41] and the use of the mid-distribution function.

Figure 3: Risk measure as a function of α for the *diversification portfolio* (PCA-based calibration).

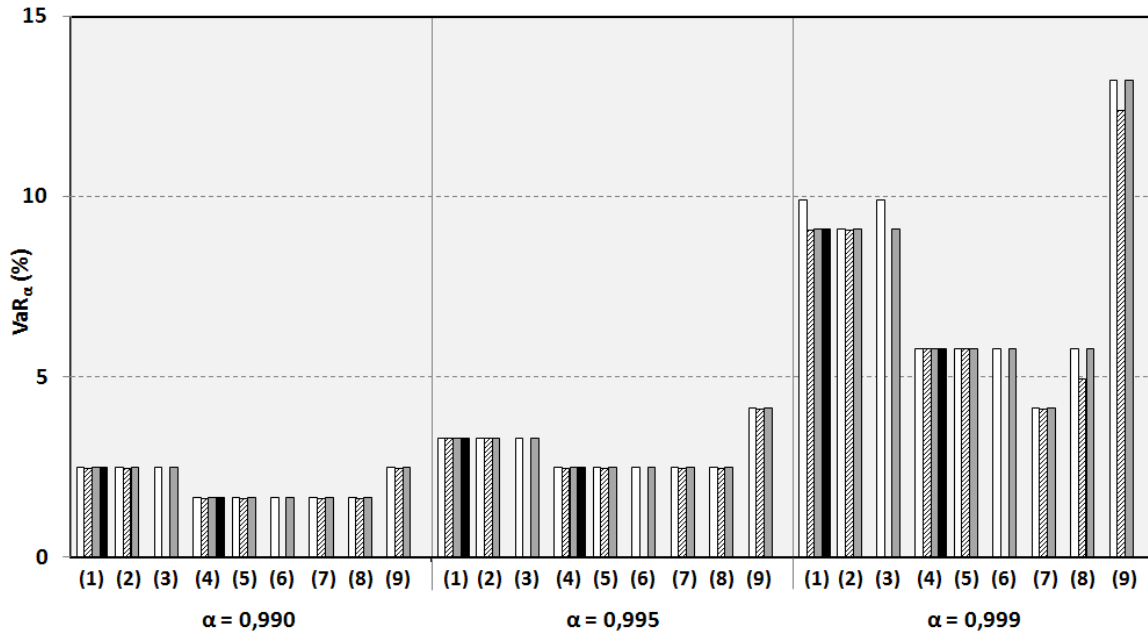
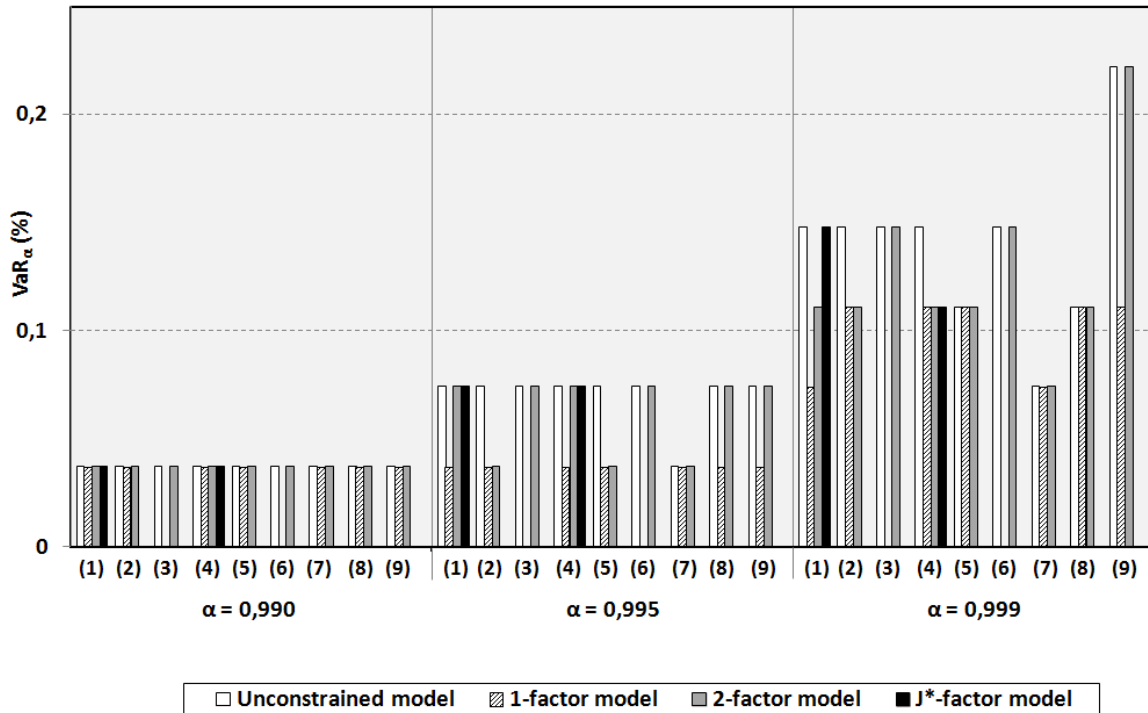


Figure 4: Risk measure as a function of α for the *hedge portfolio* (PCA-based calibration).



Configurations: (1) Equity - P1; (2) Equity - P1 - Shrunked; (3) Equity - P1 - Exogenous Factors; (4) Equity - P2; (5) Equity - P2 - Shrunked; (6) Equity - P2 - Exogenous Factors; (7) IRBA; (8) KMV - P2; (9) CDS - P2. J^* -factor model is only active for “(1) Equity – P1” and “(2) Equity – P2” configurations.

Figure (3) and Figure (4) provide guidances on the relevance of the Committee’s prescribed model and calibration procedure for reducing the RWAs variability and improving comparability between financial institutions. These numerical simulations clearly show that the VaR variability among configurations crucially depends to the level of α : increasing the confidence level results in higher variability.

In addition, for each level of α , the main source of VaR dispersion are the differences of average correlation levels among configurations. For instance, considering the regulatory confidence level ($\alpha = 0.999$), the prescribed “(1) Equity - P1” and “(9) CDS - P2” configurations lead to considerable VaR level differences.

With these equally weighted portfolios, the constrained J -factor models (including the regulatory two-factor model) tend to produce lower tail risk measures (high level of α) than the unconstrained model. This phenomenon is even more pronounced when considering dispersed correlation matrices (such as the configurations “(1) Equity - P1” and “(9) CDS – P2”) and particularly in the hedge portfolio where constrained models (generating less dispersed correlation matrices) may provoke a substantial risk mitigation.

Overall, based on our numerical simulations, we see that the principal sources of DRC variability are (i) the high confidence level of the regulatory risk measure and (ii) the differences in average correlations among configurations. Moreover, the two-factor constraint seems to be of low interest for reducing the DRC variability and tends to underestimate risk measures in comparison with the unconstrained model.

4.3. Systematic and idiosyncratic contributions to regulatory VaR

In this Subsection, we use the Hoeffding-based representation of the loss (Equations (8) and (9)) to provide insights on the contributions of the systematic factors to the overall portfolio risk and to assess the quality of the Large Pool Approximation.

We note $\phi_S^{(n)}$ the realization of the projected loss onto the subset of factors S on the scenario n . Given the pre-calculated risk measure $v = VaR_\alpha[L]$, the contribution³² estimator is:

$$C_{\phi_S}^{VaR}[L, \alpha] = \mathbb{E}[\phi_S | L = v] \implies \hat{C}_{\phi_S}^{VaR}[L, \alpha] = \frac{\sum_{n=1}^{MC} \phi_S^{(n)} \mathbf{1}_{\{L^{(n)}=v\}}}{\sum_{n=1}^{MC} \mathbf{1}_{\{L^{(n)}=v\}}} \quad (13)$$

Since the conditional expectation defining the risk contribution is conditioned on rare events, this estimator requires intensive simulations to reach an acceptable confidence interval³³. Tasche (2009) [54] and Glasserman 2005 [30] have already addressed the issue of computing credit risk contributions of individual exposures or sub-portfolios from numerical simulations. Our framework is similar to theirs, except that we focus on the contributions of the different terms involved in the Hoeffding decomposition of the aggregate risk. We are thus able to derive contribution of factors, idiosyncratic risks and interaction³⁴.

For both the *diversification portfolio* (cf. Figure (5)) and the *hedge portfolio* (cf. Figure (6)), we illustrate the influence of α on the systematic contribution to the risk by considering $\hat{C}_{\phi_1}^{VaR}[L, \alpha]/v$ with $\alpha \in \{0.99, 0.995, 0.999\}$ for each of the nine configurations and the unconstrained and factor-based models.

³²Remark that given the discrete nature of considered distributions, and similarly of the risk measure, the mapping $\alpha \mapsto C_{\phi_S}^{VaR}[L, \alpha]$ is piecewise constant. Note also that negative risk contributions may arise within the *hedge portfolio*.

³³Numerical experiments demonstrate that complex correlation structures (such as in the “Equity – P1” configuration) may induce noisy contribution estimation. This phenomenon is even more pronounced in the presence of a large number of loss combinations which implies frequent changes in value for the mapping $\alpha \mapsto \hat{C}_{\phi_S}^{VaR}[L, \alpha]/v$. Indeed, for a given loss level, there may only be a few simulated scenarios such that $L^{(n)} = v$, causing more volatile estimators.

³⁴While we could think of various optimized Monte Carlo simulation methods, we have not implemented any in the current version of the article. Yet, several papers are worth noticing and could represent a basis for extensions of our work. Glasserman (2005) [30] develops efficient methods based on importance sampling, though not directly applicable to Hoeffding representation of the discrete loss. Martin, Thompson and Browne (2001) [43] pioneer the Saddle Point Approximation of the unconditional Moment Generating Function (MGF) for the calculation of VaR and VaR contributions. Huang et al. (2007) [34] computes risk measures (VaR and ES) and contributions with the saddle point method applied to the conditional MGF, while Huang et al. (2007) [33] presents a comparative study for the calculation of VaR and VaR contributions with the Saddle Point method, the Importance Sampling method and the Normal Approximation (i.e., ASRF) method. Takano and Hashiba (2008) [51] proposes to calculate marginal contributions using a numerical Laplace transform inversion of the MGF. Recently, Masdemont and Ortiz-Gracia (2014) [44] applies a Fast Expansion Wavelet Approximation to the unconditional MGF for the calculation of VaR, ES and contributions, through numerically optimized techniques.

Figure 5: Systematic contribution as a function of α for the *diversification portfolio* (PCA-based calibration).

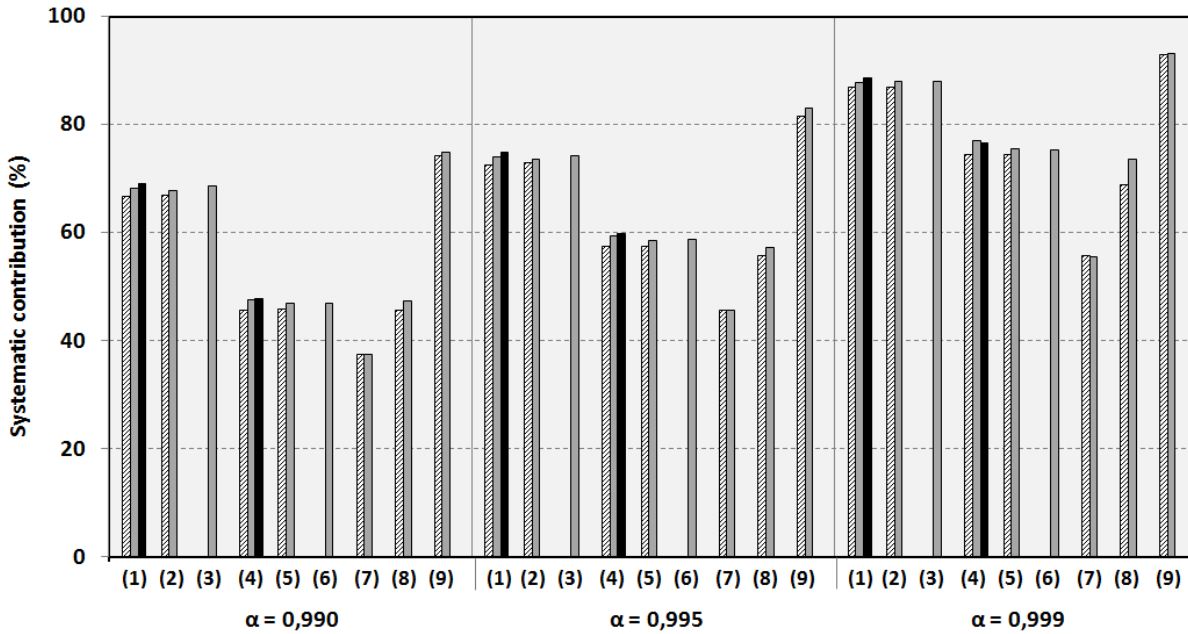
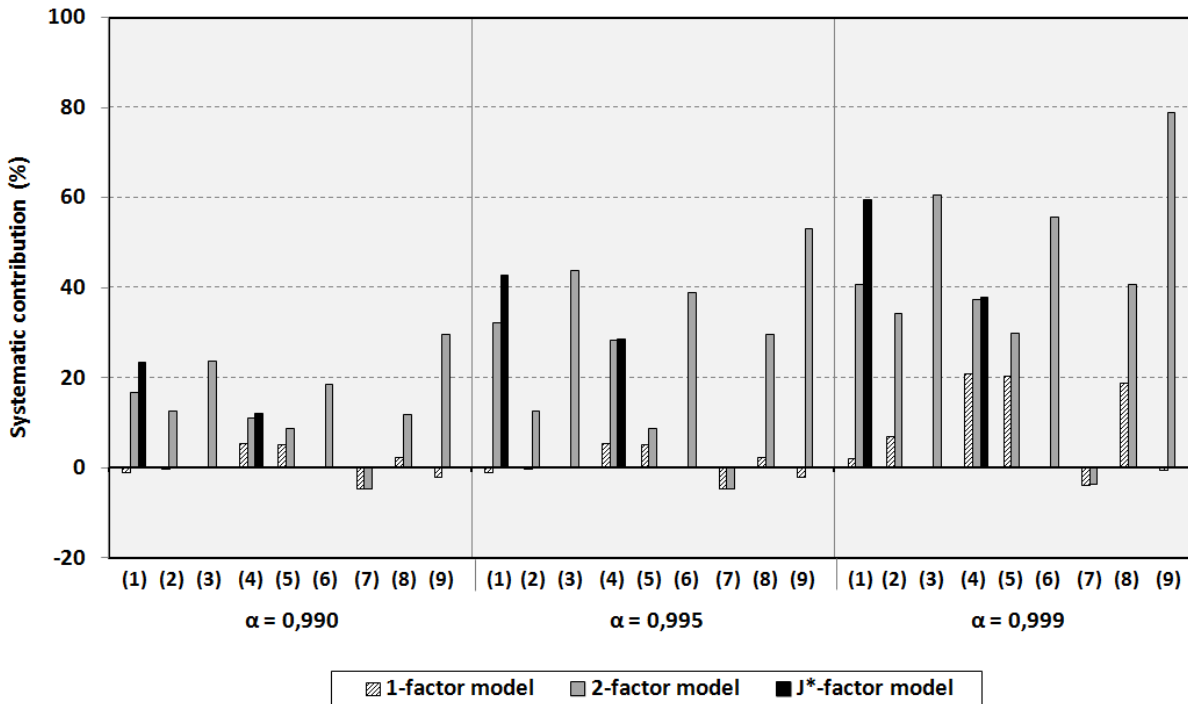


Figure 6: Systematic contribution as a function of α for the *hedge portfolio* (PCA-based calibration).



Configurations: (1) Equity - P1; (2) Equity - P1 - Shrunked; (3) Equity - P1 - Exogenous Factors; (4) Equity - P2; (5) Equity - P2 - Shrunked; (6) Equity - P2 - Exogenous Factors; (7) IRBA; (8) KMV - P2; (9) CDS - P2. J^* -factor model is only active for “(1) Equity – P1” and “(2) Equity – P2” configurations.

In the *diversification portfolio*, regarding the systematic risk as a function of α (cf. Figure (5)), we observe a high level of systematic contribution to the overall risk for configurations with a high level of average pairwise correlations (cf. Figure (2)). Moreover, for all configurations, since extreme values of the systematic factors lead to the simultaneous default of numerous dependent issuers, $C_{\phi_S}^{VaR}[L, \alpha]$ is an increasing function of α (also true for the *hedge portfolio*).

In the *hedge portfolio*, regarding the systematic risk as a function of α (cf. Figure (6)), we observe much smaller levels of systematic contributions than in the *diversification portfolio*. Strikingly, the one-factor and two-factor approximations may be inoperable and misleading: the majority of the risk being explained by the other terms of the Hoeffding decomposition.

Overall, the risk contribution analysis provides informative conclusions on the modeling assumptions. Based on our numerical simulations, the conditional approach (the Large Pool Approximation) used for the banking book should not be transposed for the trading book where typical long-short portfolios may contain significant non-systematic risk components.

5. Conclusions

Assessment of default risk in the trading book is a key point in the Fundamental Review of the Trading Book. Within the current Committee's approach, the dependence structure of defaults has to be modeled through a systematic factor model with constraints on (i) the calibration data of the initial correlation matrix and (ii) on the number of factors in the underlying correlation structure.

Based on representative long-only and long-short portfolios, this paper has considered the practical implications of such modeling constraints for both the future DRC prescription and the current Basel 2.5 IRC built on constrained and unconstrained factor models. Various correlation structures have been considered. Based on a structural-type credit model, we assessed the impacts on the VaR (at various confidence levels) of the calibration data as well as those of the estimation method of the correlation structures and the chosen number of factors. Hoeffding decomposition of portfolio exposures to factor and specific risks has been introduced to quantify systematic and specific contributions to the credit VaR.

The comparative analysis of risk factors modeling allows us to gauge the relevance of the Committee's proposals of prescribing model and calibration procedure, to reduce the RWA variability and to increase comparability among financial institutions. The key insights of our empirical analysis are the following:

- The main source of DRC variability is the high confidence level of the regulatory risk measure. As expected, $\alpha = 0.999$ gives rise to significant discrepancies among configurations (i.e. calibration data) and among the constrained models we tested. It is noteworthy that for $\alpha = 0.99$, risk measures are less dispersed and therefore less prone to model risk. Using this confidence level might be counterbalanced by a multiplier adjustment factor³⁵. We think this should be accounted for in future benchmarking exercises.
- Another important source of DRC variability are the disparities among correlation matrices. This may be due to the type of data (Equity returns, CDS spread returns,...) and/or the calibration period. Therefore, within the current Committee's approach, financial institutions could be led to take arbitrary choices regarding the calibration of the default correlation structure. This may then cause an ill-favored variability in the DRC making the comparison among institutions harder. Apart from disclosure of the data retained for correlation estimates, one possibility could be to use IRB type regulatory prescribed correlations as in the banking book approach. A possibility would be a disclosure of RWA computed with such prescribed correlations and would mitigate the issue of moving assets across the trading/booking books boundary.
- The strength of the two-factor constraint depends on the smoothness of the pairwise correlations frequencies in the initial correlation matrix: the more dispersed the underlying correlation structure, the greater the number of factors needed to approximate it. On the contrary, the estimation methods for both the initial correlation (standard or shrunk estimators) and the factor-based correlation matrices (SPG-based or PCA-based algorithms) have smaller effects, at least on the *diversification portfolio* (long-only exposures).

³⁵In the earlier charges of the Basel II agreements, a confidence level of $\alpha = 0.995$ has been envisaged. But, then a confidence level of $\alpha = 0.999$ was retained. The motivation was that Tier 1 was most restricted to hard capital, which is no longer the case in the Basel III agreements.

- The impact of the correlation structure on the risk measure mainly depends on the composition of the portfolio (long-only or long-short). For the particular case of a *diversification portfolio* (long-only exposures) with a smooth correlation structure (e.g. estimated on non-stressed equity returns), constrained factor models (mostly when considering at least two factors) and unconstrained model produce almost similar risk measure. For the specific case of a *hedge portfolio* (long-short exposures) for which widely dispersed pairwise equity or CDS-spread correlations and far tail risks (99.9%-VaR) are jointly considered, cliff effects arise from discreteness of loss: small changes in exposures or other parameters (default probabilities) may lead to significant changes in the credit VaR, jeopardizing the comparability of RWA.

Overall, the usefulness of the two-factor constraint can be challenged:

- In our case study, it drives down the VaR. Moreover, it is unclear whether it would enhance model comparability and reduce the RWAs variability.
- On the other hand, the Committee's prescriptions might prove quite useful when dealing with a large number of assets. In such a framework, reasonably standard for large financial institutions with active credit trading activities, the unconstrained empirical correlation matrix would be associated with zero or small eigenvalues. This would ease the building of opportunistic portfolios, seemingly with low risk and would jeopardize the reliance on internal models.

Eventually, when it comes to long/short portfolios, the use of Large Pool Approximation is questionable, since it contributes poorly to the overall credit VaR.

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