Trading book and credit risk : how fundamental is the Basel review ?

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Regulatory capital requirement

- **Minimum Capital Requirement in the Basel Framework**
  - Based on the concept of Risk Weighted Assets. BCBS (2004) [1]
  - RWA: bank's asset exposure, weighted by its risk.

$$\text{Minimum Required Capital} = X\% \times \text{RWA} \tag{1}$$

- **Trading book vs. Banking book**
  - Trading book: regroups actively traded assets.
  - Banking book: regroups MT & LT transactions, kept until maturity.

⇒ Proposals for distinction between the two portfolios in the FRTB. BCBS (2013) [2]

- **RWA for the Banking and the Trading books**

$$\text{RWA} = \text{RWA}_{\text{Banking book}} + \text{RWA}_{\text{Trading book}} \tag{2}$$

  - $\text{RWA}_{\text{Banking book}}$: focused on credit risk.
  - $\text{RWA}_{\text{Trading book}}$: essentially focused on market risk (but also includes an incremental capital charge for credit risk).

Basel framework for credit risk capital charge

- **In the Banking book**
  - Basel II (2004): 3 available approaches. BCBS (2004) [1]
    - 1 standard approach.
    - 2 internal-model-based approaches (*Internal Rating Based*)
      ⇒ IRB-Advanced (IRBA): banks calibrate the model parameters: PD, LGD, EAD.
  - Prescribed model for default risk: the *Asymptotic Single Risk Factor Model* (ASFR).
  - Correlation matrix is constrained (prescribed function of PDs).
- **In the Trading book**
  - Before the 2008-2009 crisis, the credit risk was not monitored in the Trading book.
  - Basel 2.5 & Basel III: *Incremental Risk Charge* (IRC) for default and migration risks. BCBS (2009) [3]
  - Initially created for credit derivatives . . . but also impacts bond portfolios.
  - Based on internal models (often multi-factor models): no prescribed model.

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Risk weighted assets variability - 1/2

- **RWA comparison?**

- RCAP: *Regulatory Consistency Assessment Program*
- For the Banking & the Trading books.

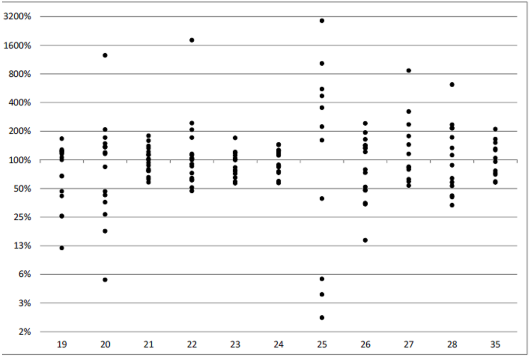
⇒ High variability between financial institutions and jurisdictions.

- **$\text{RWA}_{\text{Trading book}}$ variability**

- RWA analysis in the Trading book: RCAP1 & RCAP2. BCBS (2013) [4]
- Internal models in cause . . . especially for the IRC calculation (cf. next slide).
- IRC main variability sources:
  - Overall modelling approach;
  - Probability of Default calibration;
  - Correlation assumptions.

Risk weighted assets variability - 2/2

Dispersion of normalised IRC results for credit spread portfolios



Portfolio	Description
P19	Sovereign CDS portfolio
P20	Sovereign bond/CDS portfolio
P21	Sector concentration portfolio
P22	Diversified index portfolio
P23	Diversified index portfolio (higher concentration)
P24	Diversified corporate portfolio
P25	Index basis trade on iTraxx 5-year Europe index
P26	CDS bond basis on 5 financials
P27	Short index put on iTraxx Europe Crossover
P28	Quanto CDS on Spain with delta hedge
P35	All-in portfolio comprising portfolios P19–P28

Note: Normalisation is defined as dividing it by its median;
The vertical axis in each panel is a base 2 log scale.

Source: RCAP 2. BCBS (2013) [4]

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FRTB - Main propositions

- **Trading book - Banking book boundary**
  - Evidence-based approach.
- **Standardized approach**
  - Greater recognition of hedges and diversification benefits.
- **Internal models approach**
  - Approval at the desk-level.
  - All banks that have received internal approval would have to use the Expected Shortfall approach to calculate their market risk requirement measured at 97,5% confidence level and calibrated to a period of significant financial stress.
  - Credit exposure would be subject to a stand-alone model using a Incremental Default Risk Charge (IDR). The credit spread risk charge for migration risk will be modelled as part of the total capital charge within the ES measure.

Fundamental Review of the Trading book and IDR

- **Replace the IRC (default and migration risk) by a IDR charge (default risk only).**
  - *Incremental Default Risk* (IDR) charge.
  - May be seen as an IRC charge with deactivated migration feature.

- **Incremental Default Risk charge** BCBS (2012-2013) [5]

  " To maintain consistency with the banking book treatment, the Committee has decided to propose an incremental capital charge for default risk based on a VaR calculation using a one-year time horizon and calibrated to a 99.9th percentile confidence level (consistent with the holding period and confidence level in the banking book)".

- **Prescribed benchmark model**

  " The Committee has decided to develop a more prescriptive IDR charge in the models-based framework. Banks using the internal model approach to calculate a default risk charge must use a two-factor default simulation model, which the Committee believes will reduce variation in market risk-weighted assets but be sufficiently risk sensitive as compared to multifactor models."

Questions - Problematics

- **Impact of factor models on the risk? Impact of the factor number?**
- **Two-factor model? What model?**
- **Calibration parameters and methods? Impacts on the risk?**

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Portfolio Loss

- **One period portfolio loss**

$$L = \sum_k EAD_k \times LGD_k \times \text{DefaultIndicator}_k \qquad (3)$$

- EAD_k and LGD_k are supposed to be constant.
- Positions may be loans (Banking book), CDS or bonds (Trading book).

- **Diversification or hedge portfolio**

- EAD_k may be long (sign +) or short (sign -).
- Long portfolio (= diversification portfolio), long-short portfolio (= hedge portfolio).
- The Trading book often contains long-only and long-short portfolios.

- **Discrete or continuous Loss distribution?**

- Depends on the modelling assumptions on $\text{DefaultIndicator}_k$
- Model for $\text{DefaultIndicator}_k$? (cf. next slide)

Portfolio credit risk models: $\text{DefaultIndicator}_k$

- **Model for** $\text{DefaultIndicator}_k$ **?**
- **Latent variable models**

- Default occurs if a latent variable, X_k, lies below a threshold.

$$\text{DefaultIndicator}_k = 1_{\{X_k \leq \text{threshold}_k\}} \tag{4}$$

- Asset value of the obligor k:

$$X_k = \beta_k Z + \sqrt{1 - \beta_k' \beta_k} \epsilon_k \tag{5}$$

  - $Z \in \mathbb{R}^J$; $Z_j \sim N(0,1)$: systematic factor (sectors, regions ...).
  - $\epsilon_k \sim N(0,1)$: idiosyncratic factors.
  - $\beta \in \mathbb{R}^{K,J}$: systematic factor loadings.

- $\text{threshold}_k = \Phi^{-1}(p_k)$ where p_k is the probability of default of the obligor k and Φ the standard normal cdf.

- MERTON (1974) [6] ,BCBS (IRB) (2004) [1], ROSEN & SAUNDERS (2010) [7].

Asymptotic Single Risk Factor - Banking book (Basel 2)

- **Example of latent variable model: the IRBA prescribed 1-factor model**
  - 1 systematic factor Z: $\beta \in \mathbb{R}^{K,1}$.
  - Credit state of the obligor k: $X_k = \beta_k Z + \sqrt{1-\beta_k^2}\epsilon_k$
  - Portfolio loss:

$$L = \sum_k EAD_k \times LGD_k \times 1_{\{\beta_k Z + \sqrt{1-\beta_k^2}\epsilon_k \leq \Phi^{-1}(p_k)\}} \quad (6)$$

$\Rightarrow$ L is a discrete random variable.

  - Systematic factor conditioning (Large Pool Approximation).

$$L_Z = \mathbb{E}[L|Z] = \sum_{k=1} EAD_k \times LGD_k \times \Phi\left(\frac{\Phi^{-1}(p_k) - \beta_k Z}{\sqrt{1-\beta_k^2}}\right) \quad (7)$$

$\Rightarrow$ L_Z is a continuous random variable.

- **Portfolio invariance property**
  - Homogeneous portfolio assumption: $EAD_k = EAD, p_k = p$. WILDE (2001) [8]
  - The capital required for any given loan does not depend on the portfolio it is added to.
  - Additive capital requirement (no diversification).

Distinction between Trading book and Banking book

- **book positions**
  - The Banking book positions:
    - are supposed to be held until maturity.
    - are often long credit risk.
    - are often enough, and nearly homogeneous, to make the Large Pool Approximation a good one (up to a granularity adjustment).
  - The Trading book positions:
    - are actively traded.
    - are long or short credit risk.
    - are inhomogeneous and may be few.
- **Impact on the modelling**
  - Large pool assumption seems too restrictive to be applied.

  ⇒ The loss distribution is discrete.

  - Need to take into account systematic risk and specific risk.

Incremental Default Risk Charge in the Trading book

- **Prescribed two-factor model** BCBS (2012-2013) [5]

  "Banks must use a two-factor default simulation model with default correlations based on listed equity prices."

- **What type of factor model?**

- Banking book uses latent variables as underlying default process.
- It is also a standard approach for internal IRC models, validated by the regulators.
- Financial institution often uses this modeling.

- **What two-factor model really means?**

- 1 global systematic risk factor and 1 specific risk factor? (like in the Banking book?)
- 2 systematic global risk factors Z_1 and Z_2? Interpretation (cf. next slide)?
- 1 sector systematic risk factor and 1 specific risk factor?
- 1 geographical systematic risk factor and 1 specific risk factor?
- ...

⇒ This specification may have important impacts on the factor interpretation, on the correlation structure and/or on the portfolio risk.

Factor interpretation

- **1-factor model: IRB model**
  - The systematic factor is interpreted as the state of the economy.
  - It may be interpreted as a generic macroeconomic variable affecting all firms.

  $\Rightarrow$ The interpretation seems clear.

- **J-factor models**
  - We may use latent-factors models or macroeconomic-variable-based models.
  - We may refer factors to sectors, regions . . .
  - We may postulate detailed correlation structure between factors.
  - For instance, we may use inter and intra correlations between factors.

  $\Rightarrow$ The interpretation seems straightforward.

- **2-factor models**
  - With only two factors, the sectors or regions segmentation seems poor.

  $\Rightarrow$ The interpretation is not clear.

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Correlation calibration - 1/3

- **Prescribed two-factor model** BCBS (2012-2013) [5]

  "Banks must use a two-factor default simulation model with default correlations based on listed equity prices."

- In the previous latent variable model, the correlation matrix, $C(\beta)$ between the X_k is:

$$C(\beta) = Correlation(X_k, X_l)_{k,l=1,\dots,K} = \beta\beta^t + diag(Id - \beta\beta^t) \tag{8}$$

- **Constrained correlation estimation parameters.**

  "Default correlations must be based on listed equity prices and must be estimated over a one-year time horizon (based on a period of stress) using a [250] day liquidity horizon."

  "These correlations should be based on objective data and not chosen in an opportunistic way where a higher correlation is used for portfolios with a mix of long and short positions and a low correlation used for portfolios with long only exposures."

- **From those recommendations, how to estimate the betas?**

Correlation calibration - 2/3

- **Assumptions**
  - We postulate a two-factor model with two systematic risk factors (without interpretation) that impact all obligors.
  - Other correlation structures, induced by differences in factor models, may be calibrated by adding appropriate constraints in the optimization problem.
- **Objective**
  - Finding a two-factor model producing a correlation matrix closed to a pre-determined correlation matrix C_0 (computed from historical stock prices for instance).
  - Formally, we look for a two-factor modelled X_k

$$X_k = \beta_k Z + \sqrt{1 - \beta_k' \beta_k} \epsilon_k \text{ with } \beta \in \mathbb{R}^{K \times 2} \; Z \in \mathbb{R}^2 \text{ and } \epsilon \in \mathbb{R}^K \tag{9}$$

  - With correlation structure, $C(\beta)$, induced by the β matrix as closed as possible to C_0.

Correlation calibration - 3/3

- **Optimization problem**

$$\min_{\beta} f_{obj}(\beta) = \|C(\beta) - C_0\|_F \text{ subject to } \beta \in \Omega \quad (10)$$

- We recall that $C(\beta) = \beta\beta^t + diag(Id - \beta\beta^t)$.
- $\Omega = \{\beta \in \mathbb{R}^{K\times 2} | \beta_k^{'}\beta_k \leq 1, k = 1, \dots, K\}$ is a closed, convex set.
- Constraint ensures that $\beta\beta^{'}$ has diagonal elements bounded by 1 that implies that $C(\beta)$ is positive semi-definite.
- The solution is also known as the *nearest correlation matrix with two-factor structure*.
- Gradient: $\nabla f_{obj}(\beta) = 4\left(\beta(\beta^t\beta) - C_0\beta + \beta - diag(\beta\beta^t)\beta\right)$

Nearest correlation matrix with two-factor structure

- **Optimization methods**
  - Comparative study: BORSDORF, HIGHAM & RAYDAN (2010) [9]
  - Financial applications: GLASSERMAN & SUCHINTABANDID (2007) [10], and in JACKEL (2004) [11].
- **Principal Factors Method.** ANDERSEN et al. (2003) [12]
  - Already used for financial applications (credit basket securities).
  - Ignores the non-linear problem constraints.
  - Not supported by any convergence theory.
- **Spectral Projected Gradient Method.** BIRGIN et. al (2000) [13]
  - Has guaranted convergence.
  - cf. next slide.

Spectral Projected Gradient Method

- **Spectral Projected Gradient Method**
  - To minimize $f_{obj} : \mathbb{R}^k \in \mathbb{R}$ over a convex set Ω

$$\beta_{i+1} = \beta_i + \alpha_i d_i \tag{11}$$

  - $d_i = Proj_\Omega \left(\beta_i - \lambda_i \nabla f_{obj}(\beta_i)\right) - \beta_i$ is the descent direction, with $\lambda_i > 0$ a precomputed scalar.
  - $\alpha_i \in [-1, 1]$ chosen through non-monotone line search strategy.
- **Advantages**
  - Solve the full constrained problem and generates a sequence of matrices that is guaranted to converge to a stationnary point of Ω.
  - $Proj_\Omega$ is cheap to compute.
  - Fast and easy to implement.
  - Algorithm available in BIRGIN et. al (2001) [14]
- **Norm choice**
  - Common Froebenius norm: $\forall A \in \mathbb{R}^{K \times K} : \|A\|_F =< A, A >^{1/2}$ where $< A, B >= tr(B^t A)$. (Impact of the norm has not yet been studied).

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Loss - Discrete vs continuous distribution

- **Context**
  - We do not use the large pool approximation since, in the Banking book, $\mathbb{E}[L|Z]$ is not, in general, a gool approximation of L (importance of specific risks $\approx$ lack of granularity).
  - Discrete loss distribution may not be convenient for simulations and for defining and calculating marginal contribution to the risk.
  - Asymptotic distribution of sample quantiles is normal for absolutely continuous distribution. However, it is not longer true for discrete distributions.

  $\Rightarrow$ Alternatives? Solutions?

- **Kernel smoothing**

- **Definition of sample quantiles based on mid-distribution function**
  - Provide an unified framework for asymptotic properties of sample quantiles from absolutely continuous and from discrete distributions.
  - Exposed by MA et. al (2011) [15]

- **What impacts on the risk?**

Loss - The Hoeffding decomposition

- **Context**

- Portfolio loss: $L = \sum_k EAD_k \times LGD_k \times 1_{\{\beta_k Z + \sqrt{1-\beta_k^2}\epsilon_k \leq \Phi^{-1}(p_k)\}}$
- We want to analyse the contribution of the systematic factor to the risk.
- We want to be able to dissociate between specific and systematic risk.

$\Rightarrow$ Hoeffding decomposition of the loss.

- **Hoeffding decomposition** HOEFFDING (1948) [16]

- Consider $F_1, \ldots, F_M$ independent r.v such that $\forall m \in \{1, \ldots, M\}$: $\mathbb{E}\left[F_m^2\right] \leq +\infty$.
- Consider $L(F_1, \ldots, F_M)$ such that $\mathbb{E}\left[L^2\right] \leq +\infty$.
- The Hoeffding decomposition gives a unique way of writing L as a sum of uncorrelated terms involving conditional expectations of f_F given sets of the factors F.

$$\begin{aligned} L &= \sum_{S \subseteq \{1,\ldots,M\}} \Phi_S(L; F_m, m \in S) \\ &= \sum_{S \subseteq \{1,\ldots,M\}} \sum_{\tilde{S} \subseteq S} (-1)^{|S|-|\tilde{S}|} \, \mathbb{E}\left[L | F_m; m \in \tilde{S}\right] \end{aligned} \tag{12}$$

- Very flexible since we may decompose L on any subset of the M variables.
- Exhaustive presentation in: VAN DER VAART (2000) [17]

Loss - The Hoeffding decomposition and dependence

- **Dependence**
  - The Hoeffding decomposition is usually applied to independent factors.
  - The general decomposition formula is still valid for dependent factors.

  ⇒ Hoeffding decomposition for dependent macroeconomic variables is possible. In this case, each term depends on the joint distribution of the factors.

- **Example**
  - Consider $L_Z = w_1 Z_1 + w_2 Z_2$
  - (Z_1, Z_2) have a joint normal distribution with $N(0,1)$ marginals and correlation ρ.
  - Hoeffding decomposition:

$$\begin{aligned} L_Z &= \phi_\emptyset(L_Z) + \phi_1(L_Z; Z_1) + \phi_2(L_Z; Z_2) + \phi_{1,2}(L_Z; Z_1, Z_2) \\ &= (w_1 + w_2\rho)Z_1 + (w_1\rho + w_2)Z_2 - \rho(w_2 Z_1 + w_1 Z_2) \end{aligned} \tag{13}$$

  - ρ impacts the contribution to the risk of Z_1, Z_2 and their interactions.

- **Our setting**
  - We have assumed that the systematics factors and the idyosincratic factors are independent.

  ⇒ The Hoeffding elements are independents.

Loss - Specific-systematic factors decomposition

- **Portfolio Loss**: $L = \sum_k EAD_k \times LGD_k \times 1_{\{X_k \leq \Phi^{-1}(p_k)\}}$

$$\begin{aligned} L &= \phi_\emptyset(L) + \phi_1(L;Z) + \phi_2(L;\epsilon) + \phi_{1,2}(L;Z,\epsilon) \\ &= \text{"Average Loss"} + \text{"Systematic Loss"} + \text{"Specific Loss"} + \text{"Interaction Loss"} \end{aligned}$$

$\Rightarrow$ Hoeffding decomposition of the Loss. ROSEN & SAUNDERS (2010) [7]

- **Systematic loss**

$$\begin{aligned} \phi_1(L;Z) &= \mathbb{E}[L|Z] - \mathbb{E}[L] \\ &= \sum_k EAD_k \times LGD_k \times \Phi\left(\frac{\Phi^{-1}(p_k) - \beta_k Z}{\sqrt{1-\beta_k'\beta_k}}\right) - \mathbb{E}[L] \qquad (14) \end{aligned}$$

- Corresponds (up to the expected loss term) to the heterogeneous large pool approximation (or asymptotic framework in regulatory terminology.)

- **Specific loss**

$$\begin{aligned} \phi_2(L;\epsilon) &= \mathbb{E}[L|\epsilon] - \mathbb{E}[L] \\ &= \sum_k EAD_k \times LGD_k \times \Phi\left(\frac{\Phi^{-1}(p_k) - \sqrt{1-\beta_k'\beta_k}\epsilon_k}{\sqrt{\beta_k'\beta_k}}\right) - \mathbb{E}[L] \qquad (15) \end{aligned}$$

Loss - Systematic factors decomposition

- **Systematic factors decomposition**
  - We may also consider systematic risks only: $L_Z = \mathbb{E}[L|Z_1, \ldots, Z_J]$.
  - This hypothesis is used for large and homogeneous portfolios (ASRF model): $L_Z \approx L$
  - This assumption allows us to decompose the loss in terms of systematic factors.
- **Example with 2 systematic factors**
  - The loss is a function of Z_1 and Z_2.

$$\begin{aligned} L_Z &= \phi_\emptyset(L_Z) + \phi_1(L_Z; Z_1) + \phi_2(L_Z; Z_2) + \phi_{1,2}(L_Z; Z_1, Z_2) \\ &= \text{"Average Loss"} + \text{"Factor 1 Loss"} + \text{"Factor 2 Loss"} + \text{"Interaction Loss"} \end{aligned}$$

  - with $j \in 1, 2$:

$$\begin{aligned} \phi_i(L_Z; Z_j) &= \mathbb{E}\left[L_Z|Z_j\right] - \mathbb{E}\left[L_Z\right] \\ &= \sum_k EAD_k \times LGD_k \times \Phi\left(\frac{\Phi^{-1}(p_k) - \beta_{k,j} Z_j}{\sqrt{1 - \beta_{k,j}^2}}\right) - \mathbb{E}\left[L_Z\right] \quad (16) \end{aligned}$$

Portfolio risk

- **Risk measure**
  - A risk measure is defined as: $\varrho : \mathbb{R} \ni L \to \varrho[L] \in \mathbb{R}$.
  - Positive homogeneous risk measure: $\forall \lambda \in \mathbb{R}^+$, $\varrho[\lambda L] = \lambda \varrho[L]$.
- **Common risk measures**
  - Value-at-Risk: $VaR_\alpha[L] = \inf\{l \in \mathbb{R} | \mathbb{P}(L \leq l) \geq \alpha\}$
  - Conditional Tail Expectation: $CTE_\alpha[L] = \mathbb{E}\left[L | L \geq VaR_\alpha[L]\right]$
- **Continuous vs discrete Loss distribution**
  - $F_L(l) = \mathbb{P}(L \leq l)$
  - For continuous loss distributions, $F_L^{-1}(l)$ exists. In particular: $VaR_\alpha[L] = F_L^{-1}(\alpha)$
  - For discrete loss distributions, there are only a finite number of possible realizations. For a fixed α, there may be no loss realization that matches α.

Factor contribution to portfolio risk - 1/2

- **Component (sub-portfolio, position...) contribution to the risk**

- Portfolio loss: $L = \sum_{k=1}^{K} L_k$
- Contribution of L_k to the VaR : $C_k[L] = \mathbb{E}[L_k | L = Var_\alpha[L]]$
- Link with marginal contribution (Euler allocation) TASCHE (2008) [18]. If L ad L_k have continuous joint probability density function, then:

$$C_k[L] = \lim_{\delta \to 0} \frac{VaR_\alpha(L + \delta L_k) - VaR_\alpha(L)}{\delta} \tag{17}$$

⇒ **Ongoing study: extension to the case where L and L_k have discrete distributions**

The equivalence between VaR derivative and conditional expectation should remain true except for certain discontinuity points.

- **Full allocation property**

$$VaR_\alpha(L) = \sum_k C_k[L] \tag{18}$$

⇒ This property is true for any decomposition of the loss as a sum of its components.

Factor contribution to portfolio risk - 2/2

- **Specific-systematic factor contribution to the risk**

$$\begin{aligned} L &= \phi_{\emptyset}(L) + \phi_1(L;Z) + \phi_2(L;\epsilon) + \phi_{1,2}(L;Z,\epsilon) \\ &= \text{"Average Loss"} + \text{"Systematic Loss"} + \text{"Specific Loss"} + \text{"Interaction Loss"} \end{aligned}$$

- Contribution of the "Systematic Loss" to the risk: $C_{\phi_1}(L) = \mathbb{E}\left[\phi_1(L;Z) | L = VaR_\alpha[L]\right]$

⇒ Link with marginal contribution ROSEN & SAUNDERS (2010) [7]. If L and $\phi_i(L;Z)$ have continuous joint probability density function, then:

$$C_{\phi_1}(L) = \lim_{\delta \to 0} \frac{VaR_\alpha(L + \delta\phi_1(L;Z)) - VaR_\alpha(L)}{\delta} \quad (19)$$

- **Full allocation**

$$\begin{aligned} VaR_\alpha(L) &= C_{\phi_{\emptyset}}(L) + C_{\phi_1}(L) + C_{\phi_2}(L) + C_{\phi_{1,2}}(L) \\ &= \text{"Average Loss"} + \text{"Syst. Contrib."} + \text{"Spec. Contrib."} + \text{"Cross Contrib."} \end{aligned}$$

Factor contribution interpretation

- **Latent (independent) risk-factor rotation**
  - The risk-factor rotations leave the risk measures invariant but impact the risk-factor contributions.
- **Example with Large Pool Approximation $L_Z = \mathbb{E}[L|Z]$**
  - The three following cases have the same risk but not the same factor contribution.
  - **Symmetry:** $\forall k,\ \beta_{k,1} = \beta_{k,2} = 0.2 \Rightarrow C_{\phi_1} = C_{\phi_2}$
  - **1 factor:** $\forall k,\ \beta_{k,1} = \sqrt{2 \times 0.2^2}$ and $\beta_{k,2} = 0 \Rightarrow C_{\phi_1} \neq C_{\phi_2}$
  - **Rotation:** $\forall k,\ \beta_{k,1} = 0$ and $\beta_{k,2} = \sqrt{2 \times 0.2^2} \Rightarrow C_{\phi_1} \neq C_{\phi_2}$

	Symmetry	1 factor	Rotation 1 factor
VaR99.9[L] (Notional=1)	0,010	0,010	0,010
Average Loss	9,6%	9,6%	9,6%
Systemic contribution	25,2%	90,4%	0,0%
Specific contribution	25,3%	0,0%	90,4%
Cross contribution	39,9%	0,0%	0,0%

Factor contribution interpretation - Rotation - 1/4

- **Context**
  - Let us consider the Large Pool Approximation: $L_Z = \mathbb{E}[L|Z_1, Z_2]$
  - The asset value of the obligors k is given by: $X_k = \beta_k Z + \sigma \epsilon_k$.
  - The risk measures are invariant by factor rotation since the law of the vector X is unchanged.
  - Factor contributions are not invariant by factor rotation.
- **Objective**
  - In order to interpret factors, we want to maximise the contribution of the first systematic factor, the second being assimilated to a systematic adjustment.

⇒ Goal: optimizing the contribution of the first factor.

  - Which factor rotation method? The usual Varimax criterion is not suited for such an optimisation.
- **Methods**
  - In the following slides, we consider 3 methods, which are portfolio-invariant.
  - The portfolio-invariance feature seems important to avoid re-calibration.

⇒ **Ongoing study : best criterion to maximise the contribution of ϕ_1, the systematic part, to $VaR_\alpha[L_Z]$.**

Factor contribution interpretation - Rotation - 2/4

- **Method 1 - Assumptions**
  - Consider R, a 2×2 rotation matrix:

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{20}$$

  - We may write: $X_k = \beta_k Z + \sigma \epsilon_k = (\beta_k R)(R^t Z) + \sigma \epsilon_k = \tilde{\beta}_k \tilde{Z} + \sigma \epsilon_k$.
- **Method 1 - Criterion**
  - The criterion is:

$$\arg\max_{\theta} C_{\phi_1}^{VaR}[L_Z; \alpha; \theta] \tag{21}$$

with:

$$\begin{aligned} C_{\phi_1}^{VaR}[L_Z; \alpha; \theta] &= \mathbb{E}\left[\sum_k EAD_k \times LGD_k \times \Phi\left(\frac{\Phi^{-1}(p_k) - \tilde{\beta}_{k,1}\tilde{Z}_1}{\sqrt{1-\tilde{\beta}_{k,1}^2}}\right) | L = VaR_\alpha[L]\right] - \mathbb{E}[L_Z] \\ \tilde{\beta}_k &= \cos(\theta)\beta_{k,1} + \sin(\theta)\beta_{k,2} \end{aligned} \tag{22}$$

Factor contribution interpretation - Rotation - 3/4

- **Method 2 - Setting**

- $\forall k$, consider: $B_k = \beta_{k,1} Z_1 + \beta_{k_2} Z_2$.
- The initial variance of B_k is given by: $Var_k^{init} = \beta_{k,1}^2 + \beta_{k,2}^2$

- **Method 2 - Criterion**

- The criterion is:

$$\begin{cases} \arg\max_{\beta_{1:K,1}} \sum_k \beta_{k,1}^2 \\ \text{s.t } \forall k, \beta_{k,1}^2 + \beta_{k,2}^2 = Var_k^{init} \end{cases}$$

- **Remarks**

- $C_{\phi_1}^{VaR}[L_Z; \alpha; \theta]$ is not a positive function of $\beta_{1:K,1}$

⇒ This criterion does not optimize the contribution of the first systematic factor.

Factor contribution interpretation - Rotation - 4/4

- **Method 3: estimation by iterations - version 1**

1) Calibration of the 1-factor model from initial correlation matrix C_0.

   We get a vector $(K \times 1)$: β.

2) Calibration of the 2-factor model from initial correlation matrix C_0.

   We get a matrix $(K \times 2)$: β.

3) Keep first colum of 1-factor model and adjust second column to replicate the variance given by the 2-factor model.

- **Method 3: estimation by iterations - version 2**

1) Calibration of the 2-factor model.

   We get a matrix $(K \times 2)$: β.

2) Calibration of the 1-factor model from the preceding 2-factor correlation matrix.

   We get a vector $K \times 1$: β.

3) Keep first colum of 1-factor model and adjust second column to replicate the variance given by the 2-factor model.

$\Rightarrow$ These methods are easy to implement and generalisable to any J-factor model.

Contents

Factor contribution estimators - 1/2

- **Loss**

$$L = \phi_\emptyset(L) + \phi_1(L;Z) + \phi_2 L;\epsilon + \phi_{1,2}(L;Z,\epsilon) = \sum_{i \in \{\emptyset,\{1\},\{2\},\{1,2\}\}} \phi_i \tag{23}$$

- **Monte Carlo framework**
  - Replications of MC iid random variables $L^{(n)}$, $n = \{1, \ldots, MC\}$.
  - We consider $\hat{VaR}_\alpha[L]$, the VaR_α estimator of L, based on the simulation.
- **Contribution to the VaR**

$$C^{VaR}_{\phi_i}[L;\alpha] = \mathbb{E}[\phi_i | L = VaR_\alpha] \Rightarrow \hat{C}^{VaR}_{\phi_i}[L;\alpha] = \frac{\sum_{n=1}^{MC} \phi_i^{(n)} 1_{\{L^{(n)} = \hat{VaR_\alpha}[L]\}}}{\sum_{n=1}^{MC} 1_{\{L^{(n)} = \hat{VaR_\alpha}[L]\}}} \tag{24}$$

- **Contribution to the CTE**

$$C^{CTE}_{\phi_i}[L;\alpha] = \mathbb{E}[\phi_i | L \geq VaR_\alpha] \Rightarrow \hat{C}^{CTE}_{\phi_i}[L;\alpha] = \frac{\sum_{n=1}^{MC} \phi_i^{(n)} 1_{\{L^{(n)} \geq \hat{VaR_\alpha}[L]\}}}{\sum_{n=1}^{MC} 1_{\{L^{(n)} \geq \hat{VaR_\alpha}[L]\}}} \tag{25}$$

- **Estimator convergence**
  - Results available in GLASSERMAN (2006) [19].

Factor contribution estimators - 2/2

- **Risk estimator convergence**
  - Since the loss distribution is discrete, the VaR is discrete.
  - VaR estimator is less stable than CTE estimator.

$\Rightarrow$ **Ongoing study: convergence of VaR estimator with discrete distribution loss.**

- **Contribution estimator convergence**
  - Estimator convergence is faster for long-only portfolio than for long-short portfolio due to higher variance induced by positive and negative credit risk expositions.
  - This phenomenon is more pronounced for:
    - High α level of the VaR.
    - Dispersed correlation matrix.
    - Dispersed PDs vector.

$\Rightarrow$ **Ongoing study: convergence of contribution estimators.**

Diversification and Hedge Portfolio

- **Default probabilities**
  - 1 year PD / constant LGD (=1)
  - Bloomberg Issuer Default Risk Methodology (DRSK)
- **Diversification portfolio**
  - Long Itraxx Europe.
  - Equi-weighted: total exposure is 1
- **Hedge portfolio**
  - Itraxx Europe - Short non-Financial issuers & Long Financial issuers.
  - Equi-weighted inside non-Financials set, Equi-weighted inside Financials set.
  - Weights chosen such that the total exposure is 0.
- **Questions**
  - How to calibrate the betas? Are factor models good approximations?
  - What impacts on the risk measures?

PD distribution	
Mean	0,10%
Std-Dev	0,10%
Max	0,61%
Min	0,03%
Median	0,05%

Contents

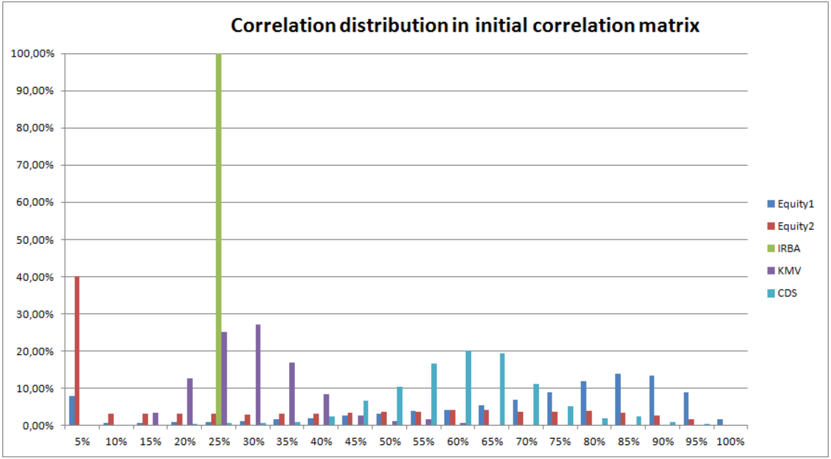
Initial correlation matrix: C_0 - 1/2

- **Listed equity correlations [Proposed by the FRTB Group for the Trading book]**
  - Equity 1: period 1, from 07/01/2008 to 07/01/2009.
  - Equity 2: period 2, from 09/02/2013 to 09/01/2014.
- **IRBA correlations [Banking book]**
  - One factor model: $\beta_{k,1} = \sqrt{\rho_k}$
  - Use of supervisory formula for correlation (function of PD):

$$\rho_k = 0.12 \times \frac{1 - \exp^{-50 \times PD_k}}{1 - \exp^{-50}} + 0.24 \times \left(1 - \frac{1 - \exp^{-50 \times PD_k}}{1 - \exp^{-50}}\right) \qquad (26)$$

  - Pairwise correlation: $Correl(X_k, X_l) = \sqrt{\rho_k \times \rho_l}$
- **KMV correlations**
  - Based on GCorr Moody's KMV methodology.
- **CDS (relative changes) correlations**
  - Period: 2013.

Initial correlation matrix: C_0 - 2/2



Calibrated J-factor model - 2/2

- **Spectral Gradient Projected Method**
  - Applied to previous initial correlations matrix.
  - Calibration of 1, 2 and 5 factor-models.

- **Main conclusions**
  - The approximation increases some correlations, decreases some others in comparison with the initial correlation matrix C_0.
  - Impact on the risk depends on the portfolio: long only portfolio (diversification portfolio), long-short portfolio (hedging portfolio).
  - The less dispersed correlation matrix are well replicated by factor models.
  - Dispersed correlations matrix requires more factors to be well replicated.
  - Example with correlation from Equity 2 and IRBA. (cf. next slides)

Equity correlation - Period 2

- **Initial matrix**
  - Low average pairwise correlation: 0.18.
  - Dispersed pairwise correlations. Standard Deviation : 0.46
- **Number of factors and correlation matrix replication?**

	Frobenius Norm	Average Correlation	Average Financial Correlation	Average Non Financial Correlation	Average Cross Correlation
Initial Matrix		0,179	0,232	0,191	0,151
1 factor	705,605	0,141	0,116	0,156	0,119
2 factors	285,119	0,144	0,190	0,155	0,118
5 factors	24,984	0,183	0,229	0,193	0,157

- **Remarks**
  - Need a 5-factor model to be closed to the initial correlation matrix (calibrated on equities).

IRBA correlation

- **Initial matrix**
  - Low average pairwise correlation: 0.24.
  - Homogeneous pairwise correlation. Standard Deviation : 0.01
- **Number of factors and correlation matrix replication?**

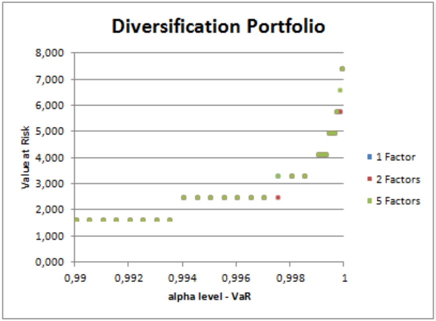
	Frobenius Norm	Average Correlation	Average Financial Correlation	Average Non Financial Correlation	Average Cross Correlation
Initial Matrix		0,24	0,26	0,24	0,23
1 factor	0,05	0,24	0,26	0,24	0,23
2 factors	0,07	0,24	0,26	0,24	0,23
5 factors	0,96	0,25	0,27	0,25	0,24

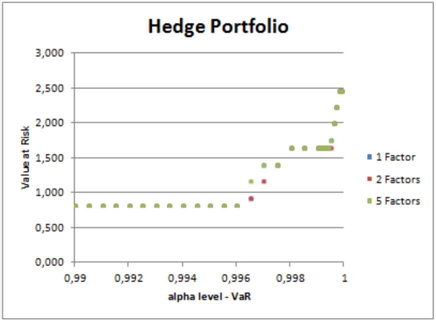
- **Remarks**
  - Of course, perfect fit to IRBA correlation matrix with 1F.
  - Froebenius norm between initial matrix and projected $\simeq 0$.

Contents

$VaR_\alpha(L)$ wrt α - Discrete loss disstribution

- **Example with IRBA correlation matrix**

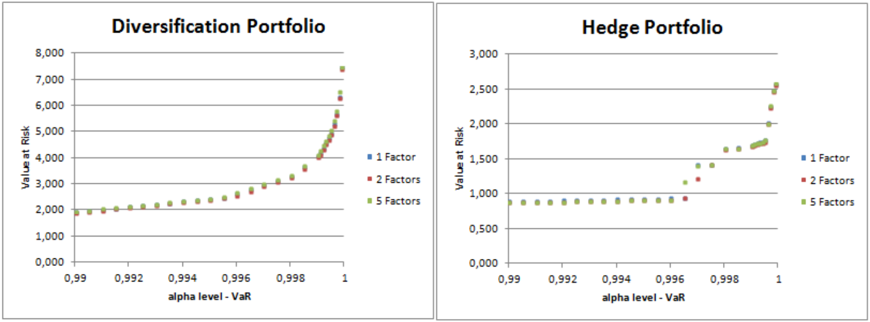




- **Remarks**
  - VaR is piecewise constant (discrete loss distribution).
  - Possible jumps in VaR for small changes in PDs and correlations.
  - Important consequences on risk and risk contribution estimates. (cf. contribution estimators)

$VaR_\alpha(L)$ wrt α - Smoothed loss disstribution - 1/2

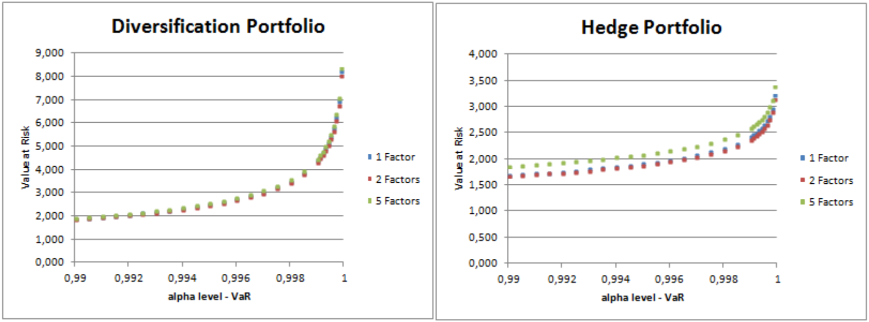
- **Mid-distribution function**



- **Remarks**
  - VaR for hedge portfolio is less smooth than the diversification portfolio

$VaR_\alpha(L)$ wrt α - Smoothed loss disstribution - 2/2

- **Kernel smoothing**



- **Remarks**

  - Smooth distribution for hedge portfolio does not replicate faithfully the initial discrete distribution.

Impacts on the risk - VaR99.9 - 1/2

- **Number of factors and risk in function of the initial correlation matrix?**

		Diversification Portfolio				Hedge Portfolio			
Initial Correlations	Risk	Initial C_0	1F	2F	5F	Initial C_0	1F	2F	5F
Equity1 (μ=0,64,σ=0,31)	VaR99.9[L]	0,198	0,190	0,190	0,198	0,054	0,042	0,042	0,052
	Relative diff. wrt initial matrix C_0		-4,2%	-4,2%	0,0%		-21,1%	-21,1%	-3,8%
Equity2 (μ=0,18,σ=0,46)	VaR99.9[L]	0,099	0,083	0,091	0,099	0,028	0,017	0,025	0,026
	Relative diff. wrt initial matrix C_0		-16,7%	-8,3%	0,0%		-41,6%	-12,4%	-8,4%
IRBA (μ=0,23,σ=0,01)	VaR99.9[L]	0,041	0,041	0,041	0,041	0,017	0,017	0,017	0,017
	Relative diff. wrt initial matrix C_0		0,0%	0,0%	0,0%		0,0%	0,0%	0,0%
KMV (μ=0,27,σ=0,08)	VaR99.9[L]	0,058	0,050	0,058	0,058	0,031	0,025	0,031	0,031
	Relative diff. wrt initial matrix C_0		-14,3%	0,0%	0,0%		-19,2%	0,0%	0,0%
CDS (μ=0,57,σ=0,11)	VaR99.9[L]	0,140	0,132	0,132	0,132	0,058	0,032	0,058	0,055
	Relative diff. wrt initial matrix C_0		-5,9%	-5,9%	-5,9%		-45,1%	0,0%	-4,1%

- **Remarks**
  - μ represents the average pairwise correlation in the considered correlation matrix, and σ, its standard deviation.
  - Notional $=1$ for diversification portfolio, whereas the sum of EAD_k is equal to 0 for hedge portfolio.
  - 1F stands for 1-factor calibrated model.
  - Risk measure estimators are sensitive to the discrete feature of the loss distribution.

Impacts on the risk - VaR99.9 - 2/2

- **General remarks**
  - 5-factor model better replicates initial risk.
  - Hedge benefit: risk in the hedge porfolio is smaller than in the diversification portfolio.
  - Hedge Portfolio: loss distribution is complex - need more factors for reflecting the risk.
  - Hedge Portfolio: 1-factor model seems inappropriate for risk measurement.
  - Hedge Portfolio: risk measures obtained with 1 and 2-factor model, calibrated on an (initial) equity correlation matrix, are far from the true (i.e. with non-constrained initial correlation matrix) risk measure.
- **Homogeneous initial correlation matrix**
  - Two factors are enough to reproduce the true initial risk measure.
- **Dispersed initial correlation matrix**
  - Bad risk estimates, even for the diversification portfolio.
  - Impact of input correlations on hedge portfolio: dispersed correlations can lead to clustering of default on long exposures, not mitigated by default on shorts.
- **IRBA initial correlation matrix**
  - 1F (hopefully!) fully explains VaR level for both diversification and hedge portfolios.

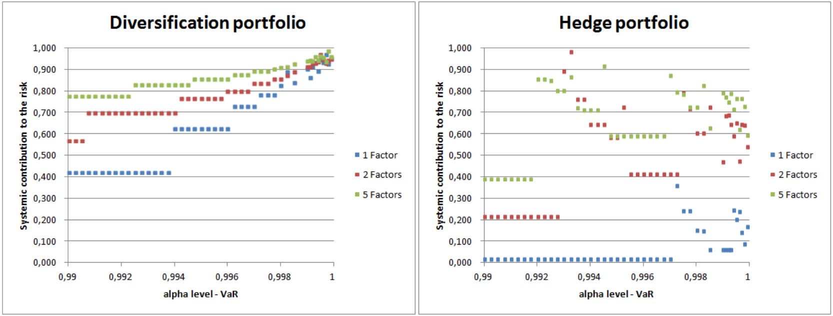
$\Rightarrow$ **Ongoing study: Gaussian vectors stochastic orders and risks.**

Contents

Systematic risk contribution in the tail

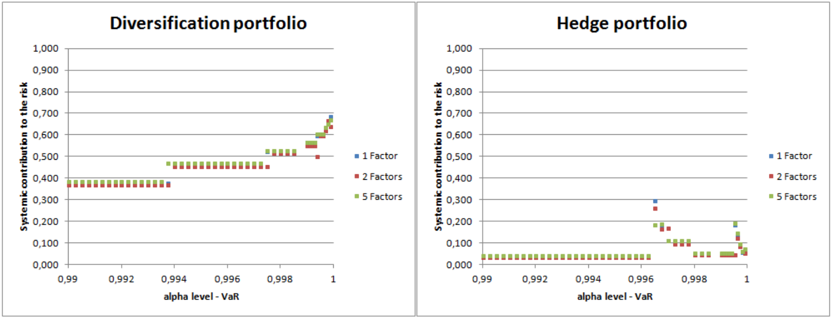
- **Impact of the Systematic risk**
  - We consider $\alpha \rightarrow C_{\phi_1}^{VaR}[L;\alpha]$, where ϕ_1 is the random variable (Hoeffding decomposition) that stands for "Systematic factors".
  - This mapping is piecewise constant since $VaR_\alpha[L]$ is piecewise constant in α.
- **General remarks**
  - For diversification portfolio: the mapping $\alpha \rightarrow C_{\phi_1}^{VaR}[L;\alpha]$ is increasing.
  - The systematic contribution is a function of: the PD, the portfolio (long-only or long-short), the correlation, the α level.
  - Ceteris paribus, loss clusters are generated by:
    - High pairwise correlations.
    - High PDs.
  - Those clusters increased the risk measure for the diversification portfolio.
  - The impact on the hedge portfolio is less obvious since gains are possible thanks to short position on credit risk.

Systematic risk contribution in the tail - Equity correlations - Period 2



- **Remarks**
  - The large pool approximation may be convenient for VaR99.9 since the systematic contribution is close to 1.
  - 1-factor model seems more stable for the Hedge portfolio.

Systematic risk contribution in the tail - IRBA correlations



- **Remarks**
  - The systematic risk is less important due to low correlation and homogeneous pairwise correlations.
  - Error convergence for the systematic risk contribution in the Hedge Portfolio is less important since the VaR function is less steepened.

Ongoing researches and extensions

- **Working with discrete loss distribution?**
  - Euler marginal contribution for discrete loss distributions.
  - Convergence of VaR estimator with discrete distribution loss.
  - Convergence of contribution estimators.
- **Model specification?**
  - Best criterion to maximise the contribution of ϕ_1, the systematic part, to $VaR_\alpha[L_Z]$?
- **Gaussian copula and tail dependence ?**
  - Use of other dependence structures (elliptical distributions) ?
  - Gaussian vectors stochastic orders and risks?
- **Risk allocation rules at the micro and the macro level ?**
- **Standardisation of risk models may lead to increased systematic risk**
- **Consistency with regulatory constraints ? Calibration of extra-parameters ? improved hedging efficiency ?**

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